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Nonlinear Analysis 62 (2005) 615–628

Nonlinear Analysis

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A relation between the domain topology and the number of minimal nodal solutions for a quasilinear elliptic problem

Marcelo F. Furtado[1]

IMECC-UNICAMP, Cx. Postal 6065, 13083-970 Campinas-SP, Brazil

Received 2 January 2004; accepted 8 March 2005

Abstract

We consider the quasilinear problem $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = |u|^{q-2}u$ in Ω, $u = 0$ on $\partial\Omega$ where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $1 < p < N$ and $p < q < p^* = Np/(N-p)$. We show that if Ω is invariant by a non-trivial orthogonal involution then, for q close to p^*, the equivariant topology of Ω is related with the number of solutions which change sign exactly once. The results complement those of Castro and Clapp [Nonlinearity 16 (2003) 579–590] since we consider subcritical nonlinearities and the quasilinear case. Without any assumption of symmetry we also extend Theorem B in Benci and Cerami [Arch. Rational. Mech. Anal. 114 (1991) 79–93] for the quasilinear case and prove that the topology of Ω affects the number of positive solutions.

Keywords: Nodal solutions; p-Laplace operator; Equivariant category; Symmetry

1. Introduction

Consider the problem

$$(P_q)\qquad \begin{cases} -\Delta_p u + |u|^{p-2}u = |u|^{q-2}u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

[1] Supported by CAPES/Brazil.

E-mail address: furtado@ime.unicamp.br.

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doi:10.1016/j.na.2005.03.073

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $1 < p < N$ and $p < q < p^* = Np/(N-p)$. It is well known that it possesses infinitely many solutions. However, when we require some properties of the nodal regions of the solutions, the problem seems to be more complicated. In the paper [2], Benci and Cerami showed that the domain topology is related with the number of positive solutions of (P_q). More specifically, they showed that if $p = 2$ and q is close to 2^*, then (P_q) has at least $\mathrm{cat}(\Omega)$ positive solutions, where $\mathrm{cat}(\Omega)$ denotes the Ljusternik–Schnirelmann category of Ω in itself. Since the work [2], multiplicity results of (P_q) with $p = 2$ have been intensively studied (see [4,7,3] for subcritical, and [17,13,21] for critical nonlinearities). To the best of our knowledge, the only work that deals with the quasilinear problem is [1], where the authors studied the critical case.

In the aforementioned works, the authors considered positive solutions. Here, motivated by Castro and Clapp [6], we are interested in solutions which change sign exactly once. This means that the solution u is such that $\Omega \backslash u^{-1}(0)$ has exactly two connected components, u is positive in one of them and negative in the other. We deal with the problem

$$(P_q^\tau) \quad \begin{cases} -\Delta_p u + |u|^{p-2}u = |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(\tau x) = -u(x) & \text{for all } x \in \Omega, \end{cases}$$

where $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is fixed, $\tau \in O(N)\backslash\{\mathrm{Id}\}$, $\tau^2 = \mathrm{Id}$, and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain such that $\tau\Omega = \Omega$. It is clear that any non-trivial solution of (P_q^τ) changes sign. We call a nodal solution minimal if it changes sign exactly once. Our existence result can be stated as

Theorem 1.1. *For any $q \in (p, p^*)$ the problem (P_q^τ) has at least one pair of solutions which change sign exactly once.*

The proof of Theorem 1.1 relies in a minimization argument. As we will see, there is a deep relation between the number of nodal regions of a solution and its energy. This relation will able us to prove that the solutions have the desired property.

The above result complements [6, Theorem 1] where the authors considered the semilinear problem

$$-\Delta u = \lambda u + |u|^{2^*-2}u, \quad u \in H_0^1(\Omega), \quad u(\tau x) = -u(x) \quad \text{in } \Omega$$

and obtain the same result for $\lambda > 0$ small enough (see also [8, Theorem A] for the existence of nodal solutions without symmetry assumptions). By taking advantage of the symmetry, the authors in [6] also studied the relation between the domain topology and the number of minimal nodal solutions. We also are able to have precise statements about this relation if we suppose that q is sufficiently close to p^*. More specifically, we prove

Theorem 1.2. *There exists $q_0 \in (p, p^*)$ such that, for all $q \in [q_0, p^*)$, the problem (P_q^τ) has at least τ-$\mathrm{cat}_\Omega(\Omega\backslash\Omega^\tau)$ pairs of solutions which change sign exactly once.*

Here, $\Omega^\tau = \{x \in \Omega : \tau x = x\}$ and τ-cat is the G_τ-equivariant Ljusternik–Schnirelmann category for the group $G_\tau = \{\mathrm{Id}, \tau\}$. There are several situations where the equivariant category turns out to be larger than the nonequivariant one. The classical example is the unit sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ with $\tau = -\mathrm{Id}$. In this case $\mathrm{cat}(\mathbb{S}^{N-1}) = 2$ whereas $\tau\text{-cat}(\mathbb{S}^{N-1}) = N$. As an easy consequence of Theorem 1.2 we have

Corollary 1.3. *Let Ω be symmetric with respect to the origin and such that $0 \notin \Omega$. Assume further that there is an odd map $\varphi : \mathbb{S}^{N-1} \to \Omega$. Then there exists $q_0 \in (p, p^*)$ such that, for all $q \in [q_0, p^*)$, the problem (P_q) has at least N pairs of odd solutions which change sign exactly once.*

It is worthwhile to mention that the above results seem to be new even for $p = 2$. We also note that the nonlinearity of the p-Laplacian, which makes the calculations more difficult, is compensated here by the homogeneity of the problem.

