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Multiplicity of nodal solutions for a critical quasilinear equation with symmetry

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Abstract

We consider the quasilinear problem $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{q-2}u + |u|^{p^*-2}u = 0$ in Ω, $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geqslant p^2$, $\lambda > 0$ and $p < q < p^* = Np/(N-p)$. We show that if Ω is invariant under a nontrivial orthogonal involution then, for λ sufficiently small, there is an effect of the equivariant topology of Ω on the number of solutions which changes sign exactly once.

Keywords: Nodal solutions; Critical problems; p-Laplace operator; Equivariant category; Symmetry

1. Introduction and statement of results

In this paper we consider the following critical problem:

$$(D_\lambda) \qquad \begin{cases} -\Delta_p u = \lambda|u|^{q-2}u + |u|^{p^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $N \geqslant p^2$, $\lambda > 0$ and $p \leqslant q < p^* = Np/(N-p)$.

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The starting point on the study of the above problem is the pioneer work of Brezis and Nirenberg [4], where the authors studied (D_λ) in the case $p=q=2$ and showed that the existence of positive solutions for (D_λ) is related with the interaction between the parameter λ and the first eigenvalue $\lambda_1(\Omega)$ of the operator $-\Delta_p$ on $W_0^{1,p}(\Omega)$, defined by

$$\lambda_1(\Omega)=\inf\left\{\int_\Omega |\nabla u|^p\,\mathrm{d}x : u\in W_0^{1,p}(\Omega),\ \int_\Omega |u|^p\,\mathrm{d}x=1\right\}. \tag{1.1}$$

Among other results, they showed that the problem (D_λ) has at least one positive solution provided $p=q=2$, $N\geqslant 4$ and $0<\lambda<\lambda_1(\Omega)$. In [10], Garcia Azorero and Peral Alonso extended the results of [4] to the p-Laplacian operator. The same authors proved in [11] that, if $p^2\leqslant N$, $q\in(p,p^*)$ and $\lambda>0$, then (D_λ) has at least one nontrivial solution (see also [12]). The main interest in (D_λ) is due to the lack of compactness of the embedding $W_0^{1,p}(\Omega)\hookrightarrow L^{p^*}(\Omega)$, which makes that the associated functional does not satisfy the Palais–Smale condition on some levels.

After the paper of Brezis and Nirenberg, a lot of papers concerning critical nonlinearities have appeared. In particular, we recall that Rey [17] and Lazzo [13] proved, for $p=q=2$, that the problem (D_λ) has at least $\mathrm{cat}(\Omega)$ positive solutions (see also the well-known paper of Benci and Cerami [2] where the subcritical case was considered), provided $\lambda>0$ is sufficiently close to 0. Here, $\mathrm{cat}(\Omega)$ stands for the usual Ljusternik–Schnirelmann category of $\overline{\Omega}$ in itself. Recently, Alves and Ding [1] extended these last results for the p-Laplacian operator and obtained a similar theorem.

We are interested here in nodal solutions of (D_λ), that is, solutions that change sign in Ω. The first result in this direction is due to Cerami, Solimini and Struwe [6], who considered the case $p=q=2$ and obtained one pair of nodal solutions, provided $N\geqslant 6$ and $0<\lambda<\lambda_1(\Omega)$. Similar results were obtained by Zhang [24] and Tarantello [21]. The question of multiplicity of nodal solutions was also discussed in [6], where the authors proved the existence of infinitely many radial solutions when Ω is a ball centered at the origin, $p=q=2$, $N\geqslant 7$ and $0<\lambda<\lambda_1(\Omega)$. For domains with some kind of symmetry Fortunato and Jannelli [9] showed the existence of solutions with arbitrarily large energy for $N\geqslant 4$ and $\lambda>0$. However, these solutions change sign many times.

In this paper we use a different approach which already appears in the work by Castro and Clapp [5]. In order to obtain nodal solutions for (D_λ) we denote by $O(N)$ the set of orthogonal linear transformations of $\mathbb{R}^N$ in $\mathbb{R}^N$ and suppose that the domain Ω has the following symmetry property:

(H) there exists $\tau\in O(N)$ such that $\tau\neq \mathrm{Id}$, $\tau^2=\mathrm{Id}$ and $\tau(\Omega)=\Omega$.

