Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Anal. Appl. 321 (2006) 705–721
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Multiplicity of positive solutions for a class of elliptic equations in divergence form

Giovany M. Figueiredo [a], Marcelo F. Furtado [b,*,1]

[a] *Departamento de Matemática, Universidade Federal do Pará, 66075-110 Belém-PA, Brazil*
[b] *IMECC–UNICAMP, Cx. Postal 6065, 13083-970 Campinas-SP, Brazil*

Received 8 June 2005
Available online 26 September 2005
Submitted by P.J. McKenna

Abstract

We prove results concerning the existence and multiplicity of positive solutions for the quasilinear equation

$$-\operatorname{div}\big(a(\varepsilon x)|\nabla u|^{p-2}\nabla u\big) + |u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N, \qquad u \in W^{1,p}\big(\mathbb{R}^N\big),$$

where $2 \leqslant p < N$, a is a positive potential and f is a superlinear function. We relate the number of solutions with the topology of the set where a attains its minimum. The results are proved by using minimax theorems and Ljusternik–Schnirelmann theory.

Keywords: Positive solutions; Ljusternik–Schnirelmann theory; Quasilinear problems

1. Introduction

The purpose of this article is to investigate the existence and multiplicity of solutions of the following quasilinear problem:

$$\begin{cases} -\operatorname{div}(a(\varepsilon x)|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = f(u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \quad u(x) > 0 & \text{for all } x \in \mathbb{R}^N, \end{cases} \tag{P_ε}$$

* Corresponding author.
E-mail addresses: giovany@ufpa.br (G.M. Figueiredo), furtado@ime.unicamp.br (M.F. Furtado).
1 Supported by post-doctoral grant 04/09232-2 from FAPESP/Brazil.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.08.084

where $\varepsilon > 0$, $2 \leqslant p < N$, $f:\mathbb{R} \to \mathbb{R}$ is a C^1-function and the potential a satisfies

(a_1) $a:\mathbb{R}^N \to \mathbb{R}$ is continuous and

$$0 < a_0 := \inf_{x\in\mathbb{R}^N} a(x) < a_\infty := \liminf_{|x|\to\infty} a(x).$$

This kind of hypothesis was introduced by Rabinowitz [23] in the study of a nonlinear Schrödinger equation.

Since we are looking for positive solutions, we suppose that

(f_1) $f(s) = 0$ for all $s < 0$.

Moreover, we assume the following growth conditions at the origin and at infinity:

(f_2) $f(s) = o(s^{p-1})$ as $s \to 0^+$,
(f_3) there exists $p < q < p^* = Np/(N-p)$ such that

$$\lim_{s\to\infty} \frac{f(s)}{s^{q-1}} = 0.$$

We call $u \in W^{1,p}(\mathbb{R}^N)$ a weak solution of the equation in (P_ε) if it verifies

$$\int_{\mathbb{R}^N} \left(a(\varepsilon x)|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + |u|^{p-2}u\varphi\right) \mathrm{d}x = \int_{\mathbb{R}^N} f(u)\varphi \, \mathrm{d}x,$$

for all $\varphi \in W^{1,p}(\mathbb{R}^N)$. If we denote by $F(t) = \int_0^t f(s)\,\mathrm{d}s$ the primitive of f, conditions (f_1)–(f_3) imply that the functional $I_\varepsilon : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I_\varepsilon(u) = \frac{1}{p}\int_{\mathbb{R}^N} \left(a(\varepsilon x)|\nabla u|^p + |u|^p\right) \mathrm{d}x - \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x$$

is well defined. Moreover, $I_\varepsilon \in C^2(W^{1,p}(\mathbb{R}^N), \mathbb{R})$ and the weak solutions of (P_ε) are precisely the positive critical points of I_ε.

In order to obtain such critical points, we use minimax theorems and Ljusternik–Schnirelmann theory. As it is known, this kind of theory is based on the existence of a linking structure and on deformation lemmas [6]. In general, to derive such deformation results, it is supposed that the functional I_ε satisfies some compactness condition. In this article, we use the classical Palais–Smale condition (see Section 2). Related with this condition we suppose that f verifies the well-known Ambrosetti–Rabinowitz superlinear condition, that is,

(f_4) there exists $\theta > p$ such that

$$0 < \theta F(s) \leqslant sf(s) \quad \text{for all } s > 0.$$

Finally, in order to localize the minimax levels of the functional I_ε, we suppose the following monotonicity condition for f:

(f_5) the function $s \mapsto f(s)/s^{p-1}$ is increasing for $s > 0$.

We recall that a solution u_0 of (P_ε) is called ground state solution if it possesses minimum energy between all solutions, that is,

$$I_\varepsilon(u_0) = \min\big\{I_\varepsilon(u):\ u \text{ is a solution of } (P_\varepsilon)\big\}.$$

In our first result we obtain, for $\varepsilon > 0$ small enough, the existence of a ground state solution of (P_ε).

Theorem 1.1. *Suppose that $2 \leqslant p < N$, a satisfies (a_1) and the function f satisfies (f_1)–(f_5). Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, the problem (P_ε) has a ground state solution.*

In the paper we also relate the number of solutions of (P_ε) with the topology of the set of minima of the potential a. In order to present our result, we introduce the set of global minima of a, given by

$$M = \big\{x \in \mathbb{R}^N:\ a(x) = a_0\big\}.$$

Note that, in view of (a_1), the set M is compact. For any $\delta > 0$, let us denote by $M_\delta = \{x \in \mathbb{R}^N:\ \operatorname{dist}(x, M) \leqslant \delta\}$ the closed δ-neighborhood of M.

We recall that, if Y is a closed set of a topological space X, $\operatorname{cat}_X(Y)$ is the Ljusternik–Schnirelmann category of Y in X, namely the least number of closed and contractible sets in X which cover Y. In our multiplicity result we assume a condition stronger than (f_5) and prove the following theorem.

Theorem 1.2. *Suppose that $2 \leqslant p < N$, a satisfies (a_1), the function f satisfies (f_1)–(f_4) and*

$(\widehat{f_5})$ *there exist $\sigma \in (p, p^*)$ and $C_\sigma > 0$ such that*

$$f'(s)s - (p-1)f(s) \geqslant C_\sigma s^{\sigma-1} \quad \text{for all } s > 0.$$

Then, for any $\delta > 0$ given, there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the problem (P_ε) has at least $\operatorname{cat}_{M_\delta}(M)$ solutions.

In the proof of Theorem 1.2 we apply a technique which was introduced by Benci and Cerami in [8]. It consists in making a comparison between the category of some sublevel sets of the energy functional I_ε, constrained on some appropriated manifold, and the category of the set M.

Several physical phenomena related to equilibrium of continuous media are modeled by the problem

$$-\operatorname{div}\big(c(x)\nabla u\big) = g(x, u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is a domain of $\mathbb{R}^N$, g is a regular function and c is a nonnegative weight. In order to be able to deal with media which possibly are somewhere "perfect" insulators or "perfect" conductors (see [16]) the coefficient c is allowed to vanish somewhere or to be unbounded.

There is a quite extensive literature about the regularity and spectral theory of the above problem when $g(x, u) \equiv g(u)$ is a linear function (see [5,7,10,15,20] and references therein). Concerning the nonlinear problem we can cite the papers [11,12,21,22,25].