Finally, without any assumption of symmetry, we can look for multiple positive solutions of (P_q) and to extend [2, Theorem B] for the quasilinear case. Since we have no symmetry in this context, we relate the number of positive solutions with the usual Ljusternik–Schnirelmann category and prove

Theorem 1.4. *There exists $q_0 \in (p, p^*)$ such that, for all $q \in [q_0, p^*)$, problem (P_q) has at least* $\mathrm{cat}(\Omega)$ *positive solutions.*

The paper is organized as follows. Section 2 is devoted to establish the notation as well as to present some technical results. In Section 3, after recalling some basic facts about the equivariant Ljusternik–Schnirelmann theory, we prove the results concerning nodal solutions. In Section 4, we present the proof of Theorem 1.4.

2. Notations and some technical results

We start by considering the space $W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega (|\nabla u|^p + |u|^p)\, \mathrm{d}x \right)^{1/p}. \tag{2.1}$$

The involution τ of Ω induces an involution of $W_0^{1,p}(\Omega)$, which we also denote by τ, in the following way: for each $u \in W_0^{1,p}(\Omega)$ we define $\tau u \in W_0^{1,p}(\Omega)$ by

$$(\tau u)(x) = -u(\tau x). \tag{2.2}$$

Thus, we can also consider the closed linear subspace of $W_0^{1,p}(\Omega)$ given by

$$W_0^{1,p}(\Omega)^\tau = \{u \in W_0^{1,p}(\Omega) : \tau u = u\}.$$

Let $E_q : W_0^{1,p}(\Omega) \to \mathbb{R}$ be given by

$$E_q(u) = \frac{1}{p} \int_\Omega (|\nabla u|^p + |u|^p)\, \mathrm{d}x - \frac{1}{q} \int_\Omega |u|^q\, \mathrm{d}x$$

and its associated Nehari manifold

$$\begin{aligned}\mathcal{N}_q &= \{u \in W_0^{1,p}(\Omega)\backslash\{0\} : \langle E_q'(u), u\rangle = 0\}\\ &= \{u \in W_0^{1,p}(\Omega)\backslash\{0\} : \|u\|^p = |u|_q^q\},\end{aligned}$$

where $|u|_s$ denote the $L^s(\Omega)$-norm for $s \geqslant 1$.

In order to obtain τ-invariant solutions we will look for critical points of the restriction of E_q to the τ-invariant Nehari manifold

$$\mathcal{N}_q^\tau = \{u \in \mathcal{N}_q : \tau u = u\} = \mathcal{N}_q \cap W_0^{1,p}(\Omega)^\tau.$$

We define the numbers

$$m_q = \inf_{u \in \mathcal{N}_q} E_q(u) \quad \text{and} \quad m_q^\tau = \inf_{u \in \mathcal{N}_q^\tau} E_q(u).$$

For any bounded τ-invariant domain $\mathscr{D} \subset \mathbb{R}^N$ we define $E_{q,\mathscr{D}}$, $\mathcal{N}_{q,\mathscr{D}}$, $\mathcal{N}_{q,\mathscr{D}}^\tau$, $m_{q,\mathscr{D}}$ and $m_{q,\mathscr{D}}^\tau$ in the same way by taking the above integrals over $\mathscr{D}$ instead of Ω. For simplicity of notation we use only $m_{q,r}$ and $m_{q,r}^\tau$ to denote $m_{q,B_r(0)}$ and $m_{q,B_r(0)}^\tau$, respectively. Also for simplicity we write $\int_{\mathscr{D}} u$ to indicate $\int_{\mathscr{D}} u(x)\,\mathrm{d}x$. For $s \geqslant 1$, we denote by $|u|_{s,\mathscr{D}}$ the $L^s(\mathscr{D})$-norm of a function u.

Lemma 2.1. *For any bounded τ-invariant domain $\mathscr{D} \subset \mathbb{R}^N$ we have that $2m_{q,\mathscr{D}} \leqslant m_{q,\mathscr{D}}^\tau$.*

Proof. Given $u \in \mathcal{N}_{q,\mathscr{D}}^\tau$ we can use (2.2) to conclude that $u^+, u^- \in \mathcal{N}_{q,\mathscr{D}}$, where $u^\pm = \max\{\pm u, 0\}$. Thus

$$E_{q,\mathscr{D}}(u) = E_{q,\mathscr{D}}(u^+) + E_{q,\mathscr{D}}(u^-) \geqslant 2m_{q,\mathscr{D}},$$

and the result follows. $\square$

As usual, we denote by S the best constant of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ given by

$$S = \inf_{u \in W_0^{1,p}(\Omega)\backslash\{0\}} \frac{\|u\|^p}{|u|_{p^*}^p}.$$

It is well known that S is independent of Ω and is never achieved in any proper subset of $\mathbb{R}^N$.

