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Multiple minimal nodal solutions for a quasilinear Schrödinger equation with symmetric potential

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Abstract

We deal with the quasilinear Schrödinger equation

$$-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) + \big(\lambda a(x)+1\big)|u|^{p-2}u = |u|^{q-2}u, \quad u \in W^{1,p}\big(\mathbb{R}^N\big),$$

where $2 \leqslant p < N$, $\lambda > 0$, and $p < q < p^\star = Np/(N-p)$. The potential $a \geqslant 0$ has a potential well and is invariant under an orthogonal involution of $\mathbb{R}^N$. We apply variational methods to obtain, for λ large, existence of solutions which change sign exactly once. We study the concentration behavior of these solutions as $\lambda \to \infty$. By taking q close $p^\star$, we also relate the number of solutions which change sign exactly once with the equivariant topology of the set where the potential a vanishes.

Keywords: Schrödinger equation; p-Laplacian; Equivariant category; Symmetry

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1. Introduction and statement of results

The goal of this article is to study the number of solutions of the quasilinear Schrödinger equation

$$\begin{cases} -\Delta_p u + (\lambda a(x)+1)|u|^{p-2}u = |u|^{q-2}u & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{$S_{\lambda,q}$}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator and $2 \leqslant p < N$. We will impose some symmetry properties and look for nodal solutions of $(S_{\lambda,q})$. The parameters λ and q are such that $\lambda > 0$ and $p < q < p^\star$, where $p^\star = Np/(N-p)$ is the critical Sobolev exponent. For the potential a we assume that

(A_1) $a \in C(\mathbb{R}^N, \mathbb{R})$ is nonnegative, $\Omega = \operatorname{int} a^{-1}(0)$ is a nonempty set with smooth boundary and $\bar{\Omega} = a^{-1}(0)$,

(A_2) there exists $M_0 > 0$ such that

$$\mathcal{L}\big(\big\{x \in \mathbb{R}^N \colon a(x) \leqslant M_0\big\}\big) < \infty,$$

where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^N$.

The above hypotheses were introduced by Bartsch and Wang in [3], where they considered the problem $(S_{\lambda,q})$ for the particular case $p = 2$. They showed that, for large values of λ, the problem $(S_{\lambda,q})$ has a positive least energy solution. Moreover, as $\lambda \to \infty$, these solutions concentrate at a positive solution of the Dirichlet problem

$$-\Delta_p u + |u|^{p-2}u = |u|^{q-2}u, \quad u \in W_0^{1,p}(\Omega). \tag{D_q}$$

Recalling that Benci and Cerami [4] showed that, for $p = 2$, q close to $2^\star$ and Ω bounded, the problem (D_q) has at least $\operatorname{cat}(\Omega)$ positive solutions, Bartsch and Wang proved in [3] that the same holds for the problem $(S_{\lambda,q})$, where $\operatorname{cat}(\Omega)$ stands the Ljusternik–Schnirelmann category of the set Ω.

Recently, using ideas from [6] and assuming that Ω has some symmetry, the author showed [11] that there is also an effect of the domain topology in the number of solutions u of (D_q) which change sign exactly once; that is, the set $\Omega \setminus u^{-1}(0)$ has exactly two connected components, u is positive in one of them and negative in the other. It is natural to ask if the same holds for the problem $(S_{\lambda,q})$. The aim of this work is to give an affirmative answer to this question.

More specifically, we deal with the problem

$$\begin{cases} -\Delta_p u + (\lambda a(x)+1)|u|^{p-2}u = |u|^{q-2}u & \text{in } \mathbb{R}^N, \\ u(\tau x) = -u(x) & \text{for all } x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{$S^\tau_{\lambda,q}$}$$

where $\lambda > 0$, $2 \leqslant p < N$, $p < q < p^\star$, and $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is an orthogonal linear function such that $\tau \neq \mathrm{Id}$ and $\tau^2 = \mathrm{Id}$, with Id being the identity of $\mathbb{R}^N$. The potential a satisfies (A_1), (A_2), and

(A_3) $a(\tau x) = a(x)$ for all $x \in \mathbb{R}^N$.

Our first result concerns the existence of solutions for $(S^\tau_{\lambda,q})$ and can be stated as follows.

Theorem 1.1. *Suppose (A_1)–(A_3) hold. Then there exists $\Lambda_0 = \Lambda_0(q) > 0$ such that, for every $\lambda \geqslant \Lambda_0$, the problem $(S^\tau_{\lambda,q})$ has at least one pair of solutions which change sign exactly once.*

The proof of the above result relies in minimizing the associated functional

$$I_{\lambda,q}(u) = \frac{1}{p} \int_{\mathbb{R}^N} \big(|\nabla u|^p + \big(\lambda a(x) + 1\big)|u|^p\big)\,dx - \frac{1}{q} \int_{\mathbb{R}^N} |u|^q\,dx$$

in some appropriated manifold of

$$X = \Big\{u \in W^{1,p}\big(\mathbb{R}^N\big):\ \int_{\mathbb{R}^N} a(x)|u|^p\,dx < \infty\Big\},$$

and relating the number of nodal regions of a critical point u_0 with its energy $I_{\lambda,q}(u_0)$. Similarly to [3], the τ-version of (D_q) acts as a limit problem for $(S^\tau_{\lambda,q})$. Thus, the following concentration result holds.

Theorem 1.2. *Let $\lambda_n \to \infty$ as $n \to \infty$ and (u_n) be a sequence of nontrivial solutions of $(S^\tau_{\lambda_n,q})$ such that $I_{\lambda_n,q}(u_n)$ is bounded. Then, up to a subsequence, $u_n \to u$ strongly in $W^{1,p}(\mathbb{R}^N)$ with u being a nontrivial solution of the Dirichlet problem*

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = |u|^{q-2}u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\\ u(\tau x) = -u(x) & \text{for all } x \in \Omega. \end{cases} \qquad \big(D^\tau_q\big)$$

By taking advantage of the symmetry and the arguments contained in [11], we can obtain, for q close to $p^\star$ and λ large enough, the following multiplicity result.

Theorem 1.3. *Suppose (A_1)–(A_3) hold and Ω is bounded. Then there exists $\tilde{q} \in (p, p^\star)$ with the property that, for each $q \in (\tilde{q}, p^\star)$, there is a number $\Lambda(q) > 0$ such that, for every $\lambda \geqslant \Lambda(q)$, the problem $(S^\tau_{\lambda,q})$ has at least $\tau\text{-cat}_\Omega(\Omega \setminus \Omega^\tau)$ pairs of solutions which change sign exactly once.*

Here, $\Omega^\tau = \{x \in \Omega:\ \tau x = x\}$ and τ-cat is the τ-equivariant Ljusternik–Schnirelmann category (see Section 4). There are several situations where the equivariant category turns out to be larger than the nonequivariant one. The classical example is the case of the unit sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ with $\tau = -\mathrm{Id}$. In this case $\mathrm{cat}(\mathbb{S}^{N-1}) = 2$, whereas $\tau\text{-cat}(\mathbb{S}^{N-1}) = N$. Consequently, as an application of Theorem 1.3 we have the following corollary.

Corollary 1.4. *Suppose (A_1) and (A_2) hold, Ω is bounded and symmetric with respect to the origin, and $0 \notin \Omega$. Assume further that the potential a is even and there is an odd map*

$\varphi : \mathbb{S}^{N-1} \to \Omega$. Then there exists $\tilde{q} \in (p, p^\star)$ with the property that, for each $q \in (\tilde{q}, p^\star)$, there is a number $\Lambda(q) > 0$ such that, for every $\lambda \geqslant \Lambda(q)$, the problem

$$\begin{cases} -\Delta_p u + (\lambda a(x) + 1)|u|^{p-2}u = |u|^{q-2}u & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases}$$

has at least N pairs of odd solutions which change sign exactly once.