This includes, e.g., domains which are symmetric with respect to the origin, as well as cylindrical or rotationally invariant domains as those considered by Fortunato and Jannelli.

We deal with the symmetric problem

$$(D_\lambda^\tau)\qquad \begin{cases} -\Delta_p u=\lambda|u|^{q-2}u+|u|^{p^*-2}u & \text{in } \Omega,\\ u(\tau x)=-u(x) & \text{for all } x\in\Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain satisfying (H) and the parameters are as before. We say that a solution u of (D_λ^τ) changes sign exactly once if the set $\Omega \backslash u^{-1}(0)$ has exactly two connected components, u is positive in one of them and negative in the other. We state below our main results.

Theorem 1.1. *Suppose that (H) holds. Then, for any $\lambda \in (0, \lambda_1(\Omega))$, the problem (D_λ^τ) has at least one pair of solutions which change sign exactly once.*

Theorem 1.2. *Suppose that (H) holds. Then there exists $\lambda_* \in (0, \lambda_1(\Omega))$ such that, for any $\lambda \in (0, \lambda_*)$, the problem (D_λ^τ) has at least $\tau\text{-cat}_\Omega(\Omega \backslash \Omega^\tau)$ pairs of solutions which change sign exactly once.*

Here, $\Omega^\tau = \{x \in \Omega : \tau x = x\}$ and τ-cat is the G_τ-equivariant Ljusternik–Schnirelmann category for the group $G_\tau = \{\mathrm{Id}, \tau\}$. There are several situations where the equivariant category turns out to be larger than the nonequivariant one. The classical example is the unit sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ with $\tau = -\mathrm{Id}$. In this case $\mathrm{cat}(\mathbb{S}^{N-1}) = 2$, whereas $\tau\text{-cat}(\mathbb{S}^{N-1}) = N$. Thus, as an easy consequence of Theorem 1.2 we have:

Corollary 1.3. *Let Ω be symmetric with respect to the origin and such that $0 \notin \Omega$. Assume further that there is an odd map $\varphi : \mathbb{S}^{N-1} \to \Omega$. Then there exists $\lambda_* \in (0, \lambda_1(\Omega))$ such that, for all $\lambda \in (0, \lambda_*)$, the problem (D_λ) has at least N pairs of odd solutions which change sign exactly once.*

The problem (D_λ^τ) was introduced by Castro and Clapp [5] in the case $p = q = 2$. They obtained existence and multiplicity of solutions which change sign exactly once. The above results improve the paper [5] in two ways: first, because we consider the p-Laplacian operator and second, because we also deal with the case $p < q < p^*$. Hence, our results seem to be new even in the semilinear case $p = 2$. In order to deal with the difference of homogeneity between $\Delta_p u$ and $\lambda |u|^{q-2} u$, we adapt some ideas introduced in [1]. Our work also complements the papers [6,24,9,21] that deal with nodal solutions and [17,13,1], where only positive solutions were considered.

The paper is organized as follows: Section 2 is devoted to establish the variational framework as well as to present some technical results. In Section 3, after recalling some facts about equivariant Ljusternik–Schnirelmann theory, we prove our main results.

2. Variational framework and some technical results

Throughout this paper we will consider the space $W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p \, \mathrm{d}x \right)^{1/p}.$$

As usual, we denote by S the best constant of the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ given by

$$S = \inf\left\{\|u\|^p : u \in W_0^{1,p}(\Omega), \int_\Omega |u|^{p^*}\,\mathrm{d}x = 1\right\}. \tag{2.1}$$

It is known that S does not depend on the set Ω and is never achieved except when $\Omega = \mathbb{R}^N$.