In [13], Chabrowski studied the problem

$$-\operatorname{div}\big(c(x)\nabla u\big) + \lambda u = K(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N \tag{1.2}$$

with $\lambda > 0$, $2 < q < 2^*$ and $c \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfying

$$0 \leqslant c(x) \leqslant \lim_{|x|\to\infty} c(x),$$

and being positive in the exterior of some ball $B_R(0)$. By using minimization arguments he obtained a nonzero solution of (1.2) belonging in some appropriated Sobolev space. In his result, it was also supposed an integrability condition for $c(x)$ and that $K \in L^\infty(\mathbb{R}^N)$ verifies either $K(x) \geqslant \lim_{|x|\to\infty} K(x)$ or K is periodic.

More recently, Lazzo [17] considered Eq. (1.2) with $K \equiv 1$ and the function c satisfying the condition (a_1) with $a(x)$ replaced by $c(x)$. She proved that, for any $\delta > 0$ given, there exists $\lambda_\delta > 0$ such that (1.2) possesses at least $\operatorname{cat}_{M_\delta}(M)$ positive solutions for any $\lambda > \lambda_\delta$.

The results of this paper extend those of [17] in two senses: first because we deal with $2 \leqslant p < N$ instead of $p = 2$, and second because, in general, our nonlinearity f is not a power. The main problem in considering $2 < p < N$ is that we need to work in a Sobolev space without Hilbertian structure. Thus, some calculations that involve the Brezis–Lieb lemma are more difficult. Since $f(u)$ may be different from $|u|^{q-2}u$, we cannot use the same arguments developed in [17]. Thus, we adapt some ideas from [3,4] and make a detailed study of the behavior of the functional I_ε restricted to its Nehari manifold. However, we would like to emphasize that our results seem to be new even in the semilinear case $p = 2$.

It is worthwhile to mention that our last result is closely related to those presented by Pomponio and Secchi in [22]. There, the authors studied positive solutions for the problem

$$-\operatorname{div}\big(J(\varepsilon x)\nabla u\big) + V(\varepsilon x)u = f(u) \quad \text{in } \mathbb{R}^N,$$

where $\varepsilon > 0$, J is a symmetric uniformly elliptic matrix and V is a positive potential. They proved some multiplicity results in the same spirit of Theorem 1.2 (see [22, Section 6]). We finally mention the paper of Cingolani and Lazzo [14], where the authors considered positive solutions for the Schrödinger equation

$$-\Delta u + a(\varepsilon x)u = |u|^{q-2}u \quad \text{in } \mathbb{R}^N$$

with $\varepsilon > 0$, $2 < q < 2^*$ and a satisfying (a_1), and obtained a multiplicity result similar to Theorem 1.2.

The paper is organized as follows. In Section 2 we present the abstract framework of the problem as well as some results about the autonomous problem. In Section 3 we obtain some compactness properties of the functional I_ε. Theorem 1.1 is proved in Section 4 and the final Section 5 is devoted to the proof of Theorem 1.2.

2. The variational framework

Throughout the paper we suppose that the functions a and f satisfy the conditions (a_1) and (f_1)–(f_4), respectively. Since $(\widehat{f_5})$ implies (f_5), we also assume hereafter that the function $s \mapsto f(s)/s^{p-1}$ is increasing for $s > 0$. We write only $\int u$ instead of $\int_{\mathbb{R}^N} u(x)\,\mathrm{d}x$.

For any $\varepsilon > 0$, let X_ε be the Sobolev space $W^{1,p}(\mathbb{R}^N)$ endowed with the norm

$$\|u\|_\varepsilon = \left\{ \int \big(a(\varepsilon x)|\nabla u|^p + |u|^p\big) \right\}^{1/p}.$$

Since the potential a is bounded and positive, the above norm is equivalent to the standard norm of $W^{1,p}(\mathbb{R}^N)$.

As stated in the introduction, we will look for critical points of the C^2-functional $I_\varepsilon : X_\varepsilon \to \mathbb{R}$ given by

$$I_\varepsilon(u) = \frac{1}{p}\int \big(a(\varepsilon x)|\nabla u|^p + |u|^p\big) - \int F(u),$$

where $F(t) = \int_0^s f(s)\,\mathrm{d}s$. We introduce the Nehari manifold of I_ε by setting

$$\mathcal{N}_\varepsilon = \big\{u \in X_\varepsilon \setminus \{0\}:\ \big\langle I'_\varepsilon(u), u\big\rangle = 0\big\} = \left\{u \in X_\varepsilon \setminus \{0\}:\ \|u\|_\varepsilon^p = \int f(u)u\right\}$$

and consider the following minimization problem:

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u).$$

We present now some properties of c_ε and $\mathcal{N}_\varepsilon$. For the proofs we refer to [27, Chapter 4]. First we observe that, for any $u \in X_\varepsilon \setminus \{0\}$ there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$. The maximum of the function $t \mapsto I_\varepsilon(tu)$ for $t \geqslant 0$ is achieved at $t = t_u$ and the function $u \mapsto t_u$ is continuous from $X_\varepsilon \setminus \{0\}$ to $(0,\infty)$. Given $\delta > 0$, we can use (f_1)–(f_3) to obtain $C_\delta > 0$ such that

$$\big|f(s)\big| \leqslant \delta|s|^{p-1} + C_\delta|s|^{q-1} \quad \text{for all } s \in \mathbb{R}. \tag{2.1}$$

Since $q > p$, the above estimate and standard calculations imply that 0 is a local minimum of I_ε. Moreover, by (f_1) and (f_4), we have that

$$F(s) \geqslant C|s|^\theta \quad \text{for all } s \in \mathbb{R}, \tag{2.2}$$

and some $C > 0$. Hence,

$$I_\varepsilon(tu) \leqslant \frac{t^p}{p}\|u\|_\varepsilon^p - Ct^\theta \int |u|^\theta,$$

and we conclude that $I_\varepsilon(tu) \to -\infty$ as $t \to \infty$, for any $u \in X_\varepsilon \setminus \{0\}$.

The above considerations show that I_ε satisfies the geometry of the mountain pass theorem. By using (f_5) and arguing as in [27, Theorem 4.2], we can prove that c_ε is positive, it coincides with the mountain pass level of I_ε and satisfies

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} I_\varepsilon\big(\gamma(t)\big) = \inf_{u \in X_\varepsilon \setminus \{0\}} \max_{t \geqslant 0} I_\varepsilon(tu) > 0, \tag{2.3}$$

where $\Gamma_\varepsilon = \{\gamma \in C([0,1], X_\varepsilon):\ \gamma(0) = 0,\ I_\varepsilon(\gamma(1)) < 0\}$.

As we will see, it is important to compare c_ε with the minimax level of the autonomous problem

$$\begin{cases} -\mu\,\mathrm{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = f(u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \quad u(x) > 0 & \text{for all } x \in \mathbb{R}^N. \end{cases} \tag{AP_μ}$$

Denote by $\|\cdot\|_{W_\mu}$ the following norm in $W^{1,p}(\mathbb{R}^N)$:

$$\|u\|_{W_\mu} = \left\{\int \big(\mu|\nabla u|^p + |u|^p\big)\right\}^{1/p}.$$

It is well defined and it is equivalent to the standard norm of $W^{1,p}(\mathbb{R}^N)$. The solutions of (AP_μ) are precisely the positive critical points of the functional $E_\mu : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$E_\mu(u) = \frac{1}{p}\int \big(\mu|\nabla u|^p + |u|^p\big) - \int F(u).$$

Let $\mathcal{M}_\mu$ be the Nehari manifold of E_μ given by

$$\mathcal{M}_\mu = \{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}: \langle E'_\mu(u), u\rangle = 0\}$$

and define $m(\mu)$ by setting

$$m(\mu) = \inf_{u \in \mathcal{M}_\mu} E_\mu(u).$$

The number $m(\mu)$ and the manifold $\mathcal{M}_\mu$ have properties similar to those of c_ε and $\mathcal{N}_\varepsilon$. We devote the rest of this section to show that $m(\mu)$ is attained by a positive function.