We point out that, for a fixed $q \in (p, p^\star)$ (or $q \in (\tilde{q}, p^\star)$ in Theorem 1.3), the energy of the solutions obtained in Theorem 1.1 (or Theorem 1.3) is bounded independently of λ. Thus, the concentration result of Theorem 1.2 holds for such solutions. Moreover, in this case, it can be proved that the limit solution u changes sign exactly once in Ω.

It is worthwhile to mention that the above results seem to be new even in the case $p = 2$. In [8] Clapp and Ding considered the problem

$$-\Delta u + \lambda a(x)u = \mu u + |u|^{2^\star-2}u \quad \text{in } \mathbb{R}^N, \qquad u(\tau x) = -u(x) \quad \forall x \in \mathbb{R}^N$$

and proved, for positive and small values of μ, results concerning the existence and concentration of solutions in $W^{1,2}(\mathbb{R}^N)$ as $\mu \to 0$. By taking $\mu \sim 0$, they also showed a relation between the number of solutions of the above problem and the topology of Ω. The results we obtain in this paper complement those of [8] since we consider subcritical powers and we deal with the quasilinear case. The nonlinearity of the p-Laplacian, which makes the calculations more difficult, is compensated here by the homogeneity of the problem. We also would like to mention the work [2] where the quasilinear critical case is studied for positive solutions. Finally, in order to overcome the lack of compactness of the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, we use ideas introduced in [3] for the semilinear case $p = 2$.

The paper is organized as follows. In Section 2 we define the abstract framework and prove Theorems 1.1 and 1.2. Section 3 is devoted to some technical results related to the limit problem (D_q). In Section 4, after recalling some basic facts about equivariant Ljusternik–Schnirelmann theory, we present the proof of Theorem 1.3.

2. Proof of Theorems 1.1 and 1.2

For $s \geqslant 1$ we denote by $|u|_s$ the $L^s(\mathbb{R}^N)$-norm of a function u. For simplicity, we write $\int_{\mathcal{D}} u$ to indicate $\int_{\mathcal{D}} u(x)\,dx$. Let X be the space

$$X = \left\{ u \in W^{1,p}\big(\mathbb{R}^N\big) \colon \int_{\mathbb{R}^N} a(x)|u|^p < \infty \right\},$$

endowed with the norm

$$\|u\|_1^p = \int_{\mathbb{R}^N} \big(|\nabla u|^p + \big(a(x) + 1\big)|u|^p\big),$$

which is clearly equivalent to each of the norms

$$\|u\|_\lambda^p = \int_{\mathbb{R}^N} \big(|\nabla u|^p + (\lambda a(x) + 1)|u|^p\big),$$

for $\lambda > 0$. Conditions (A_1), (A_2) and Sobolev theorem imply that the embedding $X \hookrightarrow L^s(\mathbb{R}^N)$ is continuous for all $p \leqslant s \leqslant p^\star$. Moreover, if $p \leqslant s < p^\star$, then X is compactly embedded in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$. As stated in the introduction, we will look for critical points of $I_{\lambda,q} : X \to \mathbb{R}$ defined by

$$I_{\lambda,q}(u) = \frac{1}{p} \int_{\mathbb{R}^N} \big(|\nabla u|^p + \big(\lambda a(x) + 1\big)|u|^p\big) - \frac{1}{q} \int_{\mathbb{R}^N} |u|^q.$$

We recall that $I_{\lambda,q}$ satisfies the Palais–Smale condition at level $c \in \mathbb{R}$, $(\mathrm{PS})_c$ for short, if any sequence $(u_n) \subset X$ such that $I_{\lambda,q}(u_n) \to c$ and $I'_{\lambda,q}(u_n) \to 0$ possesses a convergent subsequence. In order to verify the Palais–Smale condition for $I_{\lambda,q}$, we follow [3], where the authors deal with the case $p = 2$ and consider nonlinearities more general than $|u|^{q-2}u$.

Lemma 2.1 [3, Lemmas 2.2–2.4]. *Let $(u_n) \subset X$ be a $(\mathrm{PS})_c$ sequence for $I_{\lambda,q}$. Then*

(i) *(u_n) is bounded in X,*
(ii) *$\lim_{n\to\infty} \|u_n\|^p_\lambda = \lim_{n\to\infty} |u_n|^q_q = cpq/(q-p)$,*
(iii) *if $c \neq 0$, then $c \geqslant c_0 > 0$, where c_0 is independent of λ.*

Lemma 2.2 [3, Lemma 2.5]. *Let C_0 be fixed. Then, for any given $\varepsilon > 0$, there exist $\Lambda_\varepsilon > 0$ and $R_\varepsilon > 0$ such that, if (u_n) is a $(\mathrm{PS})_c$ sequence for $I_{\lambda,q}$ with $c \leqslant C_0$ and $\lambda \geqslant \Lambda_\varepsilon$, we have*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |u_n|^q \leqslant \varepsilon,$$

where $B_{R_\varepsilon}(0) = \{x \in \mathbb{R}^N\colon |x| < R_\varepsilon\}$.

The next two results will overcome the lack of Hilbertian structure.

Lemma 2.3 [1, Lemma 3]. *Let $K \geqslant 1$, $s \geqslant 2$, and $A(y) = |y|^{s-2}y$, for $y \in \mathbb{R}^K$. Consider a sequence of vector functions $\eta_n : \mathbb{R}^N \to \mathbb{R}^K$ such that $(\eta_n) \subset (L^s(\mathbb{R}^N))^K$ and $\eta_n(x) \to 0$ for a.e. $x \in \mathbb{R}^N$. Then, if $|\eta_n|_{(L^s(\mathbb{R}^N))^K}$ is bounded, we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \big|A(\eta_n) + A(w) - A(\eta_n + w)\big|^{s/(s-1)} = 0,$$

for each $w \in (L^s(\mathbb{R}^N))^K$ fixed.

Lemma 2.4. *Let $\lambda \geqslant 0$ be fixed and let (u_n) be a $(\mathrm{PS})_c$ sequence for $I_{\lambda,q}$. Then, up to a subsequence, $u_n \rightharpoonup u$ weakly in X with u being a weak solution of $(S_{\lambda,q})$. Moreover, $v_n = u_n - u$ is a $(\mathrm{PS})_{c'}$ sequence for $I_{\lambda,q}$ with $c' = c - I_{\lambda,q}(u)$.*

Proof. Lemma 2.1(i) implies that (u_n) is bounded in X and therefore, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{weakly in } X,$$
$$u_n \to u \quad \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^N) \text{ for all } p \leqslant s < p^\star,$$
$$u_n(x) \to u(x) \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{2.1}$$

We claim that we may suppose that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{for a.e. } x \in \mathbb{R}^N,$$
$$|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i} \rightharpoonup |\nabla u|^{p-2}\frac{\partial u}{\partial x_i} \quad \text{weakly in } \big(L^p(\mathbb{R}^N)\big)',\ 1 \leqslant i \leqslant N, \tag{2.2}$$

where $(L^p(\mathbb{R}^N))'$ stands the dual space of $L^p(\mathbb{R}^N)$. In order to verify the claim, we define $P_n : \mathbb{R}^N \to \mathbb{R}$ by

$$P_n(x) = \big(\big|\nabla u_n(x)\big|^{p-2}\nabla u_n(x) - \big|\nabla u(x)\big|^{p-2}\nabla u(x)\big) \cdot \nabla\big(u_n(x) - u(x)\big).$$

Let $K \subset \mathbb{R}^N$ be a fixed compact set. Given $\varepsilon > 0$ we set $K_\varepsilon = \{x \in \mathbb{R}^N\colon \operatorname{dist}(x, K) \leqslant \varepsilon\}$ and choose a cut-off function $\psi \in C^\infty(\mathbb{R}^N)$ such that $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ in K and $\psi \equiv 0$ in $\mathbb{R}^N \setminus K_\varepsilon$. Using the definition of P_n and that the function $h : \mathbb{R}^N \to \mathbb{R}$, $h(x) = |x|^p$ is strictly convex, we have

$$0 \leqslant \int_K P_n \leqslant \int_{\mathbb{R}^N} P_n\psi = \int_{\mathbb{R}^N} |\nabla u_n|^p\psi - \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}(\nabla u_n \cdot \nabla u)\psi + \int_{K_\varepsilon} |\nabla u|^{p-2}\big(\nabla u \cdot \nabla(u - u_n)\big)\psi. \tag{2.3}$$

