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Positive solutions for some quasilinear equations with critical and supercritical growth

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Abstract

We establish results concerning the existence and multiplicity of positive solutions for the problem

$$-\text{div}(a(\varepsilon x)|\nabla u|^{p-2}\nabla u) + u^{p-1} = f(u) + u^{p^*-1} \quad \text{in } \mathbb{R}^N, u \in W^{1,p}(\mathbb{R}^N),$$

where $\varepsilon > 0$ is a small parameter, $2 \le p < N$, $p^* = Np/(N-p)$, a is a positive potential and f is a superlinear function. We obtain the existence of a ground state solution and relate the number of positive solutions with the topology of the set where a attains its minimum. We also prove a multiplicity result for a supercritical version of the above problem. In the proofs we use minimax theorems and Ljusternik–Schnirelmann theory.

Keywords: Positive solutions; Critical problems; Supercritical problems; Ljusternik–Schnirelmann theory; Quasilinear equations

1. Introduction

The aim of this paper is to study the number of solutions of some quasilinear problems. Before we make precise statements, let us comment on some works which motivated this one. We start by citing the paper [6], where Chabrowski studied the problem

$$-\text{div}(a(x)\nabla u) + \lambda u = K(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

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with $\lambda > 0$, $2 < q < 2N/(N-2)$ and $a \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfying $0 \le a(x) \le \lim_{|x|\to\infty} a(x)$ and being positive in the exterior of some ball $B_R(0)$. By using minimization arguments he obtained a nonzero solution of (1.1) in some appropriated subspace of $W^{1,2}(\mathbb{R}^N)$. In his result, he assumed an integrability condition for $a(x)$ and required that $K \in L^\infty(\mathbb{R}^N)$ verify either a periodicity condition or $K(x) \ge \lim_{|x|\to\infty} K(x)$.

In [11] Lazzo considered equation (1.1) with $K \equiv 1$ and the function a satisfying $0 < \inf_{x\in\mathbb{R}^N} a(x) < \liminf_{|x|\to\infty} a(x)$. She proved that, for λ sufficiently large, there is an effect of the topology of the set $\{x \in \mathbb{R}^N : a(x) = a_0\}$ on the number of positive solutions of (1.1).

In a recent work [9], the authors extended the results of [11] to the quasilinear case with a nonlinearity $f(u)$ more general than u^{q-1} but also having subcritical growth. In the present paper, we continue the study of [9] by considering critical and supercritical nonlinearities.

In the first part of the paper we deal with the problem

$$\begin{cases} -\mathrm{div}(a(\varepsilon x)|\nabla u|^{p-2}\nabla u) + u^{p-1} = f(u) + u^{p^*-1} & \text{in } \mathbb{R}^N, \\ u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \quad u(x) > 0 & \text{for all } x \in \mathbb{R}^N, \end{cases} \tag{P_ε}$$

where $\varepsilon > 0$, $2 \le p < N$, $p^* := Np/(N-p)$, $0 < \alpha < 1$ and the potential a satisfies

(a_1) $a \in C(\mathbb{R}^N, \mathbb{R})$ and

$$0 < a_0 := \inf_{x\in\mathbb{R}^N} a(x) < a_\infty := \liminf_{|x|\to\infty} a(x).$$

We also suppose that $f \in C^1(\mathbb{R}^+, \mathbb{R})$ satisfies

(f_1) $f(s) = o(s^{p-1})$ as $s \to 0^+$,
(f_2) there exists $p < q < p^*$ such that $f(s) = o(s^{q-1})$ as $s \to \infty$,
(f_3) there exists $p < \theta < q$ such that

$$0 < \theta F(s) := \theta \int_0^s f(\tau)\mathrm{d}\tau \le s f(s) \quad \text{for all } s > 0,$$

(f_4) the function $s \mapsto f(s)/s^{p-1}$ is increasing for $s > 0$,
(f_5) $f(s) \ge \lambda s^{q_1-1}$ for all $s > 0$, with $q_1 \in (p, p^*)$ and λ satisfying
(f_5a) $\lambda > 0$ if either $N \ge p^2$, or $p < N < p^2$ and $p^* - p/(p-1) < q_1 < p^*$,
(f_5b) λ is sufficiently large if $p < N < p^2$ and $p < q_1 \le p^* - p/(p-1)$.

Under conditions (f_1)–(f_2) it is well known that the solutions of (P_ε) are precisely the positive critical points of the functional $I_\varepsilon : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I_\varepsilon(u) := \frac{1}{p}\int_{\mathbb{R}^N} (a(\varepsilon x)|\nabla u|^p + |u|^p)\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*}\mathrm{d}x.$$

We recall that a solution u_0 of (P_ε) is called a ground state solution if it possesses minimum energy among all solutions, that is,

$$I_\varepsilon(u_0) = \min\{I_\varepsilon(u) : u \text{ is a solution of } (P_\varepsilon)\}.$$

In our first result we obtain, for $\varepsilon > 0$ small enough, the existence of a ground state solution of (P_ε).

Theorem 1.1. *Suppose that a satisfies (a_1) and f satisfies (f_1)–(f_5). Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, the problem (P_ε) has a ground state solution.*

In our second result we relate the number of solutions of (P_ε) with the topology of the set of minima of the potential a. In order to present our result we introduce the set of global minima of a, given by

$$M := \{x \in \mathbb{R}^N : a(x) = a_0\}.$$

Note that, in view of (a_1), the set M is compact. For any $\delta > 0$, let us denote by $M_\delta := \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) \le \delta\}$ the closed δ-neighborhood of M.

We recall that, if Y is a closed set of a topological space X, $\operatorname{cat}_X(Y)$ is the Ljusternik–Schnirelmann category of Y in X, namely the least number of closed and contractible sets in X which cover Y. We shall prove the following result.

Theorem 1.2. *Suppose that a satisfies (a_1) and f satisfies (f_1)–(f_5). Then, for any $\delta > 0$ given, there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the problem (P_ε) has at least $\operatorname{cat}_{M_\delta}(M)$ solutions.*

The proof of the above theorem is done by applying a technique introduced by Benci and Cerami in [3]. It consists in making a comparison between the category of some sublevel sets of the energy functional I_ε and the category of the set M. Since we are considering nonhomogeneous nonlinearities, some arguments developed in [6,11] do not apply. Thus, we make a detailed study of the behavior of the functional I_ε restricted to its Nehari manifold (see also [9]). Furthermore, since critical problems present some compactness problems, we use the ideas of Brezis and Niremberg [5], (f_4) and (f_5), and some calculations from [15] in order to obtain the required compactness property.

In the last part of the paper we study a supercritical version of problem (P_ε). In this case, we deal with the sum of two homogeneous nonlinearities and add a new positive parameter λ. More specifically, we shall consider the following problem

$$\begin{cases} -\operatorname{div}(a(\varepsilon x)|\nabla u|^{p-2}\nabla u) + u^{p-1} = u^{q-1} + \lambda u^{t-1} & \text{in } \mathbb{R}^N, \\ u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \ u(x) > 0 & \text{for all } x \in \mathbb{R}^N, \end{cases} \tag{$P_{\lambda,\varepsilon}$}$$

where $\varepsilon, \lambda > 0$, $2 \le p < N$ and the powers satisfy $p < q < p^* < t$. Our multiplicity result for the supercritical case can be stated as follows.