We start by recalling the definition of the Palais–Smale condition. So, let V be a Banach space, $\mathcal{V}$ be a C^1-manifold of V and $I : V \to \mathbb{R}$ a C^1-functional. We say that $I|_{\mathcal{V}}$ satisfies the Palais–Smale condition at level c ($(\mathrm{PS})_c$ for short) if any sequence $(u_n) \subset \mathcal{V}$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_* \to 0$ contains a convergent subsequence. Here, we are denoting by $\|I'(u)\|_*$ the norm of the derivative of I restricted to $\mathcal{V}$ at the point u (see [27, Section 5.3]).

Lemma 2.1. *Let $(u_n) \subset W^{1,p}(\mathbb{R}^N)$ be a $(\mathrm{PS})_c$ sequence for E_μ. Then we have either*

(i) $\|u_n\|_{W_\mu} \to 0$, *or*
(ii) *there exist a sequence $(y_n) \subset \mathbb{R}^N$ and constants $R,\ \gamma > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^p \geqslant \gamma > 0.$$

Proof. Suppose that (ii) does not occur. Condition (f_4) and standard calculations show that (u_n) is bounded in $W^{1,p}(\mathbb{R}^N)$. Thus, it follows from a result of P.L. Lions [19, Lemma I.1] that $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for all $p < s < p^*$. Given $\delta > 0$, we can use (2.1) to get

$$0 \leqslant \left| \int f(u_n)u_n \right| \leqslant \delta \int |u_n|^p + C_\delta \int |u_n|^q.$$

Since (u_n) is bounded in $L^p(\mathbb{R}^N)$, $u_n \to 0$ in $L^q(\mathbb{R}^N)$ and δ is arbitrary, we conclude that $\int f(u_n)u_n \to 0$. Recalling that $\langle E'_\mu(u_n), u_n\rangle \to 0$, we get

$$\|u_n\|^p_{W_\mu} = \int f(u_n)u_n + o_n(1) \to 0.$$

Hence (i) holds and the lemma is proved. □

Proposition 2.2. *Suppose that $2 \leqslant p < N$, a satisfies (a_1) and the function f satisfies (f_1)–(f_5). Then, for any $\mu > 0$, the problem (AP_μ) has a ground state solution.*

Proof. Conditions (f_1)–(f_4) imply that E_μ satisfies the mountain pass geometry. Thus, there exists a sequence $(u_n) \subset W^{1,p}(\mathbb{R}^N)$ such that

$$E_\mu(u_n) \to m(\mu) \quad \text{and} \quad E'_\mu(u_n) \to 0.$$

Since (u_n) is bounded, up to a subsequence, $u_n \rightharpoonup u$ weakly in $W^{1,p}(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. By adapting standard calculations [28] (see also [24, Corollary 3.7]), we can obtain a subsequence, still denoted by (u_n), such that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{a.e. } x \in \mathbb{R}^N,$$
$$|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i} \rightharpoonup |\nabla u|^{p-2}\frac{\partial u}{\partial x_i} \quad \text{weakly in } \big(L^p\big(\mathbb{R}^N\big)\big)^*, \quad 1 \leqslant i \leqslant N.$$

The weak convergence of (u_n), the above expression and the subcritical growth of f imply that $E'_\mu(u) = 0$.

Suppose that $u \neq 0$. Then $u \in \mathcal{M}_\mu$ and, if we denote by $u^\pm = \max\{\pm u, 0\}$ the positive (negative) part of u, we get

$$0 = \langle E'_\mu(u), u^- \rangle = \|u^-\|^p_{W_\mu} - \int f(u)u^- = \|u^-\|^p_{W_\mu}$$

and therefore $u \geqslant 0$ in $\mathbb{R}^N$. Adapting arguments from [18, Theorem 1.11], we conclude that $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $0 < \alpha < 1$, and therefore it follows from Harnack's inequality [26] that $u(x) > 0$ for all $x \in \mathbb{R}^N$.

In order to prove that $E_\mu(u) = m(\mu)$, it suffices to recall that $u \in \mathcal{M}_\mu$ and apply Fatou's lemma to get

$$\begin{aligned} m(\mu) \leqslant E_\mu(u) &= E_\mu(u) - \frac{1}{p}\langle E'_\mu(u), u\rangle = \int \left(\frac{1}{p} f(u) - F(u)\right) \\ &\leqslant \liminf_{n\to\infty} \int \left(\frac{1}{p} f(u_n) - F(u_n)\right) \\ &= \liminf_{n\to\infty}\left(E_\mu(u_n) - \frac{1}{p}\langle E'_\mu(u_n), u_n\rangle\right) = m(\mu). \end{aligned}$$

We now consider the case $u = 0$. Since $m(\mu) > 0$ and E_μ is continuous, we cannot have $\|u_n\|_{W_\mu} \to 0$. Thus, we obtain from Lemma 2.1 a sequence $(y_n) \subset \mathbb{R}^N$ and constants $R, \gamma > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^p \geqslant \gamma > 0.$$

If we define $v_n(x) = u_n(x + y_n)$ we can use the invariance of $\mathbb{R}^N$ by translations to conclude that $E_\mu(v_n) \to m(\mu)$ and $E'_\mu(v_n) \to 0$. Moreover, up to a subsequence, $v_n \rightharpoonup v$ weakly in $W^{1,p}(\mathbb{R}^N)$ and $v_n \to v$ in $L^p(B_R(0))$, with v being a critical point of E_μ. Since

$$\int_{B_R(0)} |v|^p = \liminf_{n\to\infty} \int_{B_R(0)} |v_n|^p = \liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^p \geqslant \gamma > 0,$$

we conclude that $v \neq 0$, and the lemma follows as in the first part of the proof. □

Remark 2.3. The above proposition and the same argument employed in [4, Lemma 10] show that the function $\mu \mapsto m(\mu)$ is increasing for $\mu > 0$.

We finish this section by noting that $I_\varepsilon(u) \geqslant E_{a_0}(u)$ for all $u \in W^{1,p}(\mathbb{R}^N)$. Hence, the characterization of c_ε (and of $m(a_0)$) given by (2.3) implies that $c_\varepsilon \geqslant m(a_0) > 0$ for any $\varepsilon > 0$. Thus, we can easily conclude that there exists $r > 0$, independent of ε, such that

$$\|u\|_\varepsilon \geqslant r > 0 \quad \text{for any } \varepsilon > 0,\ u \in \mathcal{N}_\varepsilon. \tag{2.4}$$

3. A compactness condition

In this section we obtain some compactness properties of the functional I_ε. We start by noting that, if (u_n) is a $(\mathrm{PS})_c$ sequence for I_ε then it is bounded in X_ε. In view of (f_1) we have

$$\langle I_\varepsilon'(u_n), u_n^-\rangle = \|u_n^-\|_\varepsilon^p - \int f(u_n)u_n^- = \|u_n^-\|_\varepsilon^p.$$

The boundedness of (u_n^-) and the above expression imply that $\|u_n^-\|_\varepsilon \to 0$. Thus, we can easily compute

$$I_\varepsilon(u_n) = I_\varepsilon(u_n^+) + o_n(1) \quad \text{and} \quad I_\varepsilon'(u_n) = I_\varepsilon'(u_n^+) + o_n(1),$$

where $o_n(1)$ denotes a quantity that goes to 0 as $n \to \infty$. This shows that (u_n^+) is also a $(\mathrm{PS})_c$ sequence. Since we are always interested in the existence of convergent subsequence, we will assume hereafter that u_n is nonnegative. The same will be done for the autonomous functional E_μ.