Let V be a Banach space, M be a C^1-manifold of V and $I : V \to \mathbb{R}$ a C^1-functional. We recall that $I|_M$ satisfies the Palais–Smale condition at level c ($(\mathrm{PS})_c$ for short) if any sequence $(u_n) \subset M$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_* \to 0$ contains a convergent subsequence. Here we are denoting by $\|I'(u)\|_*$ the norm of the derivative of the restriction of I to M (see [21, Section 5.3]). The following technical result will be useful in the future:

Lemma 2.2. *Let $(v_n) \subset W_0^{1,p}(\Omega)$ be such that $|v_n|_{p^*} = 1$ and $\|v_n\|^p \to S$. Then there exists $v \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$ and $\nabla v_n(x) \to \nabla v(x)$ for a.e. $x \in \Omega$.*

Proof. Since $\|v_n\|^p \to S$ we have that (v_n) is bounded. Thus, going to a subsequence if necessary, $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$ for some $v \in W_0^{1,p}(\Omega)$. Denoting $M = \{u \in W_0^{1,p}(\Omega) : \int_\Omega |u|^{p^*} = 1\}$, by the Ekeland Variational Principle [11] (see also [21, Theorem 8.5]), we may suppose that (v_n) is a $(PS)_S$ sequence for $\varphi : M \to \mathbb{R}$ given by $\varphi(u) = \|u\|^p$, i.e., there exits $(\theta_n) \subset \mathbb{R}$ such that

$$-\Delta_p v_n + |v_n|^{p-2}v_n - \theta_n |v_n|^{p^*-2} v_n \to 0 \quad \text{in } (W_0^{1,p}(\Omega))',$$

where $(W_0^{1,p}(\Omega))'$ is the dual space of $W_0^{1,p}(\Omega)$. The above expression imply that $\varphi(v_n) - \theta_n \to 0$ and $\theta_n \to S$. Defining $\tilde{v}_n = \theta_n^{(N-p)/p^2} v_n$, an easy calculation shows that

$$E_{p^*}(\tilde{v}_n) \to \frac{1}{N} S^{N/p} \quad \text{and} \quad \|E'_{p^*}(\tilde{v}_n)\|_{(W_0^{1,p}(\Omega))'} \to 0.$$

Thus $\tilde{v}_n$ is a (PS) sequence of E_{p^*}. This fact, $\theta_n \to S$ and standard calculations [22] (see also [18, Corollary 3.7]) show that

$$\nabla \tilde{v}_n(x) \to S^{(N-p)/p^2} \nabla v(x) \quad \text{a.e. } x \in \Omega.$$

The result follows from the definition of $\tilde{v}_n$. $\square$

We present below some useful relations between $m_{q,\mathscr{D}}$, $m^\tau_{q,\mathscr{D}}$ and S.

Lemma 2.3. *For any bounded τ-invariant domain $\mathscr{D} \subset \mathbb{R}^N$ we have*

(i) $\lim_{q \to p^*} m_{q,\mathscr{D}} = m_{p^*,\mathscr{D}} = \frac{1}{N} S^{N/p}$,
(ii) $\lim_{q \to p^*} m^\tau_{q,\mathscr{D}} = m^\tau_{p^*,\mathscr{D}} = \frac{2}{N} S^{N/p}$.

Proof. The first equalities in (i) and (ii) follow from [9, Proposition 5]. Denote by $\|u\|_{\mathscr{D}}$ the norm of $u \in W_0^{1,p}(\mathscr{D})$ and let $\Sigma_{\mathscr{D}}$ be the unit sphere of $W_0^{1,p}(\mathscr{D})$. Since $\psi : u \mapsto u|u|_{p^*,\mathscr{D}}^{-N/p}$ defines a dipheomorphism between $\Sigma_{\mathscr{D}}$ and $\mathscr{N}_{p^*,\mathscr{D}}$, we have

$$\begin{aligned} Nm_{p^*,\mathscr{D}} &= \inf_{u \in \mathscr{N}_{p^*,\mathscr{D}}} \|u\|^p_{\mathscr{D}} = \inf_{u \in \Sigma_{\mathscr{D}}} \frac{\|u\|^p_{\mathscr{D}}}{|u|^N_{p^*,\mathscr{D}}} \\ &= \inf_{u \in W_0^{1,p}(\mathscr{D}) \setminus \{0\}} \left(\frac{\|u\|^p_{\mathscr{D}}}{|u|^p_{p^*,\mathscr{D}}} \right)^{N/p} = S^{N/p} \end{aligned}$$

and therefore $m_{p^*,\mathscr{D}} = \frac{1}{N} S^{N/p}$. In [6, Proposition 5] is proved that $m^\tau_{p^*,\mathscr{D}} = \frac{2}{N} S^{N/p}$. We observe that in [9,6] the authors consider only the semilinear case $p = 2$. However, taking advantage of the homogeneity, it is not difficult to see that the arguments hold for $1 < p < N$. $\square$

In what follows we denote by $\mathscr{M}(\mathbb{R}^N)$ the Banach space of finite Radon measures over $\mathbb{R}^N$ equipped with the norm

$$\|\mu\| = \sup_{\phi\in C_0(\mathbb{R}^N),|\phi|_\infty\leqslant 1} |\mu(\phi)|.$$

A sequence $(\mu_n) \subset \mathscr{M}(\mathbb{R}^N)$ is said to converge weakly to $\mu \in \mathscr{M}(\mathbb{R}^N)$ provided $\mu_n(\phi) \to \mu(\phi)$ for all $\phi \in C_0(\mathbb{R}^N)$. By the Banach–Alaoglu theorem, every bounded sequence $(\mu_n) \subset \mathscr{M}(\mathbb{R}^N)$ contains a weakly convergent subsequence.