Since (ψu_n) is bounded in X and $I'_{\lambda,q}(u_n) \to 0$, we have

$$\lim_{n\to\infty}\big\langle I'_{\lambda,q}(u_n), \psi u_n\big\rangle = \lim_{n\to\infty}\big\langle I'_{\lambda,q}(u_n), \psi u\big\rangle = 0.$$

The above expression, (2.3), $\psi \equiv 0$ in $\mathbb{R}^N \setminus K_\varepsilon$, and (2.1) give

$$0 \leqslant \int_K P_n \leqslant C_1 + C_2 + C_3 - C_4 + o(1), \tag{2.4}$$

as $n \to \infty$, with

$$C_1 := \int_{K_\varepsilon} |\nabla u_n|^{p-2}(\nabla u_n \cdot \nabla\psi)(u - u_n),$$
$$C_2 := \int_{K_\varepsilon} \lambda a(x)\psi\big(|u_n|^{p-2}u_n u - |u_n|^p\big),$$
$$C_3 := \int_{K_\varepsilon} \psi\big(|u_n|^{p-2}u_n u - |u_n|^p\big), \quad \text{and} \quad C_4 := \int_{K_\varepsilon} \psi\big(|u_n|^{q-2}u_n u - |u_n|^q\big).$$

Since (u_n) is bounded in X and $u_n \to u$ in $L^p(K_\varepsilon)$, we have that

$$|C_1| \leqslant |\nabla\psi|_\infty \int_{K_\varepsilon} |\nabla u_n|^{p-1}|u_n - u| \leqslant |\nabla\psi|_\infty \|u_n\|_1^{p-1}|u_n - u|_{p,K_\varepsilon} = o(1),$$

as $n \to \infty$. Next we observe that, up to a subsequence,

$$\int_{K_\varepsilon} |u_n|^p \to \int_{K_\varepsilon} |u|^p, \quad \text{as } n \to \infty. \tag{2.5}$$

Moreover, since $u_n(x) \to u(x)$ for a.e. $x \in K_\varepsilon$ and $(|u_n|^{p-2}u_n)$ is bounded in $L^{p/(p-1)}(K_\varepsilon)$, we have that $|u_n|^{p-2}u_n \rightharpoonup |u|^{p-2}u$ weakly in $L^{p/(p-1)}(K_\varepsilon)$. Thus,

$$\int_{K_\varepsilon} |u_n|^{p-2}u_n u \to \int_{K_\varepsilon} |u|^p, \quad \text{as } n \to \infty.$$

The above expression, (2.5), and the boundedness of $a(x)\psi$ in K_ε imply that $\lim_{n\to\infty} C_2 = 0$. In the same way we can show that $\lim_{n\to\infty} C_3 = \lim_{n\to\infty} C_4 = 0$. Therefore, we can rewrite (2.4) as

$$0 \leqslant \int_K \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right) \cdot \nabla(u_n - u) \to 0, \quad \text{as } n \to \infty.$$

Considering that $(|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b) \geqslant M_p|a - b|^p$, for every $a, b \in \mathbb{R}^N$ (see [15, p. 210]), we get

$$\lim_{n\to\infty} \int_K |\nabla u_n - \nabla u|^p = 0,$$

i.e., $\nabla u_n \to \nabla u$ strongly in $(L^p(K))^N$. Since K is arbitrary and (u_n) is bounded in X, we may suppose that (2.2) holds.

By using (2.2) and (2.1), we conclude that $I'_{\lambda,q}(u) = 0$. The boundedness of (u_n), the pointwise convergences and the Brezis and Lieb's lemma [5] imply

$$I_{\lambda,q}(v_n) = I_{\lambda,q}(u_n) - I_{\lambda,q}(u) + o(1),$$

as $n \to \infty$. Thus $\lim_{n\to\infty} I_{\lambda,q}(v_n) = c - I_{\lambda,q}(u)$.

In order to verify that $I'_{\lambda,q}(v_n) \to 0$, we note that, for any $\phi \in X$, we have

$$\left\langle I'_{\lambda,q}(v_n), \phi\right\rangle = \left\langle I'_{\lambda,q}(u_n), \phi\right\rangle - \left\langle I'_{\lambda,q}(u), \phi\right\rangle + C_5 + C_6 - C_7, \tag{2.6}$$

where

$$C_5 := \int_{\mathbb{R}^N} \left(|\nabla v_n|^{p-2}\nabla v_n + |\nabla u|^{p-2}\nabla u - |\nabla u_n|^{p-2}\nabla u_n\right) \cdot \nabla\phi,$$

$$C_6 := \int_{\mathbb{R}^N} \left(\lambda a(x) + 1\right)\left(|v_n|^{p-2}v_n + |u|^{p-2}u - |u_n|^{p-2}u_n\right)\phi, \quad \text{and}$$

$$C_7 := \int_{\mathbb{R}^N} \left(|v_n|^{q-2}v_n + |u|^{q-2}u - |u_n|^{q-2}u_n\right)\phi.$$

Using Hölder's inequality and Lemma 2.3 with $\eta_n = \nabla v_n$ and $w = \nabla u$, we get

$$|C_5| \leqslant \left(\int_{\mathbb{R}^N} \big| |\nabla v_n|^{p-2}\nabla v_n + |\nabla u|^{p-2}\nabla u - |\nabla u_n|^{p-2}\nabla u_n \big|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} |\phi|_p$$
$$\leqslant o(1)\, \|\phi\|_\lambda ,$$

as $n \to \infty$. In the same way we can see that the above estimate holds also for C_6 and C_7. Therefore, since $I'_{\lambda,q}(u_n) \to 0$ and $I'_{\lambda,q}(u) = 0$, we obtain from (2.6) that

$$\big|\big\langle I'_{\lambda,q}(v_n), \phi\big\rangle\big| \leqslant o(1)\, \|\phi\|_\lambda, \quad \text{as } n \to \infty,$$

for all $\phi \in X$. This implies that $I'_{\lambda,q}(v_n) \to 0$ and concludes the proof of the lemma. □

We are now ready to state the compactness condition we will need.

Proposition 2.5. *For any $C_0 > 0$ given, there exists $\Lambda_0 = \Lambda_0(q) > 0$ such that $I_{\lambda,q}$ satisfies $(\mathrm{PS})_c$ for all $c \leqslant C_0$ and $\lambda \geqslant \Lambda_0$.*

Proof. The proof is similar to that of [3, Proposition 2.1] and will be presented here by the sake of completeness. Let c_0 be given by Lemma 2.1(iii) and fix $\varepsilon > 0$ such that $2\varepsilon < c_0 pq/(q-p)$. For any $C_0 > 0$ we take Λ_ε and R_ε given by Lemma 2.2 and we will prove that the proposition holds for $\Lambda_0 = \Lambda_\varepsilon$. Let (u_n) be a $(\mathrm{PS})_c$ sequence of $I_{\lambda,q}$ with $c \leqslant C_0$ and $\lambda \geqslant \Lambda_0$. By Lemma 2.4, we may suppose that $u_n \rightharpoonup u$ weakly in X and $v_n = u_n - u$ is a $(\mathrm{PS})_{c'}$ sequence for $I_{\lambda,q}$, with $c' = c - I_{\lambda,q}(u)$. We claim that $c' = 0$ and therefore Lemma 2.1(ii) implies that $\lim_{n\to\infty} \|v_n\|_\lambda^p = c' pq/(p-q) = 0$, i.e., $u_n \to u$ strongly in X.