Lemma 3.1. *Let $(v_n) \subset X_\varepsilon$ be a $(\mathrm{PS})_d$ sequence for I_ε such that $v_n \rightharpoonup 0$ weakly in X_ε. Then,*

$$\limsup_{n\to\infty} \int \big(s_n a_\infty - a(\varepsilon x)\big)|\nabla v_n|^p \leqslant 0$$

for any sequence $(s_n) \subset \mathbb{R}$ satisfying $s_n \to 1$.

Proof. Let $C > 0$ be such that $\int |\nabla v_n|^p \leqslant C$. Since $s_n \to 1$ and

$$\int \big(s_n a_\infty - a(\varepsilon x)\big)|\nabla v_n|^p = \int \big(a_\infty - a(\varepsilon x)\big)|\nabla v_n|^p + a_\infty(s_n - 1)\int |\nabla v_n|^p,$$

it suffices to consider the case $s_n \equiv 1$.

Given $\delta > 0$, we can use condition (a_1) to obtain $R = R(\delta) > 0$ such that $a(\varepsilon x) \geqslant a_\infty - \delta$ for any $|x| \geqslant R$. We claim that $\int_{B_R(0)} |\nabla v_n|^p \to 0$ as $n \to \infty$. Assuming the claim, we get

$$\int \big(a_\infty - a(\varepsilon x)\big)|\nabla v_n|^p \leqslant \int_{B_R(0)} \big(a_\infty - a(\varepsilon x)\big)|\nabla v_n|^p + \delta C = o_n(1) + \delta C$$

for any $\delta > 0$, and the lemma follows.

In order to prove the claim, we take $\psi \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $\psi \equiv 1$ in $B_R(0)$ and $\operatorname{supp}\psi \subset B_{2R}(0)$. By using condition (a_1) and the definition of I_ε and ψ, we get

$$a_0 \int_{B_R(0)} |\nabla v_n|^p \leqslant \int_{B_R(0)} a(\varepsilon x)|\nabla v_n|^p\psi \leqslant \int a(\varepsilon x)|\nabla v_n|^p\psi = A_n + B_n, \tag{3.1}$$

where

$$A_n = -\int a(\varepsilon x)|\nabla v_n|^{p-2} v_n(\nabla v_n \cdot \nabla\psi)$$

and

$$B_n = \langle I_\varepsilon'(v_n), v_n\psi\rangle - \int |v_n|^p\psi + \int f(v_n)v_n\psi.$$

The boundedness of a and Hölder's inequality imply that

$$|A_n| \leqslant C_1\left(\int |\nabla v_n|^p\right)^{(p-1)/p}\left(\int |v_n|^p|\nabla\psi|^p\right)^{1/p} \leqslant C_2\left(\int_{B_{2R}(0)} |v_n|^p|\nabla\psi|^p\right)^{1/p}.$$

Since $v_n \to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ and ψ is regular, we conclude that $A_n \to 0$. Recalling that (v_n) is a Palais–Smale sequence, we can use the boundedness of $(v_n\psi)$, the convergence of v_n in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ and (2.1) as in the proof of Lemma 2.1 to conclude that $B_n \to 0$. It follows from (3.1) that $\int_{B_R(0)} |\nabla v_n|^p \to 0$. $\square$

Lemma 3.2. *Let $(v_n) \subset X_\varepsilon$ be a* $(\mathrm{PS})_d$ *sequence for I_ε such that $v_n \rightharpoonup 0$ weakly in X_ε. If $v_n \nrightarrow 0$ in X_ε, then $d \geqslant m(a_\infty)$.*

Proof. Let $(t_n) \subset (0, +\infty)$ be such that $(t_n v_n) \subset \mathcal{M}_{a_\infty}$. We start by proving that

$$t_0 = \limsup_{n\to\infty} t_n \leqslant 1.$$

Arguing by contradiction, we suppose that there exist $\delta > 0$ and a subsequence, which we also denote by (t_n), such that

$$t_n \geqslant 1 + \delta \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Since (v_n) is bounded in X_ε, $\langle I'_\varepsilon(v_n), v_n\rangle \to 0$, that is,

$$\int \left(a(\varepsilon x)|\nabla v_n|^p + |v_n|^p\right) = \int f(v_n)v_n + o_n(1).$$

Moreover, recalling that $(t_n v_n) \subset \mathcal{M}_{a_\infty}$, we get

$$t_n^p \int \left(a_\infty|\nabla v_n|^p + |v_n|^p\right) = \int f(t_n v_n)(t_n v_n).$$

Since $s \mapsto f(s)/s^{p-1}$ is increasing, we can use the above equalities and (3.2) to get

$$\begin{aligned} \int \left(a_\infty - a(\varepsilon x)\right)|\nabla v_n|^p &= \int \left(\frac{f(t_n v_n)}{(t_n v_n)^{p-1}} - \frac{f(v_n)}{(v_n)^{p-1}}\right)(v_n)^p + o_n(1) \\ &\geqslant \int \left(\frac{f((1+\delta)v_n)}{((1+\delta)v_n)^{p-1}} - \frac{f(v_n)}{(v_n)^{p-1}}\right)(v_n)^p + o_n(1). \end{aligned} \tag{3.3}$$

Since $\|v_n\|_\varepsilon \nrightarrow 0$, we can argue as in the proof of Lemma 2.1 to obtain $(y_n) \subset \mathbb{R}^N$ and $R, \gamma > 0$ such that

$$\int_{B_R(y_n)} |v_n|^p \geqslant \gamma > 0. \tag{3.4}$$

If $\tilde{v}_n(x) = v_n(x + y_n)$, then there exists a nonnegative function $\tilde{v}$ such that, up to a subsequence, $\tilde{v}_n \rightharpoonup \tilde{v}$ weakly in X_ε, $\tilde{v}_n \to \tilde{v}$ in $L^p(B_R(0))$ and $\tilde{v}_n(x) \to \tilde{v}(x)$ a.e. in $\mathbb{R}^N$. Moreover, in view of (3.4), there exists a subset $\Omega \subset B_R(0)$ with positive measure such that $\tilde{v}(x) > 0$ for a.e. $x \in \Omega$.

On the other hand, by changing variables in (3.3), we can use Fatou's lemma and Lemma 3.1 to obtain

$$\int \left(\frac{f((1+\delta)\tilde{v})}{((1+\delta)\tilde{v})^{p-1}} - \frac{f(\tilde{v})}{(\tilde{v})^{p-1}}\right)\tilde{v}^p \leqslant 0.$$

Since the integrand is nonnegative, the above expression contradicts the positiveness of $\tilde{v}$ in Ω. This contradiction shows that $t_0 \leqslant 1$, as claimed.

If $t_0 < 1$ we may suppose, without loss of generality, that $t_n < 1$ for all $n \in \mathbb{N}$. Conditions (f_1) and (f_5) imply that the function $s \mapsto \frac{1}{p} f(s)s - F(s)$ is nondecreasing. Thus,

$$\begin{aligned}m(a_\infty) &\leqslant E_{a_\infty}(t_n v_n) - \frac{1}{p}\langle E'_{a_\infty}(t_n v_n), t_n v_n\rangle = \int \left\{\frac{1}{p} f(t_n v_n)(t_n v_n) - F(t_n v_n)\right\} \\ &\leqslant \int \left\{\frac{1}{p} f(v_n)(v_n) - F(v_n)\right\} = I_\varepsilon(v_n) - \frac{1}{p}\langle I'_\varepsilon(v_n), v_n\rangle = d + o_n(1).\end{aligned}$$

Taking the limit, we conclude that $m(a_\infty) \leqslant d$.