Theorem 1.3. *Suppose that a satisfies (a_1). Then there exists $\lambda_0 > 0$ with the following property: for any $\lambda \in (0, \lambda_0)$ and $\delta > 0$ given, there exists $\varepsilon_{\lambda,\delta} > 0$ such that, for any $\varepsilon \in (0, \varepsilon_{\lambda,\delta})$, the problem $(P_{\lambda,\varepsilon})$ has at least $\operatorname{cat}_{M_\delta}(M)$ solutions.*

For the proof of this theorem we follow Chabrowski and Yang [7], where a technique introduced by Rabinowitz [17] was utilized. The main idea is, first, to consider a truncated problem with subcritical growth and, then, to apply a result of [9] to get a multiplicity of solutions for the truncated problem. After obtaining a priori bounds for these solutions, we use Moser's iteration method [16] to prove that, if λ is small enough, the solutions of the truncated problem also satisfy the original problem $(P_{\lambda,\varepsilon})$. To the best of our knowledge, in the literature there are no multiplicity results for supercritical problems via Ljusternik–Schnirelmann theory.

The results of this paper complement those of [6,7,11] in several senses. First, because we consider the quasilinear case $2 \le p < N$. Second, because we deal with critical and supercritical growth. Finally, at least in the critical case, we consider nonhomogeneous nonlinearities. They also complement the results of [9], where only the subcritical case is considered. We finish this introduction by emphasizing that our results seem to be new even in the semilinear case $p = 2$.

The paper is organized as follows. in the next section we present some results concerning the autonomous problem associated to (P_ε). In Section 3 we obtain a local compactness property for I_ε. Theorems 1.1 and 1.2 are proved in Section 4 and the final Section 5 is devoted to the proof of Theorem 1.3.

2. The autonomous problem

Throughout the paper we suppose that the functions a and f satisfy (a_1) and (f_1)–(f_5), respectively. Since we are interested in positive solutions, we extend f to the whole real line by setting $f(s) := 0$ for $s \leq 0$. To save notation, we write only $\int u$ instead of $\int_{\mathbb{R}^N} u(x)\mathrm{d}x$. For any $1 \leq s \leq \infty$, $|u|_s$ denotes the $L^s(\mathbb{R}^N)$-norm of a function $u \in L^s(\mathbb{R}^N)$.

In this section we make a detailed study of the autonomous problem associated to (P_ε), namely

$$\begin{cases} -\mu \operatorname{div}(|\nabla u|^{p-2}\nabla u) + u^{p-1} = f(u) + u^{p^*-1} & \text{in } \mathbb{R}^N, \\ u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \ u(x) > 0 & \text{for all } x \in \mathbb{R}^N. \end{cases} \tag{AP_μ}$$

We denote by W_μ the Sobolev space $W^{1,p}(\mathbb{R}^N)$ endowed with the norm

$$\|u\|_{W_\mu} := \left\{ \int (\mu|\nabla u|^p + |u|^p) \right\}^{1/p}.$$

The solutions of (AP_μ) are precisely the positive critical points of the functional $E_\mu : W_\mu \to \mathbb{R}$ given by

$$E_\mu(u) := \frac{1}{p}\int (\mu|\nabla u|^p + |u|^p) - \int F(u) - \frac{1}{p^*}\int |u|^{p^*}.$$

Let $\mathcal{M}_\mu := \left\{ u \in W_\mu \setminus \{0\} : \langle E'_\mu(u), u\rangle = 0 \right\}$ be the Nehari manifold of E_μ and define $m(\mu)$ by setting

$$m(\mu) := \inf_{u \in \mathcal{M}_\mu} E_\mu(u).$$

In view of conditions (f_1)–(f_3), we can easily check that E_μ satisfies the Mountain Pass geometry. Moreover, since $f(s)/s^{p-1}$ is increasing, we have the following characterization (see [21, Chapter 4])

$$m(\mu) = \inf_{\gamma \in \Gamma_\mu} \max_{t \in [0,1]} E_\mu(\gamma(t)) = \inf_{u \in W_\mu \setminus \{0\}} \max_{t \geq 0} E_\mu(tu) > 0,$$

where $\Gamma_\mu := \{\gamma \in C([0,1], W^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0, E_\mu(\gamma(1)) < 0\}$.

We devote the rest of this section to show that $m(\mu)$ is attained by a positive function. We start by defining the best constant of the Sobolev embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ as

$$S := \inf\left\{ \int_{\mathbb{R}^N} |\nabla u|^p : u \in W^{1,p}(\mathbb{R}^N), \ \int |u|^{p^*} = 1 \right\}. \tag{2.1}$$

As in [5,10], we are able to compare the minimax level $m(\mu)$ with a suitable number which involves the constant S.

Lemma 2.1. *For any $\mu > 0$ there exists $v \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$ such that*

$$\max_{t \geq 0} E_\mu(tv) < \frac{1}{N}(\mu S)^{\frac{N}{p}}.$$

In particular, $m(\mu) < \frac{1}{N}(\mu S)^{N/p}$ for any $\mu > 0$.

Proof. For each $\xi > 0$, consider the function

$$w_\xi(x) := \left[N\left(\frac{N-p}{p-1}\right)^{p-1}\right]^{(N-p)/p^2} \left(\xi + |x|^{p/(p-1)}\right)^{(p-N)/p}.$$

We recall that w_ξ satisfies the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = u^{p^*-1} & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \ u(x) > 0 & \text{for all } x \in \mathbb{R}^N, \end{cases}$$

and by a result due to Talenti [19]

$$\int_{\mathbb{R}^N} |\nabla w_\xi|^p = \int_{\mathbb{R}^N} |w_\xi|^{p^*} = S^{N/p}.$$

Let $\eta \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\eta \equiv 1$ on $B_1(0)$ and $\eta \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$. By setting $v_\xi := \eta w_\xi |\eta w_\xi|_{p^*}^{-1}$, we can use (f_5) to get

$$E_\mu(tv_\xi) \le \frac{t^p}{p}\int_{B_2(0)} \mu|\nabla v_\xi|^p + \frac{t^p}{p}\int_{B_2(0)} |v_\xi|^p - \frac{t^{q_1}}{q_1}\lambda\int_{B_2(0)} |v_\xi|^{q_1} - \frac{t^{p^*}}{p^*}.$$

Arguing as in [15], we obtain

$$\max_{t\ge 0}\left\{\frac{t^p}{p}\int_{B_2(0)} \mu|\nabla v_\xi|^p + \frac{t^p}{p}\int_{B_2(0)} |v_\xi|^p - \frac{t^{q_1}}{q_1}\lambda\int_{B_2(0)} |v_\xi|^{q_1} - \frac{t^{p^*}}{p^*}\right\} < \frac{1}{N}(\mu S)^{N/p}.$$

Thus $\max_{t\ge 0} E_\mu(tv_\xi) < \frac{1}{N}(\mu S)^{N/p}$, as desired. $\square$

Let $I : V \to \mathbb{R}$ be a C^1-functional defined on a Banach space V. We say that I satisfies the Palais–Smale condition at level c ($(PS)_c$ for short) if any sequence $(u_n) \subset V$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ contains a convergent subsequence. The following result presents an interesting property of the Palais–Smale sequences of E_μ.