The next result is a version of the Second Concentration-Compactness Lemma of Lions [14, Lemma I.1]. The proof can be found in [21, Lemma 1.40; 19, Lemma 2.1 and Remark 2.2].

Lemma 2.4. *Let $(u_n) \subset D^{1,p}(\mathbb{R}^N)$ be a sequence such that*

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{weakly in } D^{1,p}(\mathbb{R}^N),\\
&|\nabla(u_n-u)|^p \rightharpoonup \mu \quad \text{weakly in } \mathscr{M}(\mathbb{R}^N),\\
&|u_n-u|^{p^*} \rightharpoonup \nu \quad \text{weakly in } \mathscr{M}(\mathbb{R}^N),\\
&u_n(x) \to u(x) \quad \text{a.e. } x \in \mathbb{R}^N,\\
&\nabla u_n(x) \to \nabla u(x) \quad \text{a.e. } x \in \mathbb{R}^N
\end{aligned} \tag{2.3}$$

and define

$$\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |\nabla u_n|^p, \qquad \nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^{p^*}.$$

Then

$$\|\nu\|^{p/p^*} \leqslant S^{-1}\|\mu\|, \tag{2.4}$$

$$\limsup_{n\to\infty} |\nabla u_n|^p_{p,\mathbb{R}^N} = |\nabla u|^p_{p,\mathbb{R}^N} + \|\mu\| + \mu_\infty \tag{2.5}$$

and

$$\limsup_{n\to\infty} |u_n|^{p^*}_{p^*,\mathbb{R}^N} = |u|^{p^*}_{p^*,\mathbb{R}^N} + \|\nu\| + \nu_\infty \tag{2.6}$$

Moreover, if $u = 0$ and $\|\nu\|^{p/p^*} = S^{-1}\|\mu\|$, then μ and ν are concentrated at single points.

Remark 2.5. In [21, Lemma 1.40] the author proves the above lemma for $p = 2$ without the assumption of pointwise convergence for the gradient. The proof for the general case follows the same lines of case $p = 2$ except for Eq. (2.5). As noted in [19, Example 2.3], it can fail for $p \neq 2$ if we do not impose that $\nabla u_n(x) \to \nabla u(x)$ for a.e. $x \in \mathbb{R}^N$. However, when this last assumption is assumed, Eq. (2.5) can be verified as in [19, Lemma 2.1 and Remark 2.2].

For any $r > 0$ we define the set

$$\Omega_r^+ = \{x \in \mathbb{R}^N : \mathrm{dist}(x, \Omega) < r\}. \tag{2.7}$$

We also define the barycenter map $\beta : W_0^{1,p}(\Omega)\backslash\{0\} \to \mathbb{R}^N$ by setting

$$\beta(u) = \frac{\int_{\mathbb{R}^N} |u|^{p^*} x \, \mathrm{d}x}{\int_{\mathbb{R}^N} |u|^{p^*} \, \mathrm{d}x}.$$

The following result is a version of [2, Lemma 4.2]:

Lemma 2.6. *For any $r > 0$ there exist $q_0 = q_0(r) \in (p, p^*)$ such that, for all $q \in [q_0, p^*)$, we have*

(i) $m_{q,r}^{\tau} < 2m_q^{\tau}$,
(ii) *if $u \in \mathcal{N}_q^{\tau}$ and $E_q(u) \leqslant m_{q,r}^{\tau}$, then $\beta(u^+) \in \Omega_r^+$.*

Proof. We suppose, by contradiction, that (i) is false. Then there exists a sequence $q_n \uparrow p^*$ such that $m_{q_n,r}^{\tau} \geqslant 2m_{q_n}^{\tau}$. Taking the limit and using Lemma 2.3(ii) we conclude that $S^{N/p} \geqslant 2S^{N/p}$, which does not make sense.

Arguing by contradiction once more, we suppose that (ii) is not true. Then there exist $q_n \uparrow p^*$, $(u_n) \in \mathcal{N}_{q_n}^{\tau}$ with $E_{q_n}(u_n) \leqslant m_{q_n,r}^{\tau}$ and $\beta(u_n^+) \notin \Omega_r^+$. We can use (2.2) to verify that $u_n^+ \in \mathcal{N}_{q_n}$ and $2E_{q_n}(u_n^+) = E_{q_n}(u_n)$. Thus,

$$m_{q_n} \leqslant E_{q_n}(u_n^+) = \left(\frac{1}{p} - \frac{1}{q_n}\right) \|u_n^+\|^p \leqslant 2^{-1} m_{q_n,r}^{\tau}.$$

Taking the limit, using the definition of $\mathcal{N}_{q_n}^{\tau}$ and Lemma 2.3, we conclude that

$$\lim_{n\to\infty} |u_n^+|_{q_n}^{q_n} = \lim_{n\to\infty} \|u_n^+\|^p = S^{N/p}. \tag{2.8}$$