In order to verify that $c' = 0$ we suppose, by contradiction, that $c' > 0$. In view of Lemma 2.1(iii) we have $c' \geqslant c_0 > 0$. Since $v_n \to 0$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$, we can use Lemmas 2.1(ii) and 2.2 to conclude that

$$c_0 \frac{pq}{q-p} \leqslant c' \frac{pq}{q-p} = \lim_{n\to\infty} |v_n|_q^q$$
$$\leqslant \lim_{n\to\infty} \int_{B_{R_\varepsilon}} |v_n|^q + \limsup_{n\to\infty} \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |v_n|^q \leqslant \frac{c_0}{2} \frac{pq}{q-p}.$$

This is a contradiction and the proposition is proved. □

We are now ready to take advantage of the symmetry and present our variational framework. We start by noting that τ induces an involution on X, which we also denote by τ, in the following way: for each $u \in X$ we define $\tau u \in X$ by

$$(\tau u)(x) = -u(\tau x). \tag{2.7}$$

We denote by $X^\tau = \{u \in X\colon\ \tau u = u\}$ the subspace of τ-invariant functions of X and consider the Nehari manifold

$$\mathcal{V}_{\lambda,q} = \big\{u \in X \setminus \{0\}\colon\ \big\langle I'_{\lambda,q}(u), u\big\rangle = 0\big\} = \big\{u \in X \setminus \{0\}\colon\ \|u\|_\lambda^p = |u|_p^p\big\}.$$

Since we are looking for τ-invariant solutions, we define the τ-invariant Nehari manifold by setting

$$\mathcal{V}^\tau_{\lambda,q} = \{u \in \mathcal{V}_{\lambda,q}\colon\ \tau u = u\} = \mathcal{V}_{\lambda,q} \cap X^\tau.$$

The critical points we will obtain are related with the following minimizing problems:

$$c_{\lambda,q} = \inf_{u\in\mathcal{V}_{\lambda,q}} I_{\lambda,q}(u) \quad \text{and} \quad c^{\tau}_{\lambda,q} = \inf_{u\in\mathcal{V}^{\tau}_{\lambda,q}} I_{\lambda,q}(u).$$

Now we fix some notation in order to deal with the limit problem. Given a τ-invariant domain $\mathcal{D}\subset\mathbb{R}^N$ we consider the space $W_0^{1,p}(\mathcal{D})$ endowed with the norm

$$\|u\|^p_{\mathcal{D}} = \int_{\mathcal{D}} |\nabla u|^p + |u|^p.$$

For any $p < q \leqslant p^{\star}$, we define $E_{q,\mathcal{D}} : W_0^{1,p}(\mathcal{D}) \to \mathbb{R}$ by setting

$$E_{q,\mathcal{D}}(u) = \frac{1}{p}\int_{\mathcal{D}} \left(|\nabla u|^p + |u|^p\right) - \frac{1}{q}\int_{\mathcal{D}} |u|^q,$$

and the associated Nehari manifolds

$$\mathcal{N}_{q,\mathcal{D}} = \left\{u\in W_0^{1,p}(\mathcal{D})\setminus\{0\}\colon \left\langle E'_{q,\mathcal{D}}(u), u\right\rangle = 0\right\} \quad \text{and} \quad \mathcal{N}^{\tau}_{q,\mathcal{D}} = \mathcal{N}_{q,\mathcal{D}}\cap X^{\tau}.$$

We also define the numbers

$$m_{q,\mathcal{D}} = \inf_{u\in\mathcal{N}_{q,\mathcal{D}}} E_{q,\mathcal{D}}(u) \quad \text{and} \quad m^{\tau}_{q,\mathcal{D}} = \inf_{u\in\mathcal{N}^{\tau}_{q,\mathcal{D}}} E_{q,\mathcal{D}}(u). \tag{2.8}$$

Before presenting the proof of Theorem 1.1, we note that, if u is a solution of $(S^{\tau}_{\lambda,q})$, then it is necessarily of class C^1. We say that u changes sign n times if the set $\{x\in\mathbb{R}^N\colon u(x)\neq 0\}$ has $n+1$ connected components. Obviously, if u is a nontrivial solution of problem $(S^{\tau}_{\lambda,q})$, then it changes sign an odd number of times. The relation between the number of nodal regions of a solution and its energy is given by the result below.

Proposition 2.6. *If u is a solution of problem $(S^{\tau}_{\lambda,q})$ which changes sign $2k-1$ times, then $I_{\lambda,q}(u)\geqslant kc^{\tau}_{\lambda,q}$.*

Proof. The set $\{x\in\mathbb{R}^N\colon u(x)>0\}$ has k connected components $A_1,\ldots,A_k$. Let $u_i(x) = u(x)$ if $x\in A_i\cup\tau A_i$ and $u_i(x)=0$, otherwise. Since u is a critical point of $I_{\lambda,q}$, an easy calculation show that $0 = \langle I'_{\lambda,q}(u), u_i\rangle = \|u_i\|^p_{\lambda} - |u_i|^q_q$. Thus, $u_i\in\mathcal{V}^{\tau}_{\lambda,q}$ for all $i = 1,\ldots,k$, and

$$I_{\lambda,q}(u) = I_{\lambda,q}(u_1)+\cdots+I_{\lambda,q}(u_k)\geqslant kc^{\tau}_{\lambda,q},$$

as desired. □

Proof of Theorem 1.1. Let $q\in(p,p^{\star})$ be fixed and $\Lambda_0 = \Lambda_0(q)$ be given by Proposition 2.5 with $C_0 = m^{\tau}_{q,\Omega}$. Let $\lambda\geqslant\Lambda_0$ and $(u_n)\subset\mathcal{V}^{\tau}_{\lambda,q}$ be a minimizing sequence for $c^{\tau}_{\lambda,q}$. Since $\mathcal{N}^{\tau}_{q,\Omega}\subset\mathcal{V}^{\tau}_{\lambda,q}$, we have that $c^{\tau}_{\lambda,q}\leqslant m^{\tau}_{q,\Omega}$. Moreover, by the Ekeland variational principle [10] (see also [18, Theorem 8.5]), we may suppose that (u_n) is a Palais–Smale sequence and therefore the infimum is achieved by some $u\in\mathcal{V}^{\tau}_{\lambda,q}$. The definition of X^{τ}

and the Proposition 2.6 show that u changes sign exactly once. In order to finish the proof, we note that, by the Lagrange multiplier rule, there exits $\theta \in \mathbb{R}$ such that

$$\langle I'_{\lambda,q}(u) - \theta J'_{\lambda,q}(u), \phi \rangle = 0, \quad \forall \phi \in X^\tau,$$

where $J_q(u) = \|u\|_\lambda^p - |u|_q^q$. Taking $\phi = u \in \mathcal{V}^\tau_{\lambda,q}$, we get

$$0 = \langle I'_{\lambda,q}(u), u \rangle - \theta \langle J'_q(u), u \rangle = \theta(q-p)\|u\|_\lambda^p.$$

This implies $\theta = 0$ and therefore

$$\langle I'_{\lambda,q}(u), \phi \rangle = 0, \quad \forall \phi \in X^\tau.$$

The above expression and the principle of symmetric criticality [14] (see also [13, Proposition 1]) imply that u (and also $-u$) is a solution of $(S^\tau_{\lambda,q})$ which changes sign exactly once. The theorem is proved. □

Using the above ideas and making no assumption of symmetry, we can extend the existence result in [3] for the quasilinear case $2 \leqslant p < N$ and prove:

Theorem 2.7. *Suppose (A_1) and (A_2) hold. Then there exists $\Lambda_0 = \Lambda_0(q) > 0$ such that, for every $\lambda \geqslant \Lambda_0$, the problem $(S_{\lambda,q})$ has a positive least energy solution.*

Proof. For any $q \in (p, p^\star)$ fixed we take $\Lambda_0 = \Lambda_0(q)$ given by Proposition 2.5 with $C_0 = m_{q,\Omega}$. For $\lambda \geqslant \Lambda_0$, arguing as in the proof of Theorem 1.1, we conclude that $c_{\lambda,q}$ is achieved by some $u \in \mathcal{V}_{\lambda,q}$ which is a solution of $(S_{\lambda,q})$. By [3, Lemma 3.10], u does not change sign and therefore, by the maximum principle, we may suppose that u is positive. □

For the study of the concentration of solutions we need the following technical result.