Lemma 2.2. *Let $(u_n) \subset W_\mu$ be a $(PS)_d$ sequence for E_μ with $d < \frac{1}{N}(\mu S)^{N/p}$ and $u_n \rightharpoonup 0$ weakly in W_μ. Then we have either*

(i) $\|u_n\|_{W_\mu} \to 0$, *or*
(ii) *there exist a sequence $(y_n) \subset \mathbb{R}^N$ and constants $R, \ \gamma > 0$ such that*

$$\liminf_{n\to\infty}\int_{B_R(y_n)} |u_n|^p \ge \gamma > 0.$$

Proof. Suppose that (ii) does not occur. Condition (f_4) and standard calculation show that (u_n) is bounded in $W^{1,p}(\mathbb{R}^N)$. It follows from [13, Lemma I.1] that $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for any $p < s < p^*$. Given $\delta > 0$, we can use (f_1) and (f_2) to get

$$0 \le \left|\int f(u_n)u_n\right| \le \delta\int |u_n|^p + C_\delta\int |u_n|^q,$$

for some constant $C_\delta > 0$. Since (u_n) is bounded in $L^p(\mathbb{R}^N)$, $u_n \to 0$ in $L^q(\mathbb{R}^N)$ and δ is arbitrary, we conclude that $\int f(u_n)u_n \to 0$. Thus, from $\langle E'_\mu(u_n), u_n\rangle \to 0$, we obtain

$$\|u_n\|_{W_\mu}^p = |u_n|_{p^*}^{p^*} + o_n(1),$$

where $o_n(1)$ denotes a quantity approaching zero as $n \to \infty$. Taking a subsequence, we obtain $l \geq 0$ such that

$$|u_n|_{W_\mu}^p \to l \quad \text{and} \quad |u_n|_{p^*}^{p^*} \to l.$$

By (f_3), we get $\int F(u_n) \to 0$. Since $E_\mu(u_n) = d + o_n(1)$, the above expression implies that $l = Nd$.

Recalling that

$$\|u_n\|_{W_\mu}^p \geq \mu \int |\nabla u_n|^p \geq \mu S |u_n|_{p^*}^p$$

and letting $n \to \infty$, we conclude that $l \geq \mu S l^{p/p^*}$. If $l \neq 0$ we get

$$Nd = l \geq (\mu S)^{N/p}$$

which is a contradiction. Hence $l = 0$ and therefore $u_n \to 0$ in W_μ. $\square$

As a consequence of the two above lemmas, we have the following existence result for the autonomous problem.

Proposition 2.3. *Suppose that a satisfies (a_1) and f satisfies (f_1)–(f_5). Then, for any $\mu > 0$, the problem (AP_μ) has a ground state solution.*

Proof. It suffices to argue as in the proof of [9, Proposition 2.2] by using Lemmas 2.1 and 2.2. We omit the details. $\square$

We end this section by noting that, in view of the above proposition, we can argue as in [2, Lemma 10] and show that the function $\mu \mapsto m(\mu)$ is increasing for $\mu > 0$.

3. The Palais–Smale condition for I_ε

For any $\varepsilon > 0$, let X_ε be the Sobolev space $W^{1,p}(\mathbb{R}^N)$ endowed with the norm

$$\|u\|_\varepsilon := \left\{ \int \left(a(\varepsilon x)|\nabla u|^p + |u|^p \right) \right\}^{1/p}.$$

As stated in the introduction, we will look for critical points of the C^2-functional $I_\varepsilon : X_\varepsilon \to \mathbb{R}$ given by

$$I_\varepsilon(u) := \frac{1}{p} \int \left(a(\varepsilon x)|\nabla u|^p + |u|^p \right) - \int F(u) - \frac{1}{p^*} \int |u|^{p^*}.$$

We introduce the Nehari manifold of I_ε by setting

$$\mathcal{N}_\varepsilon := \left\{ u \in X_\varepsilon \setminus \{0\} : \langle I'_\varepsilon(u), u \rangle = 0 \right\}$$

and consider the following minimization problem

$$c_\varepsilon := \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u).$$

As in the previous section, the functional I_ε satisfies the Mountain Pass geometry. Hence, we can prove that c_ε verifies

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} I_\varepsilon(\gamma(t)) = \inf_{u \in X_\varepsilon \setminus \{0\}} \max_{t \geq 0} I_\varepsilon(tu) > 0, \tag{3.1}$$

where $\Gamma_\varepsilon := \{\gamma \in C([0,1], X_\varepsilon) : \gamma(0) = 0,\ I_\varepsilon(\gamma(1)) < 0\}$. Moreover, since $I_\varepsilon(u) \geq E_{a_0}(u)$ for all $u \in W^{1,p}(\mathbb{R}^N)$, we have that $c_\varepsilon \geq m(a_0) > 0$ for any $\varepsilon > 0$. Thus, we can easily obtain $r > 0$, independent of ε, such that

$$\|u\|_\varepsilon \geq r > 0 \quad \text{for any } \varepsilon > 0,\ u \in \mathcal{N}_\varepsilon. \tag{3.2}$$

From now on we are interested in establishing a compactness property for I_ε. We start with two technical results. The first is in the same spirit of Lemma 2.2 and the second is a version of [9, Lemma 3.1].

Lemma 3.1. *Let $(v_n) \subset X_\varepsilon$ be a $(PS)_d$ sequence for I_ε with $d < \frac{1}{N}(a_0 S)^{N/p}$ and $v_n \rightharpoonup 0$ weakly in X_ε. Then we have either*

(i) $\|v_n\|_\varepsilon \to 0$, *or*
(ii) *there exists a sequence $(y_n) \subset \mathbb{R}^N$ and constants $R,\ \gamma > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |v_n|^p \geq \gamma > 0.$$

Proof. It suffices to note that $\|v_n\|_\varepsilon^p \geq a_0 \int |\nabla v_n|^p$ and argue as in the proof of Lemma 2.2. □

Lemma 3.2. *Let $(v_n) \subset X_\varepsilon$ be a $(PS)_d$ sequence for I_ε with $d < \frac{1}{N}(a_0 S)^{N/p}$ and $v_n \rightharpoonup 0$ weakly in X_ε. Then, up to a subsequence,*

$$\limsup_{n\to\infty} \int (s_n a_\infty - a(\varepsilon x))|\nabla v_n|^p \leq 0,$$

for any sequence $(s_n) \subset \mathbb{R}$ satisfying $s_n \to 1$.

Proof. Let $C > 0$ be such that $\int |\nabla v_n|^p \leq C$. Since $s_n \to 1$ and

$$\int (s_n a_\infty - a(\varepsilon x))|\nabla v_n|^p = \int (a_\infty - a(\varepsilon x))|\nabla v_n|^p + a_\infty (s_n - 1) \int |\nabla v_n|^p,$$

it suffices to consider the case $s_n \equiv 1$.