By Hölder's inequality we have

$$\int_{\Omega} (u_n^+)^{q_n} \leqslant \mathscr{L}(\Omega)^{(p^*-q_n)/p^*} \left(\int_{\Omega} (u_n^+)^{p^*} \right)^{q_n/p^*},$$

where $\mathscr{L}$ denotes the Lebesgue measure in $\mathbb{R}^N$. The above expression and (2.8) imply that $\liminf_{n\to\infty} |u_n^+|_{p^*}^{p^*} \geqslant S^{N/p}$. On the other hand, recalling that $|u_n^+|_{p^*}^p \leqslant S^{-1} \|u_n^+\|^p$, we get $\limsup_{n\to\infty} |u_n^+|_{p^*}^{p^*} \leqslant S^{N/p}$. Hence,

$$\lim_{n\to\infty} |u_n^+|_{p^*}^{p^*} = S^{N/p}. \tag{2.9}$$

This and (2.8) imply that $v_n := \frac{u_n^+}{|u_n^+|_{p^*}}$ satisfies the hypotheses of Lemma 2.2 and therefore, up to subsequence, we have

$$\nabla u_n^+(x) \to \nabla u(x) \quad \text{a.e. } x \in \Omega,$$

where u is the weak limit of u_n^+ in $W_0^{1,p}(\Omega)$. By going if necessary to a subsequence, we may assume that (2.3) holds with (u_n) replaced by (u_n^+). We may also assume that $u_n^+ \to u$

in $L^p(\Omega)$. The Lemma 2.4, and Eqs. (2.8) and (2.9) provide

$$S^{N/p} = \|u\|^p + \|\mu\|, \quad S^{N/p} = |u|_{p^*}^{p^*} + \|v\|$$

and

$$\|v\|^{p/p^*} \leqslant S^{-1}\|\mu\|, \quad |u|_{p^*}^{p} \leqslant S^{-1}\|u\|^p.$$

Note that, since Ω is bounded, the terms μ_∞ and v_∞ do not appear in the above expressions.

The inequality $(a+b)^t < a^t + b^t$ for $a, b > 0$ and $0 < t < 1$, and the above expressions imply that $\|v\|$ and $|u|_{p^*}^{p^*}$ are equal either to 0 or $S^{N/p}$. Indeed, if this is not the case, we get

$$\begin{aligned} S^{(N-p)/p} &= S^{-1}(\|u\|^p + \|\mu\|) \geqslant (|u|_{p^*}^{p^*})^{p/p^*} + \|v\|^{p/p^*} \\ &> (|u|_{p^*}^{p^*} + \|v\|)^{p/p^*} = S^{(N-p)/p}, \end{aligned}$$

which is absurd. Suppose $|u|_{p^*}^{p^*} = S^{N/p}$. Since $u_n^+ \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$, we have that $\|u\|^p \leqslant \liminf_{n\to\infty} \|u_n^+\|^p = S^{N/p}$. Hence

$$\frac{\|u\|^p}{|u|_{p^*}^{p}} \leqslant \frac{S^{N/p}}{S^{(N-p)/p}} = S$$

and we conclude that S is attained by $u \in W_0^{1,p}(\Omega)$, which does not make sense. This shows that $u = 0$ and therefore $\|v\| = S^{N/p}$ and v is concentrated at a single point y. Since $(u_n^+) \subset W_0^{1,p}(\Omega)$, we conclude that $y \in \overline{\Omega}$. Hence

$$\beta(u_n^+) = \frac{\int_{\mathbb{R}^N} (u_n^+)^{p^*} x \, \mathrm{d}x}{\int_{\mathbb{R}^N} (u_n^+)^{p^*} \, \mathrm{d}x} \to S^{-N/p} \int_{\Omega} x \, \mathrm{d}v = y \in \overline{\Omega},$$

which contradicts $\beta(u_n^+) \notin \Omega_r^+$. The lemma is proved. $\square$

3. Minimal nodal solutions

We start this section by noting that, if u is a solution of (P_q), then it is of class C^1. We say it changes sign n times if the set $\{x \in \Omega : u(x) \neq 0\}$ has $n+1$ connected components. Obviously, if u is a non-trivial solution of problem (P_q^τ), then it changes sign an odd number of times. The relation between the nodal regions of a solution and its energy is given by the result below (see [6] for $p = 2$).

Proposition 3.1. *If u is a solution of problem (P_q^τ) which changes sign $2k-1$ times, then $E_q(u) \geqslant k m_q^\tau$.*

Proof. The set $\{x \in \Omega : u(x) > 0\}$ has k connected components $A_1, \ldots, A_k$. Let $u_i(x) = u(x)$ if $x \in A_i \cup \tau A_i$ and $u_i(x) = 0$, otherwise. Since u is a critical point of E_q,

$$0 = \langle E_q'(u), u_i\rangle = \int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla u_i + |u|^{p-2}uu_i - |u|^{q-2}uu_i)$$
$$= \|u_i\|^p - |u_i|_q^q.$$