If $t_0 = 1$ then, up to a subsequence, we may suppose that $t_n \to 1$. Thus,

$$\begin{aligned}m(a_\infty) &\leqslant E_{a_\infty}(t_n v_n) - I_\varepsilon(v_n) + I_\varepsilon(v_n) \\ &= \frac{1}{p}\int \left(t_n^p a_\infty - a(\varepsilon x)\right)|\nabla v_n|^p + \int \left(F(v_n) - F(t_n v_n)\right) + d + o_n(1).\end{aligned}$$

By using an straightforward application of the mean value theorem, (2.1) and the Lebesgue theorem we can check that $\int (F(v_n) - F(t_n v_n)) = o_n(1)$. Hence, the above expression and Lemma 3.1 imply that $m(a_\infty) \leqslant d$. The lemma is proved. □

We present below the two compactness results which we will need for the proof of the main theorems.

Proposition 3.3. *The functional I_ε satisfies the* $(\mathrm{PS})_c$ *condition at any level $c < m(a_\infty)$.*

Proof. Let $(u_n) \subset X_\varepsilon$ be such that $I_\varepsilon(u_n) \to c$ and $I'_\varepsilon(u_n) \to 0$ in X_ε^*. Up to a subsequence, $u_n \rightharpoonup u$ weakly in X_ε with u being a critical point of I_ε. Thus, we can use (f_4) to get

$$I_\varepsilon(u) = I_\varepsilon(u) - \frac{1}{p}\langle I'_\varepsilon(u), u\rangle = \int \left(\frac{1}{p} f(u)u - F(u)\right) \geqslant 0.$$

Let $v_n = u_n - u$. Arguing as in [2, Lemma 3.3] we can show that $I'_\varepsilon(v_n) \to 0$ and

$$I_\varepsilon(v_n) \to c - I_\varepsilon(u) = d < m(a_\infty),$$

where we used that $c < m(a_\infty)$ and $I_\varepsilon(u) \geqslant 0$. Since $v_n \rightharpoonup 0$ weakly in X_ε and $d < m(a_\infty)$, it follows from Lemma 3.2 that $v_n \to 0$, i.e., $u_n \to u$ in X_ε. This concludes the proof of the proposition. □

Proposition 3.4. *If f verifies $(\widehat{f_5})$ then the functional I_ε restricted to $\mathcal{N}_\varepsilon$ satisfies the* $(\mathrm{PS})_c$ *condition at any level $c < m(a_\infty)$.*

Proof. Let $(u_n) \subset \mathcal{N}_\varepsilon$ be such that $I_\varepsilon(u_n) \to c$ and $\|I'_\varepsilon(u_n)\|_* \to 0$. Then there exists $(\lambda_n) \subset \mathbb{R}$ such that

$$I'_\varepsilon(u_n) = \lambda_n J'_\varepsilon(u_n) + o_n(1), \tag{3.5}$$

where $J_\varepsilon : X_\varepsilon \to \mathbb{R}$ is defined as

$$J_\varepsilon(u) = \|u\|_\varepsilon^p - \int f(u)u.$$

By $(\widehat{f_5})$,

$$\langle J_\varepsilon'(u_n), u_n\rangle = \int \left((p-1)f(u_n)u_n - f'(u_n)u_n^2\right) \leqslant -C_\sigma \int |u_n|^\sigma < 0$$

and therefore we may suppose that $\langle J_\varepsilon'(u_n), u_n\rangle \to l \leqslant 0$. If $l = 0$, it follows from $|\langle J_\varepsilon'(u_n), u_n\rangle| \geqslant C_\sigma \int |u_n|^\sigma$ that $u_n \to 0$ in $L^\sigma(\mathbb{R}^N)$. Recalling that (u_n) is bounded, we can use interpolation and argue as in the proof of Lemma 2.1 to get $\|u_n\|_\varepsilon \to 0$, which contradicts (2.4). Thus, $l < 0$ and we have that $\lambda_n \to 0$. By using (3.5) we conclude that $I_\varepsilon'(u_n) \to 0$ in X_ε^*, that is, (u_n) is a $(\mathrm{PS})_c$ sequence for I_ε. The result follows from Proposition 3.3. □

Remark 3.5. Arguing along the same lines of the above proof we can show that, if u is a critical point of I_ε restricted to $\mathcal{N}_\varepsilon$, then u is also a critical point of the unconstrained functional, that is, $I_\varepsilon'(u) = 0$ in X_ε^*.

4. Existence of a ground state solution

In order to prove our existence result, we need the following auxiliar result.

Lemma 4.1. *There exists $\varepsilon_0 > 0$ such that $c_\varepsilon < m(a_\infty)$ for any $\varepsilon \in (0, \varepsilon_0)$.*

Proof. Let us fix $\mu \in \mathbb{R}$ such that $a_0 < \mu < a_\infty$. Denote by $\omega \equiv \omega_\mu$ a ground state solution of the problem (AP_μ). For any given $r > 0$, let $\eta_r \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\eta_r(x) = 1$ if $|x| < r$ and $\eta_r(x) = 0$ if $|x| \geqslant 2r$. We also define $v_r(x) = \eta_r(x)\omega(x)$ and take $t_r > 0$ such that $\tilde{v}_r \equiv t_r v_r \in \mathcal{M}_\mu$.

We claim that there exists $r_0 > 0$ such that $\tilde{v} \equiv \tilde{v}_{r_0}$ satisfies $E_\mu(\tilde{v}) < m(a_\infty)$. Indeed, if this were not true, then $E_\mu(t_r v_r) \geqslant m(a_\infty)$ for all $r > 0$. Since $\omega \in \mathcal{M}_\mu$ and $v_r \to \omega$ in $W^{1,p}(\mathbb{R})$ as $r \to \infty$, we conclude that $t_r \to 1$. Hence, the monotonicity of the function $s \mapsto m(s)$ implies that

$$m(a_\infty) \leqslant \liminf_{r\to\infty} E_\mu(t_r v_r) = E_\mu(\omega) = m(\mu) < m(a_\infty),$$

which does not make sense.

Without loss of generality, we may suppose that $a(0) = a_0$. Recalling that a is continuous and the support of $\tilde{v}$ is compact, we obtain ε_0 such that $a(\varepsilon x) \leqslant \mu$ for any $\varepsilon \in (0, \varepsilon_0)$ and $x \in \operatorname{supp} \tilde{v}$. Thus,

$$\int a(\varepsilon x)|\nabla \tilde{v}|^p \leqslant \int \mu |\nabla \tilde{v}|^p \quad \text{for any } \varepsilon \in (0, \varepsilon_0)$$

and therefore

$$I_\varepsilon(t\tilde{v}) \leqslant E_\mu(t\tilde{v}) \quad \text{for any } \varepsilon \in (0, \varepsilon_0),\ t \geqslant 0.$$

Hence

$$\max_{t\geqslant 0} I_\varepsilon(t\tilde{v}) \leqslant \max_{t\geqslant 0} E_\mu(t\tilde{v}) = E_\mu(\tilde{v}) < m(a_\infty) \quad \text{for any } \varepsilon \in (0, \varepsilon_0)$$

and it follows from (2.3) that $c_\varepsilon < m(a_\infty)$ for any $\varepsilon \in (0, \varepsilon_0)$, as desired. □

We are now ready to present the proof of our existence theorem.