Given $\varrho > 0$, we can use condition (a_1) to obtain $R = R(\varrho) > 0$ such that $a(\varepsilon x) \geq a_\infty - \varrho$ for any $|x| \geq R$. We claim that $\int_{B_R(0)} |\nabla v_n|^p \to 0$ as $n \to \infty$. Assuming the claim, we get

$$\int (a_\infty - a(\varepsilon x))|\nabla v_n|^p \leq \int_{B_R(0)} (a_\infty - a(\varepsilon x))|\nabla v_n|^p + \varrho C = o_n(1) + \varrho C,$$

for any $\varrho > 0$, and the lemma follows.

It order to prove the claim we note that, taking a subsequence, we may suppose that

$$|\nabla v_n|^p \rightharpoonup \mu \ \text{ and } \ |v_n|^{p^*} \rightharpoonup \nu \ (\text{weak}^*\text{-sense of measures}).$$

Using the concentration compactness principle due to Lions (cf. [14, Lemma 1.2]), we obtain an at most countable index set Λ, sequences $(x_i) \subset \mathbb{R}^N$, $(\mu_i), (\nu_i) \subset (0, \infty)$, such that

$$\nu = \sum_{i\in\Lambda} \nu_i \delta_{x_i},\ \ \mu \geq \sum_{i\in\Lambda} \mu_i \delta_{x_i} \ \text{ and } \ S\nu_i^{p/p^*} \leq \mu_i, \tag{3.3}$$

for all $i \in \Lambda$, where δ_{x_i} is the Dirac mass at $x_i \in \mathbb{R}^N$.

Now, for every $\varrho > 0$, we set $\psi_\varrho(x) := \psi((x - x_i)/\varrho)$ where $\psi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ is such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$ and $|\nabla \psi|_\infty \leq 2$. Since $(\psi_\varrho v_n)$ is bounded, $\langle I'_\varepsilon(v_n), \psi_\varrho v_n \rangle \to 0$, that is,

$$\int a(\varepsilon x)|\nabla v_n|^{p-2} v_n(\nabla v_n \cdot \nabla \psi_\varrho) = -\int a(\varepsilon x)\psi_\varrho |\nabla v_n|^p - \int \psi_\varrho |v_n|^p + \int f(v_n)\psi_\varrho v_n + \int \psi_\varrho |v_n|^{p^*} + o_n(1).$$

Since $v_n \to 0$ in $L^s_{\text{loc}}(\mathbb{R}^N)$ for all $p \leq s < p^*$, f has subcritical growth and ψ_ϱ has compact support, we can let $n \to \infty$ in the above expression to obtain

$$\int \psi_\varrho \mathrm{d}\nu = \int a(\varepsilon x)\psi_\varrho \mathrm{d}\mu \geq a_0 \int \psi_\varrho \mathrm{d}\mu.$$

Letting $\varrho \to 0$ we conclude that $\nu_i \geq a_0 \mu_i$. It follows from (3.3) that

$$\nu_i \geq (a_0 S)^{N/p}. \tag{3.4}$$

Now we shall prove that the above expression cannot occur, and therefore the set Λ is empty. Indeed, arguing by contradiction, let us suppose that $\nu_i \geq (a_0 S)^{N/p}$ for some $i \in \Lambda$. Thus,

$$\begin{aligned} d &= I_\varepsilon(v_n) - \frac{1}{p}\langle I'_\varepsilon(v_n), v_n \rangle + o_n(1) \\ &= \int \left(\frac{1}{p} f(v_n)v_n - F(v_n) \right) + \frac{1}{N} \int |v_n|^{p^*} + o_n(1) \\ &\geq \frac{1}{N} \int \psi_\varrho |v_n|^{p^*} + o_n(1). \end{aligned}$$

Letting $n \to \infty$, we get

$$d \geq \frac{1}{N} \sum_{i \in \Lambda} \psi_\varrho(x_i)\nu_i = \frac{1}{N} \sum_{i \in \Lambda} \nu_i \geq \frac{1}{N}(a_0 S)^{N/p},$$

which does not make sense. Hence Λ is empty and it follows from the same arguments employed in [18, Lemmas 3.5 and 3.6] that $v_n \to 0$ in $W^{1,p}_{\text{loc}}(\mathbb{R}^N)$. □

The following lemma is a keystone to our compactness result.

Lemma 3.3. *Let $(v_n) \subset X_\varepsilon$ be a $(PS)_d$ sequence for I_ε with $d < \frac{1}{N}(a_0 S)^{N/p}$ and $v_n \rightharpoonup 0$ weakly in X_ε. If $v_n \not\to 0$ strongly in X_ε, then $d \geq m(a_\infty)$.*

Proof. The proof is an adaptation of that presented in [9, Lemma 3.2]. For the reader's convenience, we sketch it here. Let $t_n \in (0, +\infty)$ be such that $t_n v_n \in \mathcal{M}_{a_\infty}$. By using Lemma 3.1 and arguing as in [9, Lemma 3.2] we conclude that

$$t_0 := \limsup_{n \to \infty} t_n \leq 1.$$

If $t_0 < 1$ we may suppose, without loss of generality, that $t_n < 1$ for all $n \in \mathbb{N}$. Thus $t_n^{p^*} \int |v_n|^{p^*} \leq \int |v_n|^{p^*}$ and we can argue exactly as in [9, Lemma 3.2] to conclude that $d \geq m(a_\infty)$.

Let us now consider the complementary case $t_0 = 1$. Taking a subsequence if necessary, we may suppose that $t_n \to 1$. Hence

$$\begin{aligned} d + o_n(1) &\geq m(a_\infty) + I_\varepsilon(v_n) - E_{a_\infty}(t_n v_n) \\ &= m(a_\infty) + \frac{1}{p}\int (a(\varepsilon x) - t_n^p a_\infty)|\nabla v_n|^p \\ &\quad - \int (F(t_n v_n) - F(v_n)) + \frac{(t_n^{p^*} - 1)}{p^*}\int |v_n|^{p^*}. \end{aligned} \tag{3.5}$$

Since (v_n) is bounded in $L^{p^*}(\mathbb{R}^N)$, we have that $(t_n^{p^*} - 1)\int |v_n|^{p^*} = o_n(1)$. Moreover, a straightforward application of the Mean Value theorem, (f_1)–(f_2) and the Lebesgue theorem imply that $\int (F(t_n v_n) - F(v_n)) = o_n(1)$. Recalling that $t_n \to 1$, we can use these remarks, (3.5) and Lemma 3.2 to obtain, for any $\delta > 0$, a number $N_\delta > 0$ such that

$$d + o_n(1) \geq m(a_\infty) - \delta/p + o_n(1),$$

for any $n \geq N_\delta$. By taking $n \to \infty$ and $\delta \to 0$, we conclude that $d \geq m(a_\infty)$. This finishes the proof of the lemma. $\square$

We end this section by proving a local compactness condition for I_ε.