Lemma 2.8. *Let $M > 0$, $\lambda_n \geqslant 1$, and $(u_n) \subset X$ be such that $\lambda_n \to \infty$ and $\|u_n\|_{\lambda_n} \leqslant M$. Then there exists a function $u \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in X and $u_n \to u$ in $L^s(\mathbb{R}^N)$, for any $p \leqslant s < p^\star$.*

Proof. Since $\|u_n\|_1 \leqslant \|u_n\|_{\lambda_n} \leqslant M$, there exists $u \in X$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in X. It is proved in [8, Lemma 4] (see also [2, Lemma 1]) that, in fact, $u \in W_0^{1,p}(\Omega)$ and $u_n \to u$ in $L^p(\mathbb{R}^N)$. Let $p < s < p^\star$ be fixed and choose $\gamma > 0$ such that $1/s = \gamma/p + (1-\gamma)/p^\star$. By using the Hölder's inequality and the continuous embedding $X \hookrightarrow L^{p^\star}(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N} |u_n - u|^s \leqslant \left(\int_{\mathbb{R}^N} |u_n - u|^{p^\star} \right)^{(1-\gamma)s/p^\star} \left(\int_{\mathbb{R}^N} |u_n - u|^p \right)^{\gamma s/p}$$
$$\leqslant C\|u_n - u\|_1^{(1-\gamma)s} |u_n - u|_p^{\gamma s},$$

and therefore $u_n \to u$ in $L^s(\mathbb{R}^N)$. The lemma is proved. □

Proof of Theorem 1.2. Let (u_n) be a sequence of nontrivial solutions of $(S^{\tau}_{\lambda_n,q})$ such that $\lambda_n \to \infty$ and $pqI_{\lambda_n,q}(u_n) = (q-p)\|u_n\|^p_{\lambda_n}$ is bounded. We will prove the theorem for $u \in W^{1,p}_0(\Omega)$ given by Lemma 2.8. Since $I'_{\lambda_n,q}(u_n) = 0$ and $a \equiv 0$ in Ω, we can proceed as in the proof of (2.2) and suppose that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{for a.e. } x \in \Omega, \tag{2.9}$$

$$|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i} \rightharpoonup |\nabla u|^{p-2}\frac{\partial u}{\partial x_i} \quad \text{weakly in } \big(L^p(\Omega)\big)',\ 1 \leqslant i \leqslant N, \tag{2.10}$$

and

$$\int_{\Omega} \big(|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\phi + |u_n|^{p-2}u_n\phi\big) = \int_{\Omega} |u_n|^{q-2}u_n\phi, \quad \forall \phi \in W^{1,p}_0(\Omega).$$

In view of Lemma 2.8, (2.10), and Lemma 2.1(iii), we can take the limit in the above expression and conclude that $u \neq 0$ satisfies the first equation in (D^{τ}_q). Since X^{τ} is a closed subspace of X, we need only to show that $u_n \to u$ strongly in $W^{1,p}(\mathbb{R}^N)$.

By using (2.9), $u \in W^{1,p}_0(\Omega)$, and Brezis and Liebs' lemma, we get

$$\begin{aligned}\int_{\mathbb{R}^N} \big|\nabla(u_n-u)\big|^p &= \int_{\mathbb{R}^N\setminus\Omega} |\nabla u_n|^p + \int_{\Omega} \big|\nabla(u_n-u)\big|^p \\ &= \int_{\mathbb{R}^N\setminus\Omega} |\nabla u_n|^p + \int_{\Omega} |\nabla u_n|^p - \int_{\Omega} |\nabla u|^p + o(1), \end{aligned}\tag{2.11}$$

as $n \to \infty$. Moreover, using $u \in W^{1,p}_0(\Omega)$ once more, we obtain

$$\int_{\mathbb{R}^N} a(x)|u_n-u|^p = \int_{\mathbb{R}^N} a(x)|u_n|^p.$$

This, (2.11), Lemma 2.8, and the fact that u_n and u lie on the Nehari manifold $\mathcal{V}^{\tau}_{\lambda_n,q}$ imply that

$$\begin{aligned}\|u_n-u\|^p_{\lambda_n} &= \int_{\mathbb{R}^N} \big|\nabla(u_n)\big|^p + \int_{\mathbb{R}^N} \lambda_n a(x)|u_n|^p - \int_{\mathbb{R}^N} |\nabla u|^p + o(1) \\ &= \int_{\mathbb{R}^N} |u_n|^q - \int_{\mathbb{R}^N} |u_n|^p - \int_{\mathbb{R}^N} |\nabla u|^p + o(1) \\ &= \int_{\mathbb{R}^N} |u|^q - \int_{\mathbb{R}^N} |u|^p - \int_{\mathbb{R}^N} |\nabla u|^p + o(1) = o(1),\end{aligned}$$

as $n \to \infty$. Thus, $\|u_n-u\|^p_0 \leqslant \|u_n-u\|^p_{\lambda_n} \to 0$, as $n \to \infty$ and the theorem is proved. □

The next result gives the asymptotic behavior of positive solutions of $(S_{\lambda,q})$. The proof is equal to that of Theorem 1.2 and will be omitted.

Theorem 2.9. *Let $\lambda_n \to \infty$ as $n \to \infty$ and (u_n) be a sequence of nontrivial solutions of $(S_{\lambda_n,q})$ such that $I_{\lambda_n,q}(u_n)$ is bounded. Then, up to a subsequence, $u_n \to u$ strongly in $W^{1,p}(\mathbb{R}^N)$ with u being a positive solution of (D_q).*

3. The limit problem (D_q)

In this section we present some technical results that are related with the limit problem (D_q). As usual, we denote by S the best constant of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^\star}(\Omega)$ given by

$$S = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p + |u|^p}{|u|^p_{p^\star,\Omega}},$$

where $|u|_{s,\mathcal{D}}$ stands the $L^s(\mathcal{D})$-norm. It is well known that S is independent of Ω and is never achieved in any proper subset of $\mathbb{R}^N$. We start with the relation between $m_{q,\mathcal{D}}$ defined in (2.8) and S.

Lemma 3.1. *For any bounded domain $\mathcal{D} \subset \mathbb{R}^N$ we have*

$$\lim_{q \to p^\star} m_{q,\mathcal{D}} = m_{p^\star,\mathcal{D}} = \frac{1}{N} S^{N/p}.$$

Proof. The first equality is proved in [7, Proposition 5]. Let $\Sigma_{\mathcal{D}}$ be the unit sphere of $W_0^{1,p}(\mathcal{D})$. Since $\psi : u \mapsto u|u|^{-N/p}_{p^\star,\mathcal{D}}$ defines a dipheomorphism between $\Sigma_{\mathcal{D}}$ and $\mathcal{N}_{p^\star,\mathcal{D}}$, we have

$$Nm_{p^\star,\mathcal{D}} = \inf_{u \in \mathcal{N}_{p^\star,\mathcal{D}}} \|u\|^p_{\mathcal{D}} = \inf_{u \in \Sigma_{\mathcal{D}}} \frac{\|u\|^p_{\mathcal{D}}}{|u|^N_{p^\star,\mathcal{D}}} = \inf_{u \in W_0^{1,p}(\mathcal{D})\setminus\{0\}} \left(\frac{\|u\|^p_{\mathcal{D}}}{|u|^p_{p^\star,\mathcal{D}}} \right)^{N/p} = S^{N/p},$$

and therefore $m_{p^\star,\mathcal{D}} = \frac{1}{N} S^{N/p}$. □

In what follows we denote by $\mathcal{M}(\mathbb{R}^N)$ the Banach space of finite Radon measures over $\mathbb{R}^N$ equipped with the norm

$$|\mu| = \sup_{\phi \in C_0(\mathbb{R}^N),\, |\phi|_\infty \leqslant 1} |\mu(\phi)|.$$

A sequence $(\mu_n) \subset \mathcal{M}(\mathbb{R}^N)$ is said to converge weakly to $\mu \in \mathcal{M}(\mathbb{R}^N)$ provided $\mu_n(\phi) \to \mu(\phi)$ for all $\phi \in C_0(\mathbb{R}^N)$. By the Banach–Alaoglu theorem, every bounded sequence $(\mu_n) \subset \mathcal{M}(\mathbb{R}^N)$ contains a weakly convergent subsequence.