Proof of Theorem 1.1. Let ε_0 be given by the above lemma and fix $\varepsilon \in (0, \varepsilon_0)$. Since I_ε has the mountain pass geometry, we can use (2.3) to obtain $(u_n) \subset X_\varepsilon$ such that

$$I_\varepsilon(u_n) \to c_\varepsilon \quad \text{and} \quad I'_\varepsilon(u_n) \to 0.$$

Recalling that $c_\varepsilon < m(a_\infty)$, we may invoke Proposition 3.3 to guarantee that, along a subsequence, $u_n \to u$ with u being such that $I_\varepsilon(u) = c_\varepsilon$ and $I'_\varepsilon(u) = 0$. Arguing as in the proof of Proposition 2.2 we can check that u is positive in $\mathbb{R}^N$ and therefore it is a ground state solution of the problem (P_ε). The theorem is proved. □

5. Multiplicity of solutions

Let $\omega \equiv \omega_{a_0}$ be a ground state solution of the problem (AP_{a_0}) and consider $\eta : [0, \infty) \to \mathbb{R}$ a cut-off function such that $0 \leqslant \eta \leqslant 1$, $\eta(s) = 1$ if $0 \leqslant s \leqslant 1/2$ and $\eta(s) = 0$ if $s \geqslant 1$. We recall that M denotes the set of global minima points of a and define, for each $y \in M$, $\psi_{\varepsilon,y} : \mathbb{R}^N \to \mathbb{R}$ by setting

$$\psi_{\varepsilon,y}(x) = \eta\big(|\varepsilon x - y|\big)\omega\left(\frac{\varepsilon x - y}{\varepsilon}\right).$$

Let t_ε be the unique positive number satisfying

$$\max_{t \geqslant 0} I_\varepsilon(t\psi_{\varepsilon,y}) = I_\varepsilon(t_\varepsilon \psi_{\varepsilon,y})$$

and define the map $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ in the following way:

$$\Phi_\varepsilon(y) = \Phi_{\varepsilon,y} = t_\varepsilon \psi_{\varepsilon,y}. \tag{5.1}$$

The definition of t_ε shows that Φ_ε is well defined. Moreover, the following holds.

Lemma 5.1. $\lim_{\varepsilon \to 0} I_\varepsilon(\Phi_{\varepsilon,y}) = m(a_0)$ *uniformly for* $y \in M$.

Proof. Suppose, by contradiction, that the lemma is false. Then there exist $\delta > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0$ such that

$$\big|I_{\varepsilon_n}(\Phi_{\varepsilon_n,y_n}) - m(a_0)\big| \geqslant \delta > 0. \tag{5.2}$$

In order to simplify the notation, we write only Φ_n, ψ_n and t_n to denote Φ_{ε_n,y_n}, ψ_{ε_n,y_n} and t_{ε_n}, respectively.

Since $\langle I'_{\varepsilon_n}(\Phi_n), \Phi_n \rangle = 0$, we have that $\|\Phi_n\|^p_{\varepsilon_n} = \int f(\Phi_n)\Phi_n$. Thus, we can use (5.1) and the change of variables $z = (\varepsilon_n x - y_n)/\varepsilon_n$, to get

$$\begin{aligned} \|\psi_n\|^p_{\varepsilon_n} &= \int \Big(a(\varepsilon_n z + y_n)\big|\nabla\big(\eta\big(|\varepsilon_n z|\big)w(z)\big)\big|^p + \big|\eta\big(|\varepsilon_n z|\big)w(z)\big|^p\Big)\,\mathrm{d}z \\ &= \int \frac{f(t_n\eta(|\varepsilon_n z|)\omega(z))}{(t_n\eta(|\varepsilon_n z|)\omega(z))^{p-1}}\big|\eta\big(|\varepsilon_n z|\big)\omega(z)\big|^p\,\mathrm{d}z. \end{aligned} \tag{5.3}$$

By using the Lebesgue theorem, we can check that

$$\|\psi_n\|^p_{\varepsilon_n} \to \|\omega\|^p_{W_{a_0}}, \quad \int f(\psi_n)\psi_n \to \int f(\omega)\omega \quad \text{and} \quad \int F(\psi_n) \to \int F(\omega). \tag{5.4}$$

For n large we have that $B_{1/2}(0) \subset B_{1/(2\varepsilon_n)}(0)$. Thus, if we set $\alpha = \min\{w(z): |z| \leqslant 1/2\} > 0$, we infer from (5.3), the definition of η and (f_5) that

$$\|\psi_n\|_{\varepsilon_n}^p \geqslant \int_{B_{1/2}(0)} \frac{f(t_n\omega(z))}{(t_n\omega(z))^{p-1}} \big|\omega(z)\big|^p \, dz \geqslant \frac{f(t_n\alpha)}{(t_n\alpha)^{p-1}} \int_{B_{1/2}(0)} \big|\omega(z)\big|^p \, dz.$$

We claim that (t_n) has a bounded subsequence. Indeed, if this is not true, then $|t_n| \to \infty$, and therefore we can use the last estimate, (2.2) and (f_4) to conclude that $\|\psi_n\|_{\varepsilon_n}^p \to +\infty$, contradicting the first assertion in (5.4). Thus, up to a subsequence, we have $t_n \to t_0 \geqslant 0$. If $t_0 = 0$, we conclude from (5.4) that $\|t_n\psi_n\|_{\varepsilon_n} \to 0$, contradicting (2.4). Thus we have that $t_0 > 0$.

Since $t_n \to t_0 > 0$, we can take the limit in (5.3) to obtain

$$\int \big(a_0|\nabla\omega|^p + |\omega|^p\big) = \int \frac{f(t_0\omega)\omega}{t_0^{p-1}},$$

from which follows that $t_0\omega \in \mathcal{M}_{a_0}$. Since ω also belongs in $\mathcal{M}_{a_0}$, we conclude that $t_0 = 1$.

Now we note that

$$I_{\varepsilon_n}(\Phi_n) = \frac{t_n^p}{p} \int \big(a(\varepsilon_n z + y_n)\big|\nabla\big(\eta\big(|\varepsilon_n z|\big)\big)\omega(z)\big|^p + \big|\eta\big(|\varepsilon_n z|\big)\omega(z)\big|^p\big) \, dz$$
$$- \int F\big(t_n\eta\big(|\varepsilon_n z|\big)\omega(z)\big) \, dz.$$

Letting $n \to \infty$, recalling that $t_n \to 1$, using (5.4) and recovering the original notation, we get

$$\lim_{n\to\infty} I_{\varepsilon_n}(\Phi_{\varepsilon_n, y_n}) = E_{a_0}(\omega) = m(a_0),$$

which contradicts (5.2) and proves the lemma. □

Lemma 5.2. *Let $(u_n) \subset \mathcal{M}_\mu$ be such that $E_\mu(u_n) \to m(\mu)$ and $u_n \rightharpoonup u \neq 0$ weakly in $W^{1,p}(\mathbb{R}^N)$. Then, up to a subsequence, $u_n \to u$ in $W^{1,p}(\mathbb{R}^N)$.*

Proof. This proof follows quite similar lines as the proof of [1, Theorem 3.1]. We omit the details. □

Lemma 5.3. *Let $(\varepsilon_n) \subset \mathbb{R}^+$ and $(u_n) \subset \mathcal{N}_{\varepsilon_n}$ be such that $\varepsilon_n \to 0$ and $I_{\varepsilon_n}(u_n) \to m(a_0)$. Then there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $v_n(x) = u_n(x + \tilde{y}_n)$ has a convergent subsequence in $W^{1,p}(\mathbb{R}^N)$. Moreover, up to a subsequence, $(y_n) = (\varepsilon_n\tilde{y}_n)$ is such that $y_n \to y \in M$.*

Proof. By standard arguments we have that (u_n) is bounded in $W^{1,p}(\mathbb{R}^N)$. Since $m(a_0) > 0$, and since $\|u_n\|_{\varepsilon_n} \to 0$ would imply $I_{\varepsilon_n}(u_n) \to 0$, we can argue as in the proof of Lemma 2.1 to obtain a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ and constants $R, \gamma > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} |u_n|^p \geqslant \gamma > 0.$$

If we define $v_n(x) = u_n(x + \tilde{y}_n)$ we have that, up to a subsequence, $v_n \rightharpoonup v \neq 0$ weakly in $W^{1,p}(\mathbb{R}^N)$.