Proposition 3.4. *Let*

$$c^* := \min\left\{m(a_\infty), \frac{1}{N}(a_0 S)^{N/p}\right\}. \tag{3.6}$$

Then the functional I_ε satisfies the $(PS)_c$ condition at any level $c < c^*$.

Proof. Let $(u_n) \subset X_\varepsilon$ be such that $I_\varepsilon(u_n) \to c$ and $I'_\varepsilon(u_n) \to 0$. Since (u_n) is bounded, up to a subsequence, $u_n \rightharpoonup u$ weakly in X_ε. Moreover, u is a critical point of I_ε and it follows from (f_4) that

$$I_\varepsilon(u) = I_\varepsilon(u) - \frac{1}{p}\langle I'_\varepsilon(u), u\rangle = \int \left(\frac{1}{p} f(u)u - F(u)\right) + \frac{1}{N}\int |v_n|^{p^*} \geq 0.$$

Setting $v_n := u_n - u$ and arguing as in the proof of [1, Lemma 3.2] we can show that $I'_\varepsilon(v_n) \to 0$ and

$$I_\varepsilon(v_n) \to c - I_\varepsilon(u) = d < c^* \leq \frac{1}{N}(a_0 S)^{N/p},$$

where we have used $c < c^*$ and $I_\varepsilon(u) \geq 0$. It follows from Lemma 3.3 that $v_n \to 0$, i.e., $u_n \to u$ in X_ε. The proposition is proved. $\square$

Remark 3.5. For future reference we note that, since $m(a_0) < m(a_\infty)$, we can use Lemma 2.1 to conclude that $m(a_0) < c^*$.

4. Proof of Theorems 1.1 and 1.2

We start this section with the following auxiliary result.

Lemma 4.1. $\lim_{\varepsilon \to 0^+} c_\varepsilon = m(a_0)$.

Proof. We follow the arguments of [2, Lemma 3]. Since $c_\varepsilon \geq m(a_0)$ for all $\varepsilon > 0$, it suffices to check that $\limsup_{\varepsilon\to 0^+} c_\varepsilon \leq m(a_0)$. Let $\eta \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\eta \equiv 1$ on $B_1(0)$ and $\eta \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$. For any given $r > 0$ we define $v_r(x) := \eta(x/r)\omega(x)$, where ω is a ground state solution of the problem (AP_{a_0}).

Let $t_{\varepsilon,r} > 0$ be such that $t_{\varepsilon,r} v_r \in \mathcal{N}_\varepsilon$ and note that

$$c_\varepsilon \leq I_\varepsilon(t_{\varepsilon,r} v_r) = \frac{t_{\varepsilon,r}^p}{p} \int \left(a(\varepsilon x)|\nabla v_r|^p + |v_r|^p\right) - \int \left(F(t_{\varepsilon,r} v_r) + \frac{t_{\varepsilon,r}^{p^*}}{p^*}|v_r|^{p^*}\right).$$

It is easy to check that, for r fixed, $t_{\varepsilon,r} \to t_r > 0$ as $\varepsilon \to 0$. Moreover, without loss of generality, we may suppose that $a(0) = a_0$. Hence, since v_r has compact support, we can use Lebesgue's theorem to get

$$\limsup_{\varepsilon\to 0^+} c_\varepsilon \leq \frac{t_r^p}{p} \int \left(a_0|\nabla v_r|^p + |v_r|^p\right) - \int \left(F(t_r v_r) + \frac{t_r^{p^*}}{p^*}|v_r|^{p^*}\right) = E_{a_0}(t_r v_r).$$

Since $\omega \in \mathcal{M}_{a_0}$ and $v_r \to \omega$ in $W^{1,p}(\mathbb{R}^N)$ as $r \to \infty$, we can check that $t_r \to 1$ as $r \to \infty$. Thus, it follows from the above expression that

$$\limsup_{\varepsilon\to 0^+} c_\varepsilon \leq \lim_{r\to\infty} E_{a_0}(t_r v_r) = E_{a_0}(\omega) = m(a_0).$$

The lemma is proved. $\square$

We are now ready to present the proof of our existence theorem.

Proof of Theorem 1.1. Let c^* be the critical level defined in (3.6). Since $m(a_0) < c^*$, we can use Lemma 4.1 to obtain $\varepsilon_0 > 0$ such that $c_\varepsilon < c^*$ for any $\varepsilon \in (0, \varepsilon_0)$. For these values of ε, since I_ε has the Mountain Pass geometry, we can take a sequence $(u_n) \subset X_\varepsilon$ such that

$$I_\varepsilon(u_n) \to c_\varepsilon \quad \text{and} \quad I_\varepsilon'(u_n) \to 0.$$

By using Proposition 3.4 we guarantee that, along a subsequence, $u_n \to u$ with u being such that $I_\varepsilon(u) = c_\varepsilon$ and $I_\varepsilon'(u) = 0$. It remains to show that u is positive. So, let $u^\pm := \max\{\pm u, 0\}$ be the positive (negative) part of u. We have that

$$0 = \langle I_\varepsilon'(u), u^- \rangle = \|u^-\|_\varepsilon^p - \int f(u)u^- - \int |u|^{p^*-2}uu^- = \|u^-\|_\varepsilon^p + \int |u^-|^{p^*},$$

and therefore $u \geq 0$ in $\mathbb{R}^N$. By adapting arguments from [12, Theorem 1.11] we conclude that $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $0 < \alpha < 1$. It follows from Harnack's inequality [20] that $u(x) > 0$ for all $x \in \mathbb{R}^N$. The theorem is proved. $\square$

From now on we will denote by ω a ground state solution of the problem (AP_{a_0}). Let $\eta \in C^\infty(\mathbb{R}^+, [0,1])$ be a cut-off function such that $\eta(s) \equiv 1$ on $[0, 1/2]$ and $\eta \equiv 0$ on $[1, \infty)$. We recall that M is the set of global minima of the potential a and define, for each $y \in M$, $\psi_{\varepsilon,y} : \mathbb{R}^N \to \mathbb{R}$ by setting

$$\psi_{\varepsilon,y}(x) := \eta(|\varepsilon x - y|)\omega\left(\frac{\varepsilon x - y}{\varepsilon}\right).$$

Let t_ε be the unique positive number satisfying

$$\max_{t\geq 0} I_\varepsilon(t\psi_{\varepsilon,y}) = I_\varepsilon(t_\varepsilon \psi_{\varepsilon,y})$$

and define the map $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ in the following way

$$\Phi_\varepsilon(y) := t_\varepsilon \psi_{\varepsilon,y} \equiv \Phi_{\varepsilon,y}. \tag{4.1}$$

In view of the definition of t_ε we have that the above map is well defined. Moreover, the following holds.

Lemma 4.2. $\lim_{\varepsilon\to 0^+} I_\varepsilon(\Phi_{\varepsilon,y}) = m(a_0)$ *uniformly for* $y \in M$.