The next result is a version of [18, Lemma 1.40]. The proof is also inspired by [16, Lemma 2.1 and Remark 2.2].

Lemma 3.2. *Let $(q_n) \subset \mathbb{R}$ be such that $p \leqslant q_n \leqslant p^\star$ and $q_n \uparrow p^\star$. Let $(u_n) \subset W^{1,p}(\mathbb{R}^N)$ be such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\mathbb{R}^N)$, $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$, $\nabla u_n(x) \to \nabla u(x)$ for a.e. $x \in \mathbb{R}^N$,*

$$\left|\nabla(u_n-u)\right|^p \rightharpoonup \mu \quad \textit{weakly in } \mathcal{M}\big(\mathbb{R}^N\big),$$
$$|u_n-u|^{q_n} \rightharpoonup \nu \quad \textit{weakly in } \mathcal{M}\big(\mathbb{R}^N\big), \tag{3.1}$$

and define

$$\mu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|\nabla u_n|^p, \qquad \nu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|u_n|^{q_n}.$$

Then

$$\|\nu\|^{p/p^\star} \leqslant S^{-1}\|\mu\|, \tag{3.2}$$

$$\limsup_{n\to\infty}|\nabla u_n|_p^p = |\nabla u|_p^p + \|\mu\| + \mu_\infty, \quad \textit{and} \tag{3.3}$$

$$\limsup_{n\to\infty}|u_n|_{q_n}^{q_n} = |u|_{p^\star}^{p^\star} + \|\nu\| + \nu_\infty. \tag{3.4}$$

Moreover, if $u=0$ *and* $\|\nu\|^{p/p^\star} = S^{-1}\|\mu\|$*, then the measures* μ *and* ν *are concentrated at single points.*

Proof. We first assume that $u=0$. For any given $\phi\in C_c^\infty(\mathbb{R}^N)$ we denote $K=\operatorname{supp}\phi$ and use Hölder and Sobolev's inequalities to get

$$\left(\int_{\mathbb{R}^N}|\phi u_n|^{q_n}\right)^{1/q_n} \leqslant S^{-1/p}\mathcal{L}(K)^{\frac{p^\star-q_n}{q_n p^\star}}\left(\int_{\mathbb{R}^N}\left|\nabla(\phi u_n)\right|^p + |\phi u_n|^p\right)^{1/p}.$$

Since $|\phi|^{q_n}\to|\phi|^{p^\star}$ in $C_c^\infty(\mathbb{R}^N)$ and $u_n\to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, we can take the limit in the above expression and use (3.1) to obtain

$$\left(\int_{\mathbb{R}^N}|\phi|^{p^\star}\,d\nu\right)^{1/p^\star} \leqslant S^{-1/p}\left(\int_{\mathbb{R}^N}|\phi|^p\,d\mu\right)^{1/p}, \quad \forall\phi\in C_c^\infty\big(\mathbb{R}^N\big),$$

and (3.2) follows. Moreover, if $\|\nu\|^{p/p^\star} = S^{-1}\|\mu\|$, then it follows from [12, Lemma 1.2] that ν and μ are concentrated measures.

Considering now the general case, we write $v_n = u_n - u$. Since $\nabla u_n(x)\to\nabla u(x)$ for a.e. $x\in\mathbb{R}^N$, we can use Brezis and Lieb's lemma to get

$$|\nabla u_n|^p \rightharpoonup \mu + |\nabla u|^p, \quad \text{weakly in } \mathcal{M}\big(\mathbb{R}^N\big). \tag{3.5}$$

Furthermore, using the boundedness of (u_n) and Vitalli's theorem, we can check that

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}\phi|u_n|^{q_n} - \phi|u_n-u|^{q_n}\right) = \int_{\mathbb{R}^N}\phi|u|^{p^\star}, \quad \forall\phi\in C_c^\infty\big(\mathbb{R}^N\big)$$

and therefore

$$|u_n|^{q_n} \rightharpoonup \nu + |u|^{p^\star}, \quad \text{weakly in } \mathcal{M}\big(\mathbb{R}^N\big).$$

Inequality (3.2) follows from the above expression, (3.5), and the corresponding inequality for (v_n).

For $R > 1$, let $\psi_R \in C^\infty(\mathbb{R}^N)$ be such that $\psi_R \equiv 0$ in $B_R(0)$, $\psi_R \equiv 1$ in $\mathbb{R}^N \setminus B_{R+1}(0)$ and $0 \leqslant \psi(x) \leqslant 1$ for all $x \in \mathbb{R}^N$. Using (3.5), we obtain

$$\begin{aligned}\limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^p\,dx &= \limsup_{n\to\infty} \int_{\mathbb{R}^N} \big(\psi_R|\nabla u_n|^p + (1-\psi_R)|\nabla u_n|^p\big)\,dx \\ &= \int_{\mathbb{R}^N} (1-\psi_R)\,d\mu + \int_{\mathbb{R}^N} (1-\psi_R)|\nabla u|^p\,dx \\ &\quad + \limsup_{n\to\infty} \int_{\mathbb{R}^N} \psi_R|\nabla u_n|^p\,dx.\end{aligned}$$

Taking $R \to \infty$ and using the Lebesgue theorem, we obtain (3.3). The proof of (3.4) is similar. □

Considering Ω given by (A_1), we define, for any $r > 0$, the set

$$\Omega_r^+ = \big\{x \in \mathbb{R}^N\colon \operatorname{dist}(x, \Omega) < r\big\}. \tag{3.6}$$

We also define the barycenter map $\beta_q : W_0^{1,p}(\Omega) \setminus \{0\} \to \mathbb{R}^N$ by setting

$$\beta_q(u) = \frac{\int_{\mathbb{R}^N} |u|^q x\,dx}{\int_{\mathbb{R}^N} |u|^q\,dx}.$$

Hereafter we write only $m_{q,r}$ to denote $m_{q,B_r(0)}$. Also for simplicity of notation, when we omit the reference for the set in $m_{q,\mathcal{D}}$, $\mathcal{N}_{q,\mathcal{D}}$ and $E_{q,\mathcal{D}}$, we are assuming that $\mathcal{D} = \Omega$. The following result is a version of [4, Lemma 4.2].

Lemma 3.3. *For any $r > 0$ there exist $q_0 = q_0(r) \in (p, p^\star)$ such that, for all $q \in [q_0, p^\star)$, we have that $\beta_q(u) \in \Omega_r^+$ whenever $u \in \mathcal{N}_q$ and $E_q(u) \leqslant m_{q,r}$.*

Proof. Suppose, by contradiction, that the lemma is false. Then there exist $q_n \uparrow p^\star$, $(u_n) \in \mathcal{N}_{q_n}$ with $E_{q_n}(u_n) \leqslant m_{q_n,r}$ and $\beta_{q_n}(u_n) \notin \Omega_r^+$. Thus,

$$m_{q_n} \leqslant E_{q_n}(u_n) = \left(\frac{1}{p} - \frac{1}{q_n}\right)\|u_n\|_\Omega^p \leqslant m_{q_n,r}.$$

Taking the limit, using the definition of $\mathcal{N}_{q_n}$, and Lemma 3.1, we conclude that

$$\lim_{n\to\infty} |u_n|_{q_n,\Omega}^{q_n} = \lim_{n\to\infty} \|u_n\|_\Omega^p = S^{N/p}. \tag{3.7}$$

By Hölder's inequality, we have

$$\int_\Omega |u_n|^{q_n} \leqslant \mathcal{L}(\Omega)^{(p^\star - q_n)/p^\star}\left(\int_\Omega |u_n|^{p^\star}\right)^{q_n/p^\star}.$$