Let $(t_n) \subset (0, +\infty)$ be such that $w_n = t_n v_n \in \mathcal{M}_{a_0}$. Defining $y_n = \varepsilon_n\tilde{y}_n$, changing variables and recalling that $u_n \in \mathcal{N}_{\varepsilon_n}$, we get

$$\begin{aligned} m(a_0) \leqslant E_{a_0}(w_n) &\leqslant \frac{1}{p}\int \big(a\big(\varepsilon_n(x+\tilde{y}_n)\big)|\nabla w_n|^p + |w_n|^p\big) - \int F(w_n) \\ &= \frac{t_n^p}{p}\int \big(a(\varepsilon_n z)|\nabla u_n|^p + |u_n|^p\big)\,\mathrm{d}z - \int F(t_n u_n)\,\mathrm{d}z \\ &= I_{\varepsilon_n}(t_n u_n) \leqslant I_{\varepsilon_n}(u_n) = m(a_0) + o_n(1), \end{aligned}$$

from which follows that $E_{a_0}(w_n) \to m(a_0)$.

We claim that, up to a subsequence, $t_n \to t_0 > 0$. Indeed, since $v_n \nrightarrow 0$, there exists $\delta > 0$ such that $0 < \delta \leqslant \|v_n\|_{W_1}$. Hence, $0 < \tilde{\delta} \leqslant \|v_n\|_{W_{a_0}}$, for $\tilde{\delta} = \delta \min\{1, a_0\}$. It follows that

$$0 \leqslant t_n\tilde{\delta} \leqslant \|t_n v_n\|_{W_{a_0}} = \|w_n\|_{W_{a_0}} \leqslant C$$

for some $C > 0$. Thus (t_n) is bounded and we can suppose that $t_n \to t_0 \geqslant 0$. If $t_0 = 0$ then, since (v_n) is bounded, we conclude that $w_n = t_n v_n \to 0$. Hence $E_{a_0}(w_n) \to 0$, which contradicts $m(a_0) > 0$.

Let w be the weak limit of (w_n) in $W^{1,p}(\mathbb{R}^N)$. Since $t_n \to t_0 > 0$ and $v_n \rightharpoonup v \neq 0$, it follows from the uniqueness of the weak limit that $w = t_0 v \neq 0$. Hence, we conclude from Lemma 5.2 that $w_n \to w$, or equivalently, $v_n \to v$ in $W^{1,p}(\mathbb{R}^N)$.

Let us verify that (y_n) has a bounded subsequence. By using conditions (f_1)–(f_3) and the Lebesgue theorem, we can easily see that

$$\int F(w_n) \to \int F(w) \quad \text{and} \quad \int |w_n|^p \to \int |w|^p.$$

If $|y_n| \to \infty$, it follows from condition (a_1) and Fatou's lemma that

$$\int a_\infty |\nabla w|^p \leqslant \liminf_{n\to\infty} \int a(\varepsilon_n x + y_n)|\nabla w_n|^p.$$

Since $a_0 < a_\infty$, we infer from the above expressions that

$$\begin{aligned} m(a_0) = E_0(w) &< \frac{1}{p}\|w\|^p_{W_{a_\infty}} - \int F(w) \\ &\leqslant \liminf_{n\to\infty}\left\{\frac{1}{p}\int \big(a(\varepsilon_n x + y_n)|\nabla w_n|^p + |w_n|^p\big) - \int F(w_n)\right\} \\ &= \liminf_{n\to\infty}\left\{\frac{t_n^p}{p}\int \big(a(\varepsilon_n z)|\nabla u_n|^p + |u_n|^p\big)\,\mathrm{d}z - \int F(t_n u_n)\,\mathrm{d}z\right\} \\ &= \liminf_{n\to\infty} I_{\varepsilon_n}(t_n u_n) \leqslant \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) = m(a_0), \end{aligned}$$

which does not make sense. Hence, up to a subsequence, $y_n \to y \in \mathbb{R}^N$. If $y \notin M$ then $a(y) > a_0$ and we obtain a contradiction arguing as above. Thus, $y \in M$ and the lemma is proved. □

For any $\delta > 0$, let $\rho = \rho_\delta > 0$ be such that $M_\delta \subset B_\rho(0)$. Let $\chi : \mathbb{R}^N \to \mathbb{R}^N$ be defined as $\chi(x) = x$ for $|x| < \rho$ and $\chi(x) = \rho x/|x|$ for $|x| \geqslant \rho$. Finally, let us consider the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ given by

$$\beta_\varepsilon(u) = \frac{\int \chi(\varepsilon x)|u(x)|^p\,\mathrm{d}x}{\int |u(x)|^p\,\mathrm{d}x}.$$

Since $M \subset B_\rho(0)$, we can use the definition of χ and the Lebesgue theorem to conclude that

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\Phi_{\varepsilon,y}) = y \quad \text{uniformly for } y \in M. \tag{5.5}$$

Following [14], we introduce a subset of $\mathcal{N}_\varepsilon$ which will be useful in the future. We take a function $h:[0,\infty)\to[0,\infty)$ such that $h(\varepsilon)\to 0$ as $\varepsilon\to 0^+$ and set

$$\Sigma_\varepsilon=\left\{u\in\mathcal{N}_\varepsilon\colon\ I_\varepsilon(u)\leqslant m(a_0)+h(\varepsilon)\right\}.$$

Given $y\in M$, we can use Lemma 5.1 to conclude that $h(\varepsilon)=|I_\varepsilon(\Phi_{\varepsilon,y})-m(a_0)|$ is such that $h(\varepsilon)\to 0$ as $\varepsilon\to 0^+$. Thus, $\Phi_{\varepsilon,y}\in\Sigma_\varepsilon$ and we have that $\Sigma_\varepsilon\neq\emptyset$ for any $\varepsilon>0$.