Proof. Since the proof is quite similar to that of [9, Lemma 5.1] we only sketch it. Arguing by contradiction, we suppose that the lemma is false. Then there exist $\delta > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0^+$ such that

$$|I_{\varepsilon_n}(\Phi_{\varepsilon_n,y_n}) - m(a_0)| \geq \delta > 0. \tag{4.2}$$

Recall that $\Phi_{\varepsilon_n,y_n} = t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}$. By using the Lebesgue theorem, we can check that

$$\|\psi_{\varepsilon_n,y_n}\|^p_{\varepsilon_n} \to \|\omega\|^p_{W_{a_0}}, \quad |\psi_{\varepsilon_n,y_n}|^{p^*}_{p^*} \to |\omega|^{p^*}_{p^*}, \tag{4.3}$$

$$\int f(\psi_{\varepsilon_n,y_n})\psi_{\varepsilon_n,y_n} \to \int f(\omega)\omega \quad \text{and} \quad \int F(\psi_{\varepsilon_n,y_n}) \to \int F(\omega). \tag{4.4}$$

The above expressions and the same calculations made in [9, Lemma 5.1] show that, up to a subsequence, $t_{\varepsilon_n} \to 1$. Thus, taking the limit in

$$\begin{aligned} I_{\varepsilon_n}(\Phi_{\varepsilon_n,y_n}) = \frac{t_n^p}{p}\int \left(a(\varepsilon_n z + y_n)|\nabla(\eta(|\varepsilon_n z|))\omega(z)|^p + |\eta(|\varepsilon_n z|)\omega(z)|^p\right) \mathrm{d}z \\ - \int F(t_n\eta(|\varepsilon_n z|))\omega(z))\, \mathrm{d}z - \frac{t_n^{p^*}}{p^*}\int |\eta(|\varepsilon_n z|)\omega(z)|^{p^*} \mathrm{d}z \end{aligned}$$

and using (4.3) and (4.4) we get

$$\lim_{n\to\infty} I_{\varepsilon_n}(\Phi_{\varepsilon_n,y_n}) = E_{a_0}(\omega) = m(a_0),$$

which contradicts (4.2) and proves the lemma. □

For any $\delta > 0$, let $\rho = \rho_\delta > 0$ be such that $M_\delta \subset B_\rho(0)$. Let $\chi : \mathbb{R}^N \to \mathbb{R}^N$ be defined as $\chi(x) := x$ for $|x| < \rho$ and $\chi(x) := \rho x/|x|$ for $|x| \geq \rho$. Finally, let us consider the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ given by

$$\beta_\varepsilon(u) := \frac{\int \chi(\varepsilon x)|u(x)|^p \, \mathrm{d}x}{\int |u(x)|^p \, \mathrm{d}x}.$$

Since $M \subset B_\rho(0)$, we can use the definition of χ and the Lebesgue theorem to conclude that

$$\lim_{\varepsilon\to 0^+} \beta_\varepsilon(\Phi_{\varepsilon,y}) = y \ \text{ uniformly for } y \in M. \tag{4.5}$$

Following [8], we introduce a subset of $\mathcal{N}_\varepsilon$ which will be useful in the future. We take a function $h : [0,\infty) \to [0,\infty)$ such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ and set

$$\Sigma_\varepsilon := \{u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) \leq m(a_0) + h(\varepsilon)\}.$$

Given $y \in M$, we can use Lemma 4.2 to conclude that $h(\varepsilon) = |I_\varepsilon(\Phi_{\varepsilon,y}) - m(a_0)|$ is such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Thus, $\Phi_{\varepsilon,y} \in \Sigma_\varepsilon$ and therefore $\Sigma_\varepsilon \neq \varnothing$ for any $\varepsilon > 0$. By arguing as in [9, Lemma 5.4] we can obtain the following property of the manifold Σ_ε.

Lemma 4.3. *For any $\delta > 0$ we have that*

$$\lim_{\varepsilon\to 0^+} \sup_{u\in\Sigma_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

Since we intend to apply Ljusternik–Shcnirelmann theory for the functional I_ε constrained to $\mathcal{N}_\varepsilon$, we will denote by $\|I'_\varepsilon(u)\|_*$ the norm of the derivative of I_ε restricted to $\mathcal{N}_\varepsilon$ at the point u. The following result is a version of Proposition 3.4 for the constrained functional.

Proposition 4.4. *The functional I_ε restricted to $\mathcal{N}_\varepsilon$ satisfies the $(PS)_c$ condition at any level $c < c^*$, where c^* is defined in (3.6).*

Proof. Let $(u_n) \subset \mathcal{N}_\varepsilon$ be such that $I_\varepsilon(u_n) \to c$ and $\|I'_\varepsilon(u_n)\|_* \to 0$. Then there exists $(\lambda_n) \subset \mathbb{R}$ such that

$$I'_\varepsilon(u_n) = \lambda_n J'_\varepsilon(u_n) + o_n(1), \tag{4.6}$$

where $J_\varepsilon : X_\varepsilon \to \mathbb{R}$ is given by

$$J_\varepsilon(u) := \|u\|_\varepsilon^p - \int f(u)u - \int |u|^{p^*}.$$

By (f_4)

$$\begin{aligned}\langle J'_\varepsilon(u_n), u_n\rangle &= \int \left((p-1)f(u_n)u_n - f'(u_n)u_n^2\right) - (p^*-p)\int |v_n|^{p^*}\\ &\le -(p^*-p)\int |v_n|^{p^*} \le 0,\end{aligned}$$

and therefore we may suppose that $\langle J'_\varepsilon(u_n), u_n\rangle \to l \le 0$. If $l = 0$, it follows from $|\langle J'_\varepsilon(u_n), u_n\rangle| \ge (p^*-p)\int_{\mathbb{R}^N} |u_n|^{p^*} \ge 0$ that $u_n \to 0$ in $L^{p^*}(\mathbb{R}^N)$. Recalling that (u_n) is bounded in X_ε we can use interpolation and argue as in the proof of Lemma 2.2 to conclude that $\int f(u_n)u_n \to 0$ and $\int F(u_n) \to 0$. Thus, since $(u_n) \subset \mathcal{N}_\varepsilon$, we get

$$c = \lim_{n\to\infty} I_\varepsilon(u_n) = \lim_{n\to\infty}\left\{\left(\frac{1}{p} - \frac{1}{p^*}\right)|u_n|_{p^*}^{p^*} + \frac{1}{p}\int f(u_n)u_n - \int F(u_n)\right\} = 0,$$

which contradicts $c \ge c_\varepsilon > 0$. Hence, $l \ne 0$ and therefore $\lambda_n = o_n(1)$. By using (4.6), we conclude that $I'_\varepsilon(u_n) = o_n(1)$, that is, (u_n) is a $(PS)_c$ sequence for the unconstrained functional I_ε. The result now follows from Proposition 3.4. □

We are now ready to present the proof of the multiplicity result.