We note that the involution τ of Ω induces an involution of $W_0^{1,p}(\Omega)$, which we also denote by τ, in the following way: for each $u \in W_0^{1,p}(\Omega)$ we define $\tau u \in W_0^{1,p}(\Omega)$ by

$$(\tau u)(x) = -u(\tau x). \tag{2.2}$$

The weak solutions of the problem (D_λ) are the critical points of the C^1-functional $I_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$I_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x - \frac{\lambda}{q}\int_\Omega |u|^q\,\mathrm{d}x - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,\mathrm{d}x.$$

In order to obtain symmetric solutions, we will look for critical points that lie in the invariant space $W_0^{1,p}(\Omega)^\tau$ defined as

$$W_0^{1,p}(\Omega)^\tau = \{u \in W_0^{1,p}(\Omega) : \tau u = u\}.$$

Let us consider the Nehari manifold associated with the functional I_λ

$$\begin{aligned}\mathcal{N}_\lambda &= \{u \in W_0^{1,p}(\Omega) : \langle I_\lambda'(u), u\rangle = 0\}\\ &= \left\{u \in W_0^{1,p}(\Omega) : \|u\|^p = \lambda\int_\Omega |u|^q\,\mathrm{d}x + \int_\Omega |u|^{p^*}\,\mathrm{d}x\right\}\end{aligned}$$

and the τ-invariant Nehari manifold

$$\mathcal{N}_\lambda^\tau = \{u \in \mathcal{N}_\lambda : \tau u = u\} = \mathcal{N}_\lambda \cap W_0^{1,p}(\Omega)^\tau.$$

Note that, if $u \in \mathcal{N}_\lambda$, then

$$I_\lambda(u) = \lambda\left(\frac{1}{p} - \frac{1}{q}\right)\int_\Omega |u|^q\,\mathrm{d}x + \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_\Omega |u|^{p^*}\,\mathrm{d}x \geqslant 0, \tag{2.3}$$

and therefore the following minimization problems are well defined:

$$m_\lambda = \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u) \quad \text{and} \quad m_\lambda^\tau = \inf_{u \in \mathcal{N}_\lambda^\tau} I_\lambda(u). \tag{2.4}$$

For any τ-invariant bounded domain $\mathscr{D} \subset \mathbb{R}^N$ we define $\|\cdot\|_{\mathscr{D}}$, $I_{\lambda,\mathscr{D}}$, $\mathcal{N}_{\lambda,\mathscr{D}}$, $\mathcal{N}_{\lambda,\mathscr{D}}^\tau$, $m_{\lambda,\mathscr{D}}$ and $m_{\lambda,\mathscr{D}}^\tau$ in the same way, by taking all the integrals over $\mathscr{D}$ instead of Ω. We denote by $|u|_{s,\mathscr{D}}$ the $L^s(\mathscr{D})$-norm of a function $u \in L^s(\mathscr{D})$. In order to simplify the notation, whenever we omit the subscript reference of the set in the above notation, we are assuming that $\mathscr{D} = \Omega$. Also, for simplicity, we write only $\int_{\mathscr{D}} u$ instead of $\int_{\mathscr{D}} u(x)\,\mathrm{d}x$.

Lemma 2.1. *Suppose either $q=p$ and $\lambda \in (0, \lambda_1(\Omega))$ or $p<q<p^*$ and $\lambda>0$. Then there exists $r_{\lambda,q}>0$ such that*

$$\|u\| \geqslant r_{\lambda,q}, \tag{2.5}$$

for all $u \in \mathcal{N}_\lambda^\tau$. In particular, $m_\lambda^\tau > 0$.

Proof. If $q=p$ and $\lambda \in (0, \lambda_1(\Omega))$, we can use the definition of $\lambda_1(\Omega)$ and the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ to get

$$\left(1-\frac{\lambda}{\lambda_1(\Omega)}\right)\|u\|^p \leqslant \|u\|^p - \lambda|u|_p^p = |u|_{p^*}^{p^*} \leqslant C_1\|u\|^{p^*},$$

for some $C_1>0$. Thus,

$$\left(1-\frac{\lambda}{\lambda_1(\Omega)}\right) \leqslant C_1\|u\|^{p^*-p},$$

and (2.5) follows for $r_{\lambda,q}=\{(1/C_1)(1-(\lambda/\lambda_1(\Omega)))\}^{1/(p^*-p)}$. Consequently, if $u \in \mathcal{N}_\lambda^\tau$, we have