The above expression and (3.7) imply that $\liminf_{n\to\infty} |u_n|_{p^\star,\Omega}^{p^\star} \geqslant S^{N/p}$. On other hand, recalling that $|u_n|_{p^\star,\Omega}^p \leqslant S^{-1}\|u_n\|_\Omega^p$, we get $\limsup_{n\to\infty} |u_n|_{p^\star,\Omega}^{p^\star} \leqslant S^{N/p}$. Hence,

$$\lim_{n\to\infty} |u_n|_{p^\star,\Omega}^{p^\star} = S^{N/p}. \tag{3.8}$$

This and (3.7) imply that (u_n) is a minimizing sequence for S. Thus, up to a subsequence, $\nabla u_n(x) \to \nabla u(x)$ for a.e. $x \in \Omega$, where u is the weak limit of u_n in $W_0^{1,p}(\Omega)$. We may also suppose that (3.1) holds and $u_n \to u$ in $L^p(\Omega)$. Lemma 3.2 and Eqs. (3.7) and (3.8) provide

$$S^{N/p} = \|u\|_\Omega^p + |\mu|, \qquad S^{N/p} = |u|_{p^\star,\Omega}^{p^\star} + |\nu|$$

and

$$|\nu|^{p/p^\star} \leqslant S^{-1}|\mu|, \qquad |u|_{p^\star,\Omega}^p \leqslant S^{-1}\|u\|_\Omega^p.$$

Note that, since Ω is bounded, the terms μ_∞ and ν_∞ do not appear in the above expressions.

The inequality $(a+b)^t < a^t + b^t$ for $a, b > 0$ and $0 < t < 1$, and the above expressions imply that $|\nu|$ and $|u|_{p^\star,\Omega}^{p^\star}$ are equal either to 0 or $S^{N/p}$. In fact, if this is not the case, we get

$$\begin{aligned} S^{(N-p)/p} &= S^{-1}\bigl(\|u\|_\Omega^p + |\mu|\bigr) \geqslant \bigl(|u|_{p^\star,\Omega}^{p^\star}\bigr)^{p/p^\star} + |\nu|^{p/p^\star} > \bigl(|u|_{p^\star,\Omega}^{p^\star} + |\nu|\bigr)^{p/p^\star} \\ &= S^{(N-p)/p}, \end{aligned}$$

which is absurd. Suppose $|u|_{p^\star,\Omega}^{p^\star} = S^{N/p}$. Since $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$, we have that $\|u\|_\Omega^p \leqslant \liminf_{n\to\infty} \|u_n\|_\Omega^p = S^{N/p}$. Hence

$$\frac{\|u\|_\Omega^p}{|u|_{p^\star,\Omega}^p} \leqslant \frac{S^{N/p}}{S^{(N-p)/p}} = S,$$

and we conclude that S is attained by $u \in W_0^{1,p}(\Omega)$, which does not make sense. This shows that $u = 0$ and therefore $|\nu| = S^{N/p}$ and ν is concentrated at a single point $y \in \bar{\Omega}$. Hence,

$$\beta_{q_n}(u_n) = \frac{\int_{\mathbb{R}^N} |u_n|^{q_n} x\,dx}{\int_{\mathbb{R}^N} |u_n|^{q_n}\,dx} \to S^{-N/p} \int_\Omega x\,d\nu = y \in \bar{\Omega},$$

which contradicts $\beta_{q_n}(u_n) \notin \Omega_r^+$. The lemma is proved. □

Finally, we present below the relation between $c_{\lambda,q}$ and m_q.

Lemma 3.4. *For any* $q \in (p, p^\star)$ *we have* $\lim_{\lambda\to\infty} c_{\lambda,q} = m_q$.

Proof. Since $W_0^{1,p}(\Omega) \subset X$, we know that $0 \leqslant c_{\lambda,q} \leqslant m_q$ for all $\lambda \geqslant 0$. Suppose, by contradiction, that the lemma is false. Then there exist a sequence $\lambda_n \to \infty$ such that $c_{\lambda_n,q} \to c < m_q$. By Theorem 2.7, $c_{\lambda_n,q}$ is achieved by large values of n. So Theorem 2.9 implies that c is achieved by E_q on $\mathcal{N}_q$. Hence, $c \geqslant m_q$. This contradiction proves the lemma. □

4. Proof of Theorem 1.3

We recall some facts about equivariant theory. An involution on a topological space X is a continuous function $\tau_X : X \to X$ such that τ_X^2 is the identity map of X. A subset A of X is called τ_X-invariant if $\tau_X(A) = A$. If X and Y are topological spaces equipped with involutions τ_X and τ_Y, respectively, then an equivariant map is a continuous function $f : X \to Y$ such that $f \circ \tau_X = \tau_Y \circ f$. Two equivariant maps $f_0, f_1 : X \to Y$ are equivariantly homotopic if there is a homotopy $\Theta : X \times [0,1] \to Y$ such that $\Theta(x,0) = f_0(x)$, $\Theta(x,1) = f_1(x)$, and $\Theta(\tau_X(x), t) = \tau_Y(\Theta(x,t))$, for all $x \in X$, $t \in [0,1]$.

Definition 4.1. The equivariant category of an equivariant map $f : X \to Y$, denoted by (τ_X, τ_Y)-cat(f), is the smallest number k of open invariant subsets $X_1, \dots, X_k$ of X which cover X and which have the property that, for each $i = 1, \dots, k$, there is a point $y_i \in Y$ and a homotopy $\Theta_i : X_i \times [0,1] \to Y$ such that $\Theta_i(x,0) = f(x)$, $\Theta_i(x,1) \in \{y_i, \tau_Y(y_i)\}$ and $\Theta_i(\tau_X(x), t) = \tau_Y(\Theta_i(x,t))$ for every $x \in X_i$, $t \in [0,1]$. If no such covering exists, we define (τ_X, τ_Y)-cat$(f) = \infty$.

If A is a τ_X-invariant subset of X and $\iota : A \hookrightarrow X$ is the inclusion map, we write

$$\tau_X\text{-cat}_X(A) = (\tau_X, \tau_X)\text{-cat}(\iota) \quad \text{and} \quad \tau_X\text{-cat}(X) = \tau_X\text{-cat}_X(X).$$

In the literature τ_X-cat(X) is usually called $\mathbb{Z}_2$-cat(X). Here it is more convenient to specify the involution in the notation.

The following properties can be verified.

Lemma 4.2.

(i) *If $f : X \to Y$ and $h : Y \to Z$ are equivariant maps, then*

$$(\tau_X, \tau_Z)\text{-cat}(h \circ f) \leqslant \tau_Y\text{-cat}(Y).$$

(ii) *If $f_0, f_1 : X \to Y$ are equivariantly homotopic, then*

$$(\tau_X, \tau_Y)\text{-cat}(f_0) = (\tau_X, \tau_Y)\text{-cat}(f_1).$$

We denote by $\tau_a : V \to V$ the antipodal involution $\tau_a(u) = -u$ on a vector space V. A τ_a-invariant subset of V is usually called a symmetric subset. Equivariant Ljusternik–Schnirelmann category provides a lower bound for the number of pairs $\{u, -u\}$ of critical points of an even functional. The following well-known result (see [9, Theorem 1.1], [17, Theorem 5.7]) will be used in the proof of Theorem 1.3.

Theorem 4.3. *Let $I : M \to \mathbb{R}$ be an even C^1-functional on a complete symmetric $C^{1,1}$-submanifold M of some Banach space V. Assume that I is bounded below and satisfies $(\mathrm{PS})_c$ for all $c \leqslant d$. Then, denoting $I^d = \{u \in M\colon\ I(u) \leqslant d\}$, I has at least τ_a-cat(I^d) antipodal pairs $\{u, -u\}$ of critical points with $I(\pm u) \leqslant d$.*

Coming back to our problem, we set, for any given $r > 0$,

$$\Omega_r^- = \big\{x \in \Omega\colon\ \mathrm{dist}\big(x, \partial\Omega \cup \Omega^\tau\big) \geqslant r\big\}.$$

Throughout the rest of this section $r > 0$ sufficiently small is fixed in such way that the inclusion maps $\Omega_r^- \hookrightarrow \Omega \setminus \Omega^\tau$ and $\Omega \hookrightarrow \Omega_r^+$ are equivariant homotopy equivalences and Ω_r^+ is as defined in (3.6). Without loss of generality we suppose that $B_r(0) \subset \Omega$.