Proof of Theorem 1.2. Given $\delta > 0$ we can use (4.5), Lemmas 4.2 and 4.3, and argue as in [8, Section 6] to obtain $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the diagram

$$M \xrightarrow{\Phi_\varepsilon} \Sigma_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined and $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. Since $m(a_0) < c^*$, we can use the definition of Σ_ε and Proposition 4.4 to guarantee that I_ε satisfies the Palais–Smale condition in Σ_ε (taking ε_δ smaller if necessary). Standard Ljusternik–Schnirelmann theory provides at least $\operatorname{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon)$ critical points u_i of I_ε restricted to $\mathcal{N}_\varepsilon$. Arguing along the same lines of the proof of Proposition 4.4 we can check that u_i is a critical point of the unconstrained functional I_ε. As before, each u_i is positive in $\mathbb{R}^N$. Finally, the same ideas contained in the proof of [4, Lemma 4.3] show that $\mathrm{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon) \geq \mathrm{cat}_{M_\delta}(M)$, which concludes the proof. □

5. Proof of Theorem 1.3

Let $K > 0$ to be determined later and let $\widehat{f_\lambda} \in C(\mathbb{R}, \mathbb{R})$ be given by

$$\widehat{f_\lambda}(s) := \begin{cases} 0 & \text{if } s < 0, \\ s^{q-1} + \lambda s^{t-1} & \text{if } 0 \leq s < K, \\ s^{q-1} + \lambda K^{t-q} s^{q-1} & \text{if } s \geq K. \end{cases}$$

Consider $\alpha, \gamma \in \mathbb{R}$ such that $\alpha < 1 < \gamma$ and $\eta \in C^1(\mathbb{R}, \mathbb{R})$ satisfying

(η_1) $\eta(s) \leq \widehat{f_\lambda}(s)$ for all $s \in [\alpha K, \gamma K]$,
(η_2) $\eta(\alpha K) = \widehat{f_\lambda}(\alpha K)$, $\eta(\gamma K) = \widehat{f_\lambda}(\gamma K)$, $\eta'(\alpha K) = \widehat{f_\lambda}'(\alpha K)$ and $\eta'(\gamma K) = \widehat{f_\lambda}'(\gamma K)$,
(η_3) the map $s \mapsto \eta(s)/s^{p-1}$ is increasing for all $s \in [\alpha K, \gamma K]$.

Now, if we define $f_\lambda \in C^1(\mathbb{R}, \mathbb{R})$ as

$$f_\lambda(s) := \begin{cases} \widehat{f_\lambda}(s) & \text{if } s \notin [\alpha K, \gamma K], \\ \eta(s) & \text{if } s \in [\alpha K, \gamma K], \end{cases}$$

we have that

$$f_\lambda(s) \leq (1 + \lambda K^{t-q}) s^{q-1} \quad \text{for all } s > 0. \tag{5.1}$$

Thus, we can easily conclude that f_λ satisfies

(f_λ^1) $f_\lambda(s) = o(s^{p-1})$ as $s \to 0^+$,
(f_λ^2) $f_\lambda(s) = o(s^{q_1-1})$ as $s \to \infty$, for some $q_1 \in (q, p^*)$,
(f_λ^3) $0 < \theta \int_0^s f_\lambda(\tau) d\tau \leq s f_\lambda(s)$ for some $\theta \in (p, q_1)$ and for all $s > 0$,
(f_λ^4) $f_\lambda'(s)s - (p-1) f_\lambda(s) \geq (q-p) s^{q-1}$ for all $s > 0$.

Hence, f_λ is a superlinear function with subcritical growth. By directly applying [9, Theorem 1.2] we obtain the following multiplicity result for a truncated version of $(P_{\varepsilon,\lambda})$.

Proposition 5.1. *Let $\lambda \geq 0$ be fixed. Then, for any $\delta > 0$ given, there exists $\varepsilon_{\lambda,\delta} > 0$ such that, for any $\varepsilon \in (0, \varepsilon_{\lambda,\delta})$, the truncated problem*

$$\begin{cases} -\mathrm{div}(a(\varepsilon x)|\nabla u|^{p-2}\nabla u) + u^{p-1} = f_\lambda(u) & \text{in } \mathbb{R}^N, \\ u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \ u(x) > 0 & \text{for all } x \in \mathbb{R}^N, \end{cases} \tag{$TP_{\varepsilon,\lambda}$}$$

has at least $\mathrm{cat}_{M_\delta}(M)$ solutions.

Let u be a solution of $(TP_{\varepsilon,\lambda})$ which verifies

$$u(x) \leq \alpha K \quad \text{for all } x \in \mathbb{R}^N. \tag{5.2}$$

Then, in view of the definition of f_λ, we have that $f_\lambda(u) = u^{q-1} + \lambda u^{t-1}$ and therefore u is also a solution of the original problem $(P_{\varepsilon,\lambda})$. Thus, in order to prove Theorem 1.3, it suffices to show that, for λ small enough, the solutions obtained by Proposition 5.1 verify the above inequality.

We start by noting that the solutions of $(TP_{\varepsilon,\lambda})$ are critical points of the functional $I_{\varepsilon,\lambda} : X_\varepsilon \to \mathbb{R}$ given by

$$I_{\varepsilon,\lambda}(u) := \frac{1}{p}\|u\|_\varepsilon^p - \int_{\mathbb{R}^N} F_\lambda(u),$$

where $F_\lambda(s) := \int_0^s f_\lambda(\tau)\mathrm{d}\tau$. As in the first part of the paper, it is important to consider the autonomous problem

$$\begin{cases} -a_0 \operatorname{div}(|\nabla u|^{p-2}\nabla u) + u^{p-1} = f_\lambda(u) & \text{in } \mathbb{R}^N, \\ u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N),\ u(x) > 0 & \text{for all } x \in \mathbb{R}^N. \end{cases}$$

Let $m(0,\lambda)$ be the ground state level of the above problem. Since f_λ is nonnegative, for any $\lambda \geq 0$ there holds

$$m(0,\lambda) = \inf_{u\in X\setminus\{0\}} \sup_{t\geq 0} I_{0,\lambda}(tu) \leq \inf_{u\in X\setminus\{0\}} \sup_{t\geq 0} I_{0,0}(tu) = m(0,0).$$

Now, let $u_{\varepsilon,\lambda}$ be one of the solutions given by Proposition 5.1. A simple inspection of the proof of [9, Theorem 1.2] (which is analogous to that of Theorem 1.2 in the present paper) shows that $u_{\varepsilon,\lambda}$ satisfies the following energy estimate

$$I_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}) \leq m(0,\lambda) + h_\lambda(\varepsilon),$$

with $h_\lambda(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Thus, decreasing $\varepsilon_{\lambda,\delta}$ if necessary, we may suppose that

$$I_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}) \leq m(0,0) + 1$$

for any $\varepsilon \in (0, \varepsilon_{\lambda,\delta})$. On the other hand, it follows from (f_λ^3) that

$$I_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}) = I_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}) - \frac{1}{\theta}\left\langle I'_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}), u_{\varepsilon,\lambda}\right\rangle \geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_{\varepsilon,\lambda}\|_\varepsilon^p,$$

and therefore we conclude that, for any $\varepsilon \in (0, \varepsilon_{\lambda,\delta})$,

$$\|u_{\varepsilon,\lambda}\|_\varepsilon^p \leq (m(0,0)+1)\left(\frac{\theta p}{\theta - p}\right). \tag{5.3}$$

We are now able to use the above estimate and some ideas contained in [7] to prove Theorem 1.3 as follows.