$$\begin{aligned} I_\lambda(u) &= \left(\frac{1}{p}-\frac{1}{p^*}\right)(\|u\|^p - \lambda|u|_p^p) \\ &\geqslant \left(\frac{1}{p}-\frac{1}{p^*}\right)\left(1-\frac{\lambda}{\lambda_1(\Omega)}\right) r_{\lambda,q}^p, \end{aligned}$$

and therefore $m_\lambda^\tau>0$. Suppose now that $p<q<p^*$ and $\lambda>0$. Then there exists $C_2>0$ such that

$$\|u\|^p = \lambda|u|_q^q + |u|_{p^*}^{p^*} \leqslant \lambda C_2\|u\|^q + C_1\|u\|^{p^*},$$

that is,

$$1 \leqslant \lambda C_2\|u\|^{q-p} + C_1\|u\|^{p^*-p},$$

for all $u \in \mathcal{N}_\lambda^\tau$. Since $q>p$ and $p^*>p$, the above expression shows there cannot exist $(u_n) \subset \mathcal{N}_\lambda^\tau$ with $\|u_n\| \to 0$ and (2.5) follows. To verify that $m_\lambda^\tau>0$ we suppose, by contradiction, that $m_\lambda^\tau=0$. Then there exists $(u_n) \subset \mathcal{N}_\lambda^\tau$ such that $I_\lambda(u_n) \to 0$. It follows from (2.3) that $|u_n|_q^q \to 0$ and $|u_n|_{p^*}^{p^*} \to 0$. Hence $\|u_n\|^p \to 0$, contradicting (2.5). The lemma is proved. □

In view of the definition of $\mathcal{N}_{\lambda,\mathcal{D}}$, a standard calculation shows that $m_{0,\mathcal{D}}=(1/N)S^{N/p}$. On the other hand, if $N \geqslant p^2$, Garcia Azorero and Peral Alonso [10] showed that, for $0<\lambda<\lambda_1(\mathcal{D})$ and any bounded smooth domain $\mathcal{D}$,

$$m_{\lambda,\mathcal{D}} < \frac{1}{N}S^{N/p}$$

and $m_{\lambda,\mathcal{D}}$ is achieved by $I_{\lambda,\mathcal{D}}$ on $\mathcal{N}_{\lambda,\mathcal{D}}$. Although the number $m_{\lambda,\mathcal{D}}$ depends on the set $\mathcal{D}$ we have the following asymptotic property.

Lemma 2.2. *For any bounded smooth domain $\mathscr{D} \subset \mathbb{R}^N$ we have that*

$$\lim_{\lambda\to 0^+} m_{\lambda,\mathscr{D}} = m_{0,\mathscr{D}} = \frac{1}{N} S^{N/p}.$$

Proof. This proof follows quite similar lines as the proof of [1, Lemma 2.4] and will be omitted. □

The next auxiliary result establishes the relation between the numbers defined in (2.4) and the constant S.

Lemma 2.3. *For any $\lambda \in (0, \lambda_1(\Omega))$ we have that*

$$2m_\lambda \leqslant m_\lambda^\tau < \frac{2}{N} S^{N/p}.$$

Proof. We closely follow the arguments of [5, Proposition 5]. Given $u \in \mathscr{N}_\lambda^\tau$ we can use (2.2) to conclude that $u^+, u^- \in \mathscr{N}_\lambda$, where $u^\pm = \max\{\pm u, 0\}$. Thus

$$I_\lambda(u) = I_\lambda(u^+) + I_\lambda(u^-) \geqslant 2m_\lambda,$$

and the first inequality follows. Next, we choose $y \in \Omega$, and $r > 0$ such that $y \neq \tau y$, $B_r(y) \subset \Omega$ and $B_r(y) \cap B_r(\tau y) = \emptyset$. Since $0 < \lambda < \lambda_1(\Omega) < \lambda_1(B_r(0))$, we can take a positive radial function $v_\lambda \in \mathscr{N}_{\lambda, B_r(0)}$ such that $I_{\lambda, B_r(0)}(v_\lambda) = m_{\lambda, B_r(0)}$. By the choice of $r > 0$, we get