Now we follow [3] and choose $R > 0$ with $\bar{\Omega} \subset B_R(0)$ and set

$$\xi(t) = \begin{cases} 1, & \text{if } 0 \leqslant t \leqslant R, \\ R/t, & \text{if } t \geqslant R. \end{cases}$$

We also define, for $u \in \mathcal{V}_{\lambda,q}$, a truncated barycenter map

$$\bar{\beta}_q(u) = \frac{\int_{\mathbb{R}^N} |u|^q \xi(|x|) x \, dx}{\int_{\mathbb{R}^N} |u|^q \, dx}.$$

The following results will be useful in the proof of Theorem 1.3.

Lemma 4.4 [3, Lemmas 3.7 and 3.8]. *There exists $\tilde{q} \in (p, p^\star)$ with the property that, for each $q \in [\tilde{q}, p^\star)$, there is a number $\Lambda_1 = \Lambda_1(q)$ such that, for every $\lambda \geqslant \Lambda_1$, we have*

(i) $m_{q,r} < 2c_{\lambda,q}$,
(ii) *if $u \in \mathcal{V}_{\lambda,q}$ and $I_{\lambda,q}(u) \leqslant m_{q,r}$, then $\bar{\beta}_q(u) \in \Omega_r^+$.*

Lemma 4.5. *For any bounded τ-invariant domain $\mathcal{D} \subset \mathbb{R}^N$ we have $2c_{\lambda,q} \leqslant c_{\lambda,q}^\tau$.*

Proof. Given $u \in \mathcal{V}_{\lambda,q}^\tau$ we can use (2.7) to conclude that $u^+, u^- \in \mathcal{V}_{\lambda,q}$, where $u^\pm = \max\{\pm u, 0\}$. Thus

$$I_{\lambda,q}(u) = I_{\lambda,q}(u^+) + I_{\lambda,q}(u^-) \geqslant 2c_{\lambda,q},$$

and the result follows. □

Proof of Theorem 1.3. Let $\tilde{q}$ be given by Lemma 4.4 and fix $q \in (\tilde{q}, p^\star)$. We will show that the theorem holds for $\Lambda(q) = \max\{\Lambda_0(q), \Lambda_1(q)\}$, where $\Lambda_0(q)$ is given by applying Proposition 2.5 with $C_0 = 2m_{q,r}$ and $\Lambda_1(q)$ is given by Lemma 4.4.

For any $\lambda \geqslant \Lambda(q)$ we can use Theorem 4.3 for

$$I_{\lambda,q} : \mathcal{V}_{\lambda,q}^\tau \to \mathbb{R}$$

and obtain τ_a-cat$(\mathcal{V}_{\lambda,q}^\tau \cap I_{\lambda,q}^{2m_{q,r}})$ pairs $\pm u_i$ of critical points with

$$I_{\lambda,q}(\pm u_i) \leqslant 2m_{q,r} < 4c_{\lambda,q} < 2c_{\lambda,q}^\tau$$

(by Lemmas 4.4(i) and 4.5). The same argument employed in the proof of Theorem 1.1 show that $\pm u_i$ are solutions of $(S_{\lambda,q}^\tau)$ which change sign exactly once.

In order to finish the proof, we need only to verify that

$$\tau_a\text{-cat}\big(\mathcal{V}_{\lambda,q}^\tau \cap I_{\lambda,q}^{2m_{q,r}}\big) \geqslant \tau\text{-cat}_\Omega\big(\Omega \setminus \Omega^\tau\big). \tag{4.1}$$

With this purpose we take a nonnegative radial function $v_q \in \mathcal{N}_{q,B_r(0)}$ such that $E_{q,B_r(0)}(v_q) = m_{q,r}$ and define $\alpha_q : \Omega_r^- \to \mathcal{V}_{\lambda,q}^\tau \cap I_{\lambda,q}^{2m_{q,r}}$ by setting

$$\alpha_q(x) = v_q(\cdot - x) - v_q(\cdot - \tau x). \tag{4.2}$$

We claim that $|x - \tau x| \geqslant 2r$ for every $x \in \Omega_r^-$. Indeed, if this is not the case, then $\bar{x} = (x + \tau x)/2$ satisfies $|x - \bar{x}| < r$ and $\tau\bar{x} = \bar{x}$, contradicting the definition of Ω_r^-. Since v_q is radial and τ is an isometry, we can use the last claim to verify that α_q is well-defined.

We note that if $u \in \mathcal{V}_{\lambda,q}^\tau$, then $u^+ \in \mathcal{V}_{\lambda,q}$ and $I_{\lambda,q}(u) = 2I_{\lambda,q}(u^+)$. Thus, Lemma 3.3 implies that $\bar{\beta}_q(u^+) \in \Omega_r^+$ for all $u \in \mathcal{V}_{\lambda,q}^\tau \cap I_{\lambda,q}^{2m_{q,r}}$ and therefore the diagram

$$\Omega_r^- \xrightarrow{\alpha_q} \mathcal{V}_{\lambda,q}^\tau \cap I_{\lambda,q}^{2m_{q,r}} \xrightarrow{\gamma_q} \Omega_r^+, \tag{4.3}$$

where $\gamma_q(u) = \bar{\beta}_q(u^+)$, is well-defined. A direct computation shows that $\alpha_q(\tau x) = -\alpha_q(x)$ and $\gamma_q(-u) = \tau\gamma_q(u)$. Moreover, using (4.2) and the fact that v_q is radial, we get

$$\gamma_q\big(\alpha_q(x)\big) = \frac{\int_{B_r(x)} |v_q(y-x)|^q y\,dy}{\int_{B_r(x)} |v_q(y-x)|^q\,dy} = \frac{\int_{B_r(0)} |v_q(y)|^q (y+x)\,dy}{\int_{B_r(0)} |v_q(y)|^q\,dy} = x,$$

for any $x \in \Omega_r^-$. Now, recalling that r was chosen so that the inclusion maps $\Omega_r^- \hookrightarrow \Omega \setminus \Omega^\tau$ and $\Omega \hookrightarrow \Omega_r^+$ are equivariant homotopy equivalences, the inequality (4.1) follows from (4.3) and the properties given by Lemma 4.2. The theorem is proved. □

Proof of Corollary 1.4. Let $\tau : \mathbb{R}^N \to \mathbb{R}^N$ be given by $\tau(x) = -x$. It is proved in [6, Corollary 3] that our assumptions imply $\tau\text{-cat}(\Omega) \geqslant N$. Since $0 \notin \Omega$, $\Omega^\tau = \emptyset$. It suffices now to apply Theorem 1.3. □

Remark. Suppose $\lambda_n \to \infty$ and (u_n) is a sequence of solutions of $(S_{\lambda_n,q}^\tau)$ given by Theorem 1.1, Theorem 1.3 or Corollary 1.4. Then the limit solution u given by Theorem 1.2 changes sign exactly once. Indeed, in order to verify this assertion, it suffices to use the same notation of the proof of Theorem 1.3 and note that

$$0 < c_0 \leqslant c_{\lambda_n,q}^\tau \leqslant I_{\lambda_n,q}(u_n) \leqslant 2m_{q,r} < 4c_{\lambda_n,q} \leqslant 2c_{\lambda_n,q}^\tau \leqslant 2m_{q,\Omega}^\tau.$$

Taking the limit we conclude that $u \neq 0$ and $E_{q,\Omega}(u) < 2m_{q,\Omega}^\tau$. The same argument employed in the proof of Proposition 2.6 shows that u is a minimal nodal solution of (D_q^τ).

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