Proof of Theorem 1.3. For any $\lambda \geq 0$ and $\delta > 0$ we can apply Proposition 5.1 to obtain, for any $\varepsilon \in (0, \varepsilon_{\lambda,\delta})$, $\operatorname{cat}_{M_\delta}(M)$ solutions of $(TP_{\varepsilon,\lambda})$. Let $u_{\varepsilon,\lambda}$ be one of these solutions. We shall assume that $\varepsilon_{\lambda,\delta}$ is small in such a way that (5.3) holds. Our aim is to show that, if λ is small enough, the solution $u_{\varepsilon,\lambda}$ verifies the inequality in (5.2). To save notation, we will denote $u := u_{\varepsilon,\lambda}$.

For each $L > 0$, we define

$$u_L := \begin{cases} u & \text{if } u \leq L, \\ L & \text{if } u > L, \end{cases}$$

$$z_L := u u_L^{p(\beta-1)} \quad \text{and} \quad w_L := u u_L^{\beta-1},$$

where $\beta > 1$ is arbitrary. Taking z_L as a test function in $(TP_{\varepsilon,\lambda})$ we obtain

$$\begin{aligned} \int u_L^{p(\beta-1)} a(\varepsilon x)|\nabla u|^p &= -p(\beta-1)\int u_L^{p(\beta-1)-1} u|\nabla u|^{p-2}\nabla u\cdot\nabla u_L \\ &\quad + \int f_\lambda(u) u u_L^{p(\beta-1)} - \int u^p u_L^{p(\beta-1)} \\ &\leq (1+\lambda K^{t-q})\int u^q u_L^{p(\beta-1)}, \end{aligned}$$

where we have used $u, u_L > 0$, (5.1) and the inequality below

$$\int u_L^{p(\beta-1)-1} u |\nabla u|^{p-2} \nabla u \cdot \nabla u_L = \int_{\{u \le L\}} u^{p(\beta-1)} |\nabla u|^p \ge 0.$$

It follows from (a_1) that

$$\int u_L^{p(\beta-1)} |\nabla u|^p \le C_{\lambda,K} \int u^q u_L^{p(\beta-1)},$$

where $C_{\lambda,K} = a_0^{-1}(1 + \lambda K^{t-q})$. By using the Sobolev embedding and this inequality we get

$$\begin{aligned} |w_L|_{p^*}^p &\le C_1 \int |\nabla(w_L)|^p = C_1 \int |\nabla(u u_L^{\beta-1})|^p \\ &\le C_2(\beta-1)^p \int_{\{u \le L\}} u_L^{p(\beta-1)} |\nabla u|^p + C_2 \int u_L^{p(\beta-1)} |\nabla u|^p \\ &\le C_3 \beta^p \int u_L^{p(\beta-1)} |\nabla u|^p \le C_3 \beta^p C_{\lambda,K} \int u^q u_L^{p(\beta-1)}. \end{aligned}$$

Let $\alpha^* := \frac{pp^*}{p^*-(q-p)}$. Since $u^q u_L^{p(\beta-1)} = u^{q-p} w_L^p$, we can use the above expression, Hölder's inequality and (5.3) to conclude that, whenever $w_L \in L^{\alpha^*}(\mathbb{R}^N)$, it holds

$$\begin{aligned} |w_L|_{p^*}^p &\le C_3 \beta^p C_{\lambda,K} \left(\int u^{p^*} \right)^{(q-p)/p^*} \left(\int w_L^{\alpha^*} \right)^{p/\alpha^*} \\ &\le C_4 \beta^p C_{\lambda,K} \|u\|_\varepsilon^{q-p} |w_L|_{\alpha^*}^p \le C_5 \beta^p C_{\lambda,K} |w_L|_{\alpha^*}^p, \end{aligned}$$

where $C_5 := C_4 (m(0,0)+1)^{(q-p)/p} \left(\frac{\theta p}{\theta - p} \right)^{(q-p)/p}$ is independent of ε and λ.

Since $u_L \le u$, we conclude that $w_L \in L^{\alpha^*}(\mathbb{R}^N)$, whenever $u^\beta \in L^{\alpha^*}(\mathbb{R}^N)$. If this is the case, it follows from the above inequality that

$$\left(\int u^{p^*} u_L^{p^*(\beta-1)} \right)^{p/p^*} \le C_5 \beta^p C_{\lambda,K} \left(\int (u u_L^{\beta-1})^{\alpha^*} \right)^{p/\alpha^*} \le C_5 \beta^p C_{\lambda,K} |u|_{\beta\alpha^*}^{\beta p}.$$

By Fatou's lemma in the variable L, we get

$$|u|_{\beta p^*} \le \left(C_5 C_{\lambda,K} \right)^{1/(\beta p)} \beta^{1/\beta} |u|_{\beta\alpha^*}, \tag{5.4}$$

whenever $u^{\beta\alpha^*} \in L^1(\mathbb{R}^N)$.

We now set $\beta := p^*/\alpha^* > 1$ and note that, since $u \in L^{p^*}(\mathbb{R}^N)$, the above inequality holds for this choice of β. Thus, since $\beta^2 \alpha^* = \beta p^*$, it follows that (5.4) also holds with β replaced by β^2. Hence,

$$|u|_{\beta^2 p^*} \le (C_5 C_{\lambda,K})^{1/(\beta^2 p)} \beta^{2/\beta^2} |u|_{\beta^2\alpha^*} \le (C_5 C_{\lambda,K})^{\frac{1}{p}\left(\frac{1}{\beta} + \frac{1}{\beta^2}\right)} \beta^{\frac{1}{\beta} + \frac{2}{\beta^2}} |u|_{\beta\alpha^*}.$$

By iterating this process and using that $\beta\alpha^* = p^*$, we obtain

$$|u|_{\beta^m p^*} \le (C_5 C_{\lambda,K})^{p^{-1} \sum_{i=1}^m \beta^{-i}} \beta^{\sum_{i=1}^m i\beta^{-i}} |u|_{p^*}.$$

Taking the limit as $m \to \infty$ and using (5.3) again, we get

$$|u|_\infty \le (C_6 C_{\lambda,K})^{\sigma_1} \beta^{\sigma_2},$$

with $\sigma_1 = p^{-1} \sum_{i=1}^\infty \beta^{-i}$ and $\sigma_2 = \sum_{i=1}^\infty i\beta^{-i}$.

It remains to check that, for a suitable value of K and λ small enough, we have

$$\left(C_6 C_{\lambda,K}\right)^{\sigma_1} \beta^{\sigma_2} \leq \alpha K,$$

or equivalently

$$1 + \lambda K^{t-q} \leq a_0 C_6^{-1} \beta^{-\sigma_2/\sigma_1} \alpha^{1/\sigma_1} K^{1/\sigma_1} = C_7 K^{1/\sigma_1}.$$

So, we choose $K > 0$ such that $C_7 K^{1/\sigma_1} = 2$ and take $\lambda \geq 0$ such that $\lambda \leq \lambda_0 := K^{q-t}$. As observed before, the theorem holds for this choice of λ_0. $\square$

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