$$u_\lambda = v_\lambda(\cdot - y) - v_\lambda(\cdot - \tau y) \in \mathscr{N}_\lambda^\tau,$$

and therefore, since $m_{\lambda, B_r(0)} < (1/N) S^{N/p}$, we conclude that

$$m_\lambda^\tau \leqslant I_\lambda(u_\lambda) = 2 I_{\lambda, B_r(0)}(v_\lambda) < \frac{2}{N} S^{N/p}.$$

This concludes the proof of the lemma. □

Let $D^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N) : |\nabla u| \in L^p(\mathbb{R}^N)\}$ and denote by $\mathscr{M}(\mathbb{R}^N)$ the Banach space of finite Radon measures over $\mathbb{R}^N$ equipped with the norm

$$|\mu| = \sup_{\phi \in C_0(\mathbb{R}^N), |\phi|_\infty \leqslant 1} \left| \int_{\mathbb{R}^N} \phi \, \mathrm{d}\mu \right|.$$

We say that $\mu_n \rightharpoonup \mu$ weakly in $\mathscr{M}(\mathbb{R}^N)$ if for all $f \in C_0(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} f \, \mathrm{d}\mu_n \to \int_{\mathbb{R}^N} f \, \mathrm{d}\mu$. We state below a result that is a variant of the concentration-compactness lemma, see [14].

Lemma 2.4. *Let $(u_n) \subset D^{1,p}(\mathbb{R}^N)$ be a sequence such that*

$$\begin{array}{ll} u_n \rightharpoonup u & \textit{weakly in } D^{1,p}(\mathbb{R}^N), \\ |\nabla(u_n - u)|^p \rightharpoonup \mu & \textit{weakly in } \mathscr{M}(\mathbb{R}^N), \\ |u_n - u|^{p^*} \rightharpoonup \nu & \textit{weakly in } \mathscr{M}(\mathbb{R}^N), \\ u_n(x) \to u(x) & \textit{a.e. } x \in \mathbb{R}^N, \\ \nabla u_n(x) \to \nabla u(x) & \textit{a.e. } x \in \mathbb{R}^N, \end{array} \tag{2.6}$$

and define

$$\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |\nabla u_n|^p, \quad \nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^{p^*}.$$

Then

$$|\nu|^{p/p^*} \leqslant S^{-1}|\mu|, \tag{2.7}$$

$$\limsup_{n\to\infty} |\nabla u_n|^p_{p,\mathbb{R}^N} = |\nabla u|^p_{p,\mathbb{R}^N} + |\mu| + \mu_\infty, \tag{2.8}$$

and

$$\limsup_{n\to\infty} |u_n|^{p^*}_{p^*,\mathbb{R}^N} = |u|^{p^*}_{p^*,\mathbb{R}^N} + |\nu| + \nu_\infty. \tag{2.9}$$

Moreover, if $u = 0$ *and* $|\nu|^{p/p^*} = S^{-1}|\mu|$, *then* μ *and* ν *are concentrated at single points.*

Remark 2.5. In [22, Lemma 1.40] the above lemma is proved for $p = 2$ without the assumption of pointwise convergence for the gradient. The proof for the general case follows the same lines of case $p = 2$ except for Eq. (2.8). As noted in [19, Example 2.3], it can fail for $p \neq 2$ if we do not assume that $\nabla u_n(x) \to \nabla u(x)$ for a.e. $x \in \mathbb{R}^N$. However, when this last assumption is made, Eq. (2.8) can be verified as in [19, Lemma 2.1 and Remark 2.2].