Lemma 5.4. *For any* $\delta>0$ *we have that*

$$\lim_{\varepsilon\to 0^+}\sup_{u\in\Sigma_\varepsilon}\operatorname{dist}\big(\beta_\varepsilon(u),M_\delta\big)=0.$$

Proof. Let $(\varepsilon_n)\subset\mathbb{R}^+$ be such that $\varepsilon_n\to 0$. By definition, there exists $(u_n)\subset\Sigma_{\varepsilon_n}$ such that

$$\operatorname{dist}\big(\beta_{\varepsilon_n}(u_n),M_\delta\big)=\sup_{u\in\Sigma_{\varepsilon_n}}\operatorname{dist}\big(\beta_{\varepsilon_n}(u),M_\delta\big)+o_n(1).$$

Thus, it suffices to find a sequence $(y_n)\subset M_\delta$ such that

$$\big|\beta_{\varepsilon_n}(u_n)-y_n\big|=o_n(1). \tag{5.6}$$

In order to obtain such sequence, we note that $(u_n)\subset\Sigma_{\varepsilon_n}\subset\mathcal{N}_{\varepsilon_n}$. Thus

$$c_{\varepsilon_n}\leqslant I_{\varepsilon_n}(u_n)\leqslant m(a_0)+h(\varepsilon_n),$$

from which follows that $\limsup_{n\to\infty}c_{\varepsilon_n}\leqslant m(a_0)$. On the other hand, since $m(a_0)\leqslant c_{\varepsilon_n}$, we also have $m(a_0)\leqslant\liminf_{n\to\infty}c_{\varepsilon_n}$. Hence, taking the limit in the above expression, we conclude that $I_{\varepsilon_n}(u_n)\to m(a_0)$. We may now invoke Lemma 5.3 to obtain a sequence $(\tilde{y}_n)\subset\mathbb{R}^N$ such that $(y_n)=(\varepsilon_n\tilde{y}_n)\subset M_\delta$ for n sufficiently large. Thus,

$$\begin{aligned}\beta_{\varepsilon_n}(u_n)&=\frac{\int\chi(\varepsilon_n x)|u_n|^p\,\mathrm{d}x}{\int|u_n|^p\,\mathrm{d}x}=\frac{\int\chi(\varepsilon_n z+y_n)|u_n(z+\tilde{y}_n)|^p\,\mathrm{d}z}{\int|u_n(z+\tilde{y}_n)|^p\,\mathrm{d}z}\\&=y_n+\frac{\int(\chi(\varepsilon_n z+y_n)-y_n)|u_n(z+\tilde{y}_n)|^p\,\mathrm{d}z}{\int|u_n(z+\tilde{y}_n)|^p\,\mathrm{d}z}.\end{aligned}$$

Since $\varepsilon_n z+y_n\to y\in M$, we have that $\beta_{\varepsilon_n}(u_n)=y_n+o_n(1)$ and therefore the sequence (y_n) verifies (5.6). The lemma is proved. □

We are now ready to present the proof of the multiplicity result.

Proof of Theorem 1.2. Given $\delta>0$ we can use (5.5), Lemmas 5.1 and 5.4, and argue as in [14, Section 6] to obtain $\varepsilon_\delta>0$ such that, for any $\varepsilon\in(0,\varepsilon_\delta)$, the diagram

$$M\xrightarrow{\Phi_\varepsilon}\Sigma_\varepsilon\xrightarrow{\beta_\varepsilon}M_\delta$$

is well defined and $\beta_\varepsilon\circ\Phi_\varepsilon$ is homotopically equivalent to the embedding $\iota: M\to M_\delta$. Moreover, using the definition of Σ_ε and taking ε_δ small if necessary, we may suppose that I_ε satisfies the Palais–Smale condition in Σ_ε. Standard Ljusternik–Schnirelmann theory provides at least $\operatorname{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon)$ critical points u_i of I_ε restricted to $\mathcal{N}_\varepsilon$. The same ideas contained in the proof of [9, Lemma 4.3] show that $\operatorname{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon)\geqslant\operatorname{cat}_{M_\delta}(M)$. By using Remark 3.5 and the arguments of the proof of Proposition 2.2, we conclude that each u_i is a solution of (P_ε). The theorem is proved. □

Acknowledgment

The authors are grateful to the referee for his/her useful suggestions.

References

[1] C.O. Alves, Existence and multiplicity of solutions for a class of quasilinear equations, Adv. Non. Studies 5 (2005) 73–87.
[2] C.O. Alves, P.C. Carrião, E.S. Medeiros, Multiplicity of solutions for a class of quasilinear problems in exterior domains with Neumann conditions, Abstr. Appl. Anal. 3 (2004) 251–268.
[3] C.O. Alves, G.M. Figueiredo, Existence and multiplicity of positive solutions to a p-Laplacian equation in $\mathbb{R}^N$, preprint.
[4] C.O. Alves, M.A.S. Souto, On existence and concentration behavior of ground state solutions for a class of problems with critical growth, Comm. Pure Appl. Anal. 1 (2002) 417–431.
[5] A. Alvino, P.L. Lions, G. Trombetti, On optimization problems with prescribed rearrangements, Nonlinear Anal. 13 (1989) 185–220.
[6] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.
[7] M.S. Baouendi, C. Goulaouic, Régularité et théorie spectrale pour une classe d'opérateurs elliptiques dégénérés, Arch. Ration. Mech. Anal. 34 (1969) 361–379.
[8] V. Benci, G. Cerami, The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems, Arch. Ration. Mech. Anal. 114 (1991) 79–93.
[9] V. Benci, G. Cerami, Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology, Calc. Var. Partial Differential Equations 2 (1994) 29–48.
[10] H. Berestycki, J.P. Dias, M.J. Esteban, M. Figueira, Eigenvalue problem for some nonlinear Wheeler–DeWitt operators, J. Math. Pures Appl. 72 (1993) 493–515.
[11] P. Caldiroli, R. Musina, On the existence of extremal functions for a weighted Sobolev embedding with critical exponent, Calc. Var. Partial Differential Equations 8 (1999) 365–387.
[12] P. Caldiroli, R. Musina, On a variational degenerate elliptic problem, Nonlinear Differential Equations Appl. 7 (2000) 187–199.
[13] J. Chabrowski, Degenerate elliptic equation involving a subcritical Sobolev exponent, Port. Math. 53 (1996) 167–177.
[14] S. Cingolani, M. Lazzo, Multiple semiclassical standing waves for a class of nonlinear Schrödinger equations, Topol. Methods Nonlinear Anal. 10 (1997) 1–13.
[15] G.R. Cirmi, M.M. Porzio, L^∞-Solutions for some nonlinear degenerate elliptic and parabolic equations, Ann. Mat. Pura Appl. 169 (1995) 67–86.
[16] R. Dautray, J.L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology, vol. 1: Physical Origins and Classical Methods, Springer-Verlag, Berlin, 1985.
[17] M. Lazzo, Existence and multiplicity results for a class of nonlinear elliptic problems on $\mathbb{R}^N$, Discrete Contin. Dynam. Systems (suppl.) (2003) 526–535.
[18] G.B. Li, Some properties of weak solutions of nonlinear scalar fields equations, Ann. Acad. Sci. Fenn. Ser. AI Math. 15 (1990) 27–36.
[19] P.L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. Part II, Ann. Inst. H. Poincaré Non Linéaire 1 (1984) 223–283.
[20] M.K.V. Murthy, V. Stampacchia, Boundary problems for some degenerate elliptic operators, Ann. Mat. Pura Appl. (4) 80 (1968) 1–122.
[21] D. Passaseo, Some concentration phenomena in degenerate semilinear elliptic problems, Nonlinear Anal. 24 (1995) 1011–1025.
[22] A. Pomponio, S. Secchi, On a class of singularly perturbed elliptic equations in divergence form: Existence and multiplicity results, J. Differential Equations 207 (2004) 228–266.
[23] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. 43 (1992) 270–291.
[24] E.A.B. Silva, S.H.M. Soares, Quasilinear Dirichlet problems in $\mathbb{R}^N$ with critical growth, Nonlinear Anal. 43 (2001) 1–20.
[25] M. Squassina, Spike solutions for a class of singularly perturbed quasilinear elliptic equations, Nonlinear Anal. 54 (2003) 1307–1336.

[26] N.S. Trudinger, On Harnack type inequalities and their application to quasilinear elliptic equations, Comm. Pure Appl. Math. XX (1967) 721–747.

[27] M. Willem, Minimax Theorems, Birkhäuser, Boston, 1996.

[28] J. Yang, Positive solutions of quasilinear elliptic obstacle problems with critical exponents, Nonlinear Anal. 25 (1995) 1283–1306.