Thus, $u_i \in \mathcal{N}_q^\tau$ for all $i = 1, \ldots, k$, and

$$E_q(u) = E_q(u_1) + \cdots + E_q(u_k) \geqslant km_q^\tau,$$

as desired. □

Proof of Theorem 1.1. The compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ and standard calculations give the Palais–Smale condition for E_q restricted to $\mathcal{N}_q^\tau$. Let $(u_n) \subset \mathcal{N}_q^\tau$ be a minimizing sequence for m_q^τ. We may suppose that (u_n) is a (PS) sequence at level m_q^τ and therefore, by the (PS) condition, the infimum is achieved by some $u \in \mathcal{N}_q^\tau$. The definition of $W_0^{1,p}(\Omega)^\tau$ and Proposition 3.1 show that u changes sign exactly once. To finish the proof we note that, by the Lagrange multiplier rule, there exits $\theta \in \mathbb{R}$ such that

$$\langle E_q'(u) - \theta J_q'(u), \phi\rangle = 0, \quad \forall \phi \in W_0^{1,p}(\Omega)^\tau,$$

where $J_q(u) = \|u\|^p - |u|_q^q$. Since $u \in \mathcal{N}_q^\tau$, we have

$$0 = \langle E_q'(u), u\rangle - \theta\langle J_q'(u), u\rangle = \theta(q-p)\|u\|^p.$$

This implies $\theta = 0$ and therefore

$$\langle E_q'(u), \phi\rangle = 0, \quad \forall \phi \in W_0^{1,p}(\Omega)^\tau.$$

Hence the principle of symmetric criticality [16] (see also [15, Proposition 1]) imply that u (and also $-u$) is a solution of (P_q^τ) which changes sign exactly once. The theorem is proved. □

We recall some facts about equivariant theory. An involution on a topological space X is a continuous function $\tau_X : X \to X$ such that τ_X^2 is the identity map of X. A subset A of X is called τ_X-invariant if $\tau_X(A) = A$. If X and Y are topological spaces equipped with involutions τ_X and τ_Y, respectively, then an equivariant map is a continuous function $f : X \to Y$ such that $f \circ \tau_X = \tau_Y \circ f$. Two equivariant maps $f_0, f_1 : X \to Y$ are equivariantly homotopic if there is a homotopy $\Theta : X \times [0,1] \to Y$ such that $\Theta(x,0) = f_0(x)$, $\Theta(x,1) = f_1(x)$ and $\Theta(\tau_X(x), t) = \tau_Y(\Theta(x,t))$, for all $x \in X$, $t \in [0,1]$.

Definition 3.2. The equivariant category of an equivariant map $f : X \to Y$, denoted by (τ_X, τ_Y)-cat(f), is the smallest number k of open invariant subsets $X_1, \ldots, X_k$ of X which cover X and which have the property that, for each $i = 1, \ldots, k$, there is a point $y_i \in Y$ and a homotopy $\Theta_i : X_i \times [0,1] \to Y$ such that $\Theta_i(x,0) = f(x)$, $\Theta_i(x,1) \in \{y_i, \tau_Y(y_i)\}$ and $\Theta_i(\tau_X(x), t) = \tau_Y(\Theta_i(x,t))$ for every $x \in X_i$, $t \in [0,1]$. If no such covering exists we define (τ_X, τ_Y)-cat$(f) = \infty$.

If A is a τ_X-invariant subset of X and $\iota : A \hookrightarrow X$ is the inclusion map we write

$$\tau_X\text{-cat}_X(A) = (\tau_X, \tau_X)\text{-cat}(\iota) \quad \text{and} \quad \tau_X\text{-cat}(X) = \tau_X\text{-cat}_X(X).$$

In the literature τ_X-cat(X) is usually called $\mathbb{Z}_2$-cat(X). Here it is more convenient to specify the involution in the notation.

The following properties can be verified:

Lemma 3.3. (i) *If $f : X \to Y$ and $h : Y \to Z$ are equivariant maps then*

$$(\tau_X, \tau_Z)\text{-cat}(h \circ f) \leqslant \tau_Y\text{-cat}(Y).$$

(ii) *If $f_0, f_1 : X \to Y$ are equivariantly homotopic, then* (τ_X, τ_Y)-cat$(f_0) = (\tau_X, \tau_Y)$-cat(f_1).

We denote by $\tau_a : V \to V$ the antipodal involution $\tau_a(u) = -u$ on the vector space V. A τ_a-invariant subset of V is usually called a symmetric subset. Equivariant Ljusternik–Schnirelmann category provides a lower bound for the number of pairs $\{u, -u\}$ of critical points of an even functional, as stated in the following abstract result (see [10, Theorem 1.1; 20, Theorem 5.7; 12, Corollary 4.1]).

Theorem 3.4. *Let $I : M \to \mathbb{R}$ be an even C^1-functional on a complete symmetric C^1-submanifold M of some Banach space V. Assume that I is bounded below and satisfies* $(\mathrm{PS})_c$ *for all $c \leqslant d$. Then, denoting $I^d = \{u \in M : I(u) \leqslant d\}$, I has at least τ_a-cat(I^d) antipodal pairs $\{u, -u\}$ of critical points with $I(\pm u) \leqslant d$.*

Coming back to our problem we set, for any given $r > 0$,

$$\Omega_r^- = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega \cup \Omega^\tau) \geqslant r\}.$$

Throughout the rest of this section $r > 0$ sufficiently small is fixed in such way that the inclusion maps $\Omega_r^- \hookrightarrow \Omega \backslash \Omega^\tau$ and $\Omega \hookrightarrow \Omega_r^+$ are equivariant homotopy equivalences and Ω_r^+ is as defined in (2.7).