For any $r > 0$ we define the set

$$\Omega_r^+ = \{x \in \mathbb{R}^N : \mathrm{dist}(x, \Omega) < r\}. \tag{2.10}$$

We also define the barycenter map $\beta : W_0^{1,p}(\Omega)\backslash\{0\} \to \mathbb{R}^N$ by setting

$$\beta(u) = \frac{\int_{\mathbb{R}^N} |u|^{p^*} x \,\mathrm{d}x}{\int_{\mathbb{R}^N} |u|^{p^*} \,\mathrm{d}x}.$$

For simplicity, we write $m_{\lambda,r}$ instead of $m_{\lambda,B_r(0)}$.

The following result is a version of [5, Lemma 14]. As we will see, it is a key ingredient for the proof of the multiplicity result. Our proof is based in some ideas contained in [22].

Lemma 2.6. *For any* $r > 0$ *there exists* $\lambda_0 = \lambda_0(r) > 0$ *such that, for all* $0 < \lambda < \lambda_0$, *we have that* $\beta(u) \in \Omega_r^+$ *whenever* $u \in \mathcal{N}_\lambda$ *and* $I_\lambda(u) \leqslant m_{\lambda,r}$.

Proof. Suppose, by contradiction, that the lemma is false. Then there exist $\lambda_n \to 0^+$, $u_n \in \mathcal{N}_{\lambda_n}$ such that $I_{\lambda_n}(u_n) \leqslant m_{\lambda_n,r}$ but $\beta(u_n) \notin \Omega_r^+$. Note that

$$m_{\lambda_n} \leqslant I_{\lambda_n}(u_n) = \frac{1}{p}\|u_n\|^p - \frac{\lambda_n}{q}|u_n|_q^q - \frac{1}{p^*}|u_n|_{p^*}^{p^*} \leqslant m_{\lambda_n,r}$$

and

$$0 = \langle I'_{\lambda_n}(u_n), u_n\rangle = \|u_n\|^p - \lambda_n |u_n|_q^q - |u_n|_{p^*}^{p^*}.$$

Since $I_{\lambda_n}(u_n)$ is bounded, we have that (u_n) is bounded in $W_0^{1,p}(\Omega)$ (and also in $L^q(\Omega)$). Hence,

$$m_{\lambda_n} + o(1) \leqslant \frac{1}{p}\|u_n\|^p - \frac{1}{p^*}|u_n|_{p^*}^{p^*} \leqslant m_{\lambda_n,r} + o(1) \tag{2.11}$$

and $\|u_n\|^p - |u_n|_{p^*}^{p^*} = o(1)$, as $n \to \infty$. Thus,

$$\|u_n\|^p = b + o(1) \quad \text{and} \quad |u_n|_{p^*}^{p^*} = b + o(1), \tag{2.12}$$

for some $b \geqslant 0$. Taking the limit in (2.11) and using Lemma 2.2, we conclude that $b = S^{N/p}$.

We claim that, up to a subsequence,

$$\nabla u_n(x) \to \nabla u(x) \tag{2.13}$$

for a.e. $x \in \Omega$. Indeed, by (2.12), we have that the sequence $v_n := u_n/|u_n|_{p^*}$ satisfies $|v_n|_{p^*} = 1$ and

$$\|v_n\|^p = \frac{\|u_n\|^p}{|u_n|_{p^*}^p} \to \frac{S^{N/p}}{S^{(N/p)((N-p)/N)}} = S^{(N/p)(1-((N-p)/N))} = S.$$

Hence, (v_n) is a minimizing sequence for the best constant S defined in (2.1) and standard calculations (see [23,18, Corollary 3.7]) show that $\nabla v_n(x) \to \nabla v(x)$ for a.e. $x \in \Omega$. This convergence, (2.12) and the definition of v_n imply that (2.13) holds.

By going if necessary to a subsequence, we may assume that (2.6) holds and $u_n \to u$ in $L^p(\Omega)$. Lemma 2.4, $b = S^{N/p}$ and (2.12) provide

$$S^{N/p} = \|u\|^p + |\mu|, \quad S^{N/p} = |u|_{p^*}^{p^*} + |\nu|$$

and

$$|\nu|^{p/p^*} \leqslant S^{-1}|\mu|, \quad |u|_{p^*}^p \leqslant S^{-1}\|u\|^p.$$

Note that, since Ω is bounded, the terms μ_∞ and ν_∞ do not appear in the above expressions.