Lemma 3.5. *Let $q_0 = q_0(r)$ be given by Lemma* 2.6. *Then, for any $q \in [q_0, p^*)$, there exists two maps*

$$\Omega_r^- \xrightarrow{\alpha_q} \mathcal{N}_q^\tau \cap E_q^{m_{q,r}^\tau} \xrightarrow{\gamma_q} \Omega_r^+$$

such that $\alpha_q(\tau x) = -\alpha_q(x)$, $\gamma_q(-u) = \tau\gamma_q(u)$, and $\gamma_q \circ \alpha_q$ is equivariantly homotopic to the inclusion map $\Omega_r^- \hookrightarrow \Omega_r^+$.

Proof. We fix $q \in [q_0, p^*)$, take a non-negative radial function $v_q \in \mathcal{N}_{q,B_r(0)}$ such that $E_{q,B_r(0)}(v_q) = m_{q,r}$ (see [1, Lemma 3.2]) and define $\alpha_q : \Omega_r^- \to \mathcal{N}_q^\tau \cap E_q^{m_{q,r}^\tau}$ by setting

$$\alpha_q(x) = v_q(\cdot - x) - v_q(\cdot - \tau x). \tag{3.1}$$

It is clear from the definition that $\alpha_q(\tau x) = -\alpha_q(x)$. Furthermore, since v_q is radial and τ is an isometry, we have that $\alpha_q(x) \in W_0^{1,p}(\Omega)^\tau$. Note that, for every $x \in \Omega_r^-$, we have that $|x - \tau x| \geqslant 2r$ (if this is not true, then $\overline{x} = (x + \tau x)/2$ satisfies $|x - \overline{x}| < r$ and $\tau\overline{x} = \overline{x}$, contradicting the definition of Ω_r^-). Thus, we can check that $E_q(\alpha_q(x)) = 2m_{q,r} \leqslant m_{q,r}^\tau$ (by Lemma 2.1) and

$$\|\alpha_q(x)\|^p = 2\|v_q\|_{B_r(0)}^p = 2|v_q|_{q,B_r(0)}^q = |\alpha_q(x)|_q^q.$$

All the above considerations show that α_q is well defined.

By Lemma 2.6(ii) it follows that $\gamma_q : \mathcal{N}_q^\tau \cap E_q^{m_{q,r}^\tau} \to \Omega_r^+$ given by $\gamma_q(u) = \beta(u^+)$ is well defined. A simple calculation shows that $\gamma_q(-u) = \tau\gamma_q(u)$. Moreover, using (3.1) and the fact that v_q is radial, we get

$$\gamma_q(\alpha_q(x)) = \frac{\int_{B_r(x)} |v_q(y-x)|^{p^*} y \, \mathrm{d}y}{\int_{B_r(x)} |v_q(y-x)|^{p^*} \, \mathrm{d}y} = \frac{\int_{B_r(0)} |v_q(y)|^{p^*} (y+x) \, \mathrm{d}y}{\int_{B_r(0)} |v_q(y)|^{p^*} \, \mathrm{d}y} = x,$$

for any $x \in \Omega_r^-$. This concludes the proof of the lemma. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We will show that the theorem holds for $q_0 = q_0(r)$ given by Lemma 2.6. For $q \in [q_0, p^*)$ fixed, the Palais–Smale condition for E_q restricted to $\mathcal{N}_q^\tau$ follows as in the proof of Theorem 1.1. Since E_q is even we can apply Theorem 3.4 to obtain τ_a-cat$(\mathcal{N}_q^\tau \cap E_q^{m_{q,r}^\tau})$ pairs $\pm u_i$ of critical points with $E_q(\pm u_i) \leqslant m_{q,r}^\tau < 2m_q^\tau$ (by Lemma 2.6(i)). The definition of $W_0^{1,p}(\Omega)^\tau$, Proposition 3.1 and the same argument employed in the proof of Theorem 1.1 show that u_i is a solution of (P_q^τ) which changes sign exactly once.

To conclude the proof we need only to verify that

$$\tau_a\text{-cat}(\mathcal{N}_q^\tau \cap E_q^{m_{q,r}^\tau}) \geqslant \tau\text{-cat}_\Omega(\Omega \backslash \Omega^\tau). \tag{3.2}$$

With this aim we recall that r was chosen so that the inclusion maps $\Omega_r^- \hookrightarrow \Omega\backslash\Omega^\tau$ and $\Omega \hookrightarrow \Omega_r^+$ are equivariant homotopy equivalences. Thus, (3.2) follows from Lemma 3.5 and the properties given by Lemma 3.3. The theorem is proved. □

Proof of Corollary 1.3. Let $\tau : \mathbb{R}^N \to \mathbb{R}^N$ be given by $\tau(x) = -x$. It is proved in [6, Corollary 3] that our assumptions imply τ-cat$(\Omega) \geqslant N$. Since $0 \notin \Omega$, $\Omega^\tau = \emptyset$. It suffices now to apply Theorem 1.2. □

Remark 3.6. For any $\lambda \geqslant 0$ we know that

$$S_\lambda = \inf_{u \in W_0^{1,p}(\Omega)\backslash\{0\}} \frac{\int_\Omega |\nabla u|^p + \lambda |u|^p}{|u|_{p^*}^p}$$

its equal to S, independent of Ω and is never achieved in any proper subset of $\mathbb{R}^N$. Thus, a simple inspection of our proofs shows that Theorems 1.1 and 1.2, and Corollary 1.3 also hold for the problem

$$(P_{q,\lambda}^{\tau}) \quad \begin{cases} -\Delta_p u + \lambda|u|^{p-2}u = |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(\tau x) = -u(x) & \text{for all } x \in \Omega. \end{cases}$$

4. Positive solutions

In this section we will obtain multiple positive solutions for the problem (P_q). Since Theorem 1.4 does not require symmetry for the domain Ω we will consider the functional E_q restricted to the usual Nehari manifold $\mathcal{N}_q$. We fix $r > 0$ such that the sets Ω_r^+ and

$$\widetilde{\Omega}_r^- = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geqslant r\}$$

are homotopically equivalent to Ω and $B_r(0) \subset \Omega$. We start with a version of Lemma 2.6.

Lemma 4.1. *There exists $q_0 \in (p, p^*)$ such that, for all $q \in [q_0, p^*)$, we have*

(i) $m_{q,r} < 2m_q$,
(ii) *if $u \in \mathcal{N}_q$ and $E_q(u) \leqslant m_{q,r}$, then $\beta(u) \in \Omega_r^+$.*

Proof. Since the proof is analogous to that presented in Lemma 2.6 we only sketch the main steps of (ii). Suppose, by contradiction, that there exist $q_n \uparrow p^*$, $(u_n) \in \mathcal{N}_{q_n}$ with $E_{q_n}(u_n) \leqslant m_{q_n,r}$ and $\beta(u_n) \notin \Omega_r^+$. Then we have that

$$\lim_{n\to\infty} \|u_n\|^p = \lim_{n\to\infty} |u_n|_{p^*}^{p^*} = S^{N/p}$$

and $\nabla u_n(x) \to \nabla u(x)$ a.e. $x \in \Omega$, where u is the weak limit of (u_n) in $W_0^{1,p}(\Omega)$. Using Lemma 2.4 and arguing as in the proof of Lemma 2.6 we conclude that $u = 0$ and the measure v in (2.3) is concentrated at a single point $y \in \overline{\Omega}$. Thus,

$$\beta(u_n) = \frac{\int_{\mathbb{R}^N} |u_n|^{p^*} x \, \mathrm{d}x}{\int_{\mathbb{R}^N} |u_n|^{p^*} \, \mathrm{d}x} \to y \in \overline{\Omega},$$

which contradicts $\beta(u_n) \notin \Omega_r^+$. $\square$

Following Benci and Cerami [2] one can easily show that

Lemma 4.2. *If u is a solution of (P_q) with $E_q(u) < 2m_q$, then u does not change sign.*

Proof. Since u is a critical point of E_q we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi + |u|^{p-2}u\phi = \int_\Omega |u|^{q-2}u\phi \quad \forall \phi \in W_0^{1,p}(\Omega).$$

In particular for $\phi = u^{\pm}$. So, if both u^+ and u^- are nonzero, then $u^{\pm} \in \mathcal{N}_q$ and $E_q(u) = E_q(u^+) + E_q(u^-) \geqslant 2m_q$. This is a contradiction. $\square$

We are now able to prove Theorem 1.4.

Proof of Theorem 1.4. Let q_0 be given by Lemma 4.1. For $q \in [q_0, p^*)$ fixed, standard calculations show that the restriction of the functional E_q to $\mathcal{N}_q$ satisfies the Palais–Smale condition.

Take a non-negative radial function $v_q \in \mathcal{N}_{q,B_r(0)}$ such that $E_{q,B_r(0)}(v_q) = m_{q,r}$ and consider the diagram

$$\widetilde{\Omega}_r^- \xrightarrow{\alpha_q} \mathcal{N}_q \cap E_q^{m_{q,r}} \xrightarrow{\gamma_q} \Omega_r^+,$$

where $\alpha_q(x) = v_q(\cdot - x)$ and $\gamma_q(u) = \beta(u)$. Arguing as in the proof of Lemma 3.5 and using Lemma 4.1 we can verify that the diagram is well defined. Furthermore, since $\gamma_q(\alpha_q(x)) = x$ for every $x \in \widetilde{\Omega}_r^-$, we can proceed as in the proof of Theorem 1.2 and obtain $\mathrm{cat}_{\Omega_r^+}(\widetilde{\Omega}_r^-) = \mathrm{cat}(\Omega)$ pairs $\pm u_i$ of critical points of E_q such that $E_q(\pm u_i) \leqslant m_{q,r} < 2m_q$. By Lemma 4.2 none of these critical points changes sign. Thus we may suppose $u_i > 0$ and the theorem is proved. □

Acknowledgements

This paper is a part of my Ph.D. thesis which was done under the supervision of Professor Djairo G. de Figueiredo to whom I express my sincere thanks for the stimulating guidance. I should also thank the referee for his/her useful suggestions and comments.

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