The inequality $(a+b)^t < a^t + b^t$ for $a, b > 0$ and $0 < t < 1$, and the above expressions imply that $|\nu|$ and $|u|_{p^*}^{p^*}$ are equal either to 0 or $S^{N/p}$. Indeed, if this is not the case, we get

$$\begin{aligned} S^{(N-p)/p} &= S^{-1}(\|u\|^p + |\mu|) \geqslant (|u|_{p^*}^{p^*})^{p/p^*} + |\nu|^{p/p^*} \\ &> (|u|_{p^*}^{p^*} + |\nu|)^{p/p^*} = S^{(N-p)/p}, \end{aligned}$$

which is absurd. Suppose $|u|_{p^*}^{p^*} = S^{N/p}$. Since $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$, we have that $\|u\|^p \leqslant \liminf_{n\to\infty} \|u_n\|^p = S^{N/p}$. Hence

$$\frac{\|u\|^p}{|u|_{p^*}^p} \leqslant \frac{S^{N/p}}{S^{(N-p)/p}} = S,$$

and we conclude that S is attained by $u \in W_0^{1,p}(\Omega)$, which does not make sense. This shows that $u = 0$ and therefore $|v| = S^{N/p}$ and v is concentrated at a single point $y \in \overline{\Omega}$. Hence,

$$\beta(u_n) = \frac{\int_{\mathbb{R}^N} |u_n|^{p^*} x \, \mathrm{d}x}{\int_{\mathbb{R}^N} |u_n|^{p^*} \, \mathrm{d}x} \to S^{-N/p} \int_{\Omega} x \, \mathrm{d}v = y \in \overline{\Omega},$$

which contradicts $\beta(u_n) \notin \Omega_r^+$. The lemma is proved. □

3. Proof of the main results

We start this section by noting that, if u is a solution of (D_λ^τ), then it is of class C^1. We say it changes sign n times if the set $\{x \in \Omega : u(x) \neq 0\}$ has $n + 1$ connected components. Obviously, if u is a nontrivial solution of problem (D_λ^τ), then it changes sign an odd number of times. The relation between the number of nodal regions of a solution and its energy is given by the result below. The proof can be obtained by following the same arguments contained in [5, Proposition 6] and will be omitted.

Lemma 3.1. *If u is a solution of the problem (D_λ^τ) which changes sign $2k - 1$ times, then $I_\lambda(u) \geqslant k m_\lambda^\tau$.*

Let V be a Banach space, M be a C^1-manifold of V and $I : V \to \mathbb{R}$ a C^1-functional. We recall that $I|_M$ satisfies the Palais–Smale condition at level c if any sequence $(u_n) \subset M$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_* \to 0$ contains a convergent subsequence. Here we are denoting by $\|I'(u)\|_*$ the norm of the derivative of the restriction of I to M (see [22, Section 5.3]).

We establish below a local compactness condition for the functional I_λ on $\mathcal{N}_\lambda^\tau$. Note that the symmetry of the functions provides compactness below the critical level $(2/N)S^{N/p}$, which is exactly the double of the critical level for I_λ on $W_0^{1,p}(\Omega)$.

Lemma 3.2. *Suppose either $q = p$ and $\lambda \in (0, \lambda_1(\Omega))$ or $q < p < p^*$ and $\lambda > 0$. Let $(u_n) \subset \mathcal{N}_\lambda^\tau$ be such that $\|I_\lambda'(u_n)\|_* \to 0$ and $I_\lambda(u_n) \to c < (2/N)S^{N/p}$. Then (u_n) possesses a convergent subsequence.*

Proof. Since (u_n) is a Palais–Smale sequence, standard arguments [10,12] show that (u_n) is bounded in $W_0^{1,p}(\Omega)$. Since $\|I_\lambda'(u_n)\|_* \to 0$, there exits $(\theta_n) \subset \mathbb{R}$ such that

$$I_\lambda'(u_n) - \theta_n J_\lambda'(u_n) \to 0 \quad \text{in } (W_0^{1,p}(\Omega))^*, \tag{3.1}$$

where

$$J_\lambda(u) = \|u\|^p - \lambda |u|_q^q - |u|_{p^*}^{p^*}, \tag{3.2}$$

for all $u \in W_0^{1,p}(\Omega)$. Recalling that $(u_n) \subset \mathcal{N}_\lambda^\tau$, we get

$$\langle J_\lambda'(u_n), u_n \rangle = \lambda(p - q)|u_n|_q^q + (p - p^*)|u_n|_{p^*}^{p^*} < 0.$$

Thus, we can suppose that $\langle J_\lambda'(u_n), u_n\rangle \to l \leqslant 0$. If $l=0$ the above expression would imply that $\|u_n\| \to 0$, contradicting (2.5). Hence, $l<0$ and we infer from (3.1) that $\theta_n \to 0$, that is, $I_\lambda'(u_n) \to 0$ in $(W_0^{1,p}(\Omega))^*$.

Since (u_n) is bounded there exists $u \in W_0^{1,p}(\Omega)$ such that, up to a subsequence,

$$\begin{array}{ll} u_n \rightharpoonup u & \text{weakly } W_0^{1,p}(\Omega),\\ u_n \to u & \text{in } L^{p^*}_{loc}(\Omega) \text{ and } L^q(\Omega),\\ u_n(x) \to u(x) & \text{a.e. } x \in \Omega. \end{array}$$

Moreover, by applying the concentration compactness lemma of Lions as in [23], we can also suppose that

$$\begin{array}{ll} \nabla u_n(x) \to \nabla u(x) & \text{a.e. } x \in \Omega,\\ |\nabla u_n|^{p-2}\dfrac{\partial u_n}{\partial x_i} \rightharpoonup |\nabla u|^{p-2}\dfrac{\partial u}{\partial x_i} & \text{weakly in } (L^p(\Omega))^*,\ 1 \leqslant i \leqslant N, \end{array}$$

from which follows that $I_\lambda'(u)=0$.

Note that

$$\begin{aligned} c+o(1) &= I_\lambda(u_n) - \frac{1}{p}\langle I_\lambda'(u_n), u_n\rangle\\ &= \lambda\left(\frac{1}{p}-\frac{1}{q}\right)|u_n|_q^q + \left(\frac{1}{p}-\frac{1}{p^*}\right)|u_n|_{p^*}^{p^*}, \end{aligned}$$

where $o(1) \to 0$ as $n \to \infty$. Recalling that $u_n \to u$ in $L^q(\Omega)$, we get

$$\frac{1}{N}|u_n|_{p^*}^{p^*} = c + \lambda\left(\frac{1}{q}-\frac{1}{p}\right)|u|_q^q + o(1) \leqslant c + o(1). \tag{3.3}$$

Defining $w_n = u_n - u$, we can use the Brezis–Lieb lemma [3] to obtain

$$\|w_n\|^p - \lambda|w_n|_q^q = \|u_n\|^p - \lambda|u_n|_q^q - \|u\|^p + \lambda|u|_q^q + o(1),$$

$$|w_n|_{p^*}^{p^*} = |u_n|_{p^*}^{p^*} - |u|_{p^*}^{p^*} + o(1).$$

Hence, recalling that $(u_n) \subset \mathcal{N}_\lambda^\tau$, $I_\lambda'(u)=0$, and $w_n \to 0$ in $L^q(\Omega)$, we get

$$\|w_n\|^p = b + o(1) \quad \text{and} \quad |w_n|_{p^*}^{p^*} = b + o(1) \tag{3.4}$$

for some $b \in \mathbb{R}$. Moreover, by (3.3),

$$b + o(1) = |w_n|_{p^*}^{p^*} = |u_n|_{p^*}^{p^*} - |u|_{p^*}^{p^*} + o(1) \leqslant Nc + o(1),$$

and therefore

$$b \leqslant Nc < 2S^{N/p}. \tag{3.5}$$

Since $(w_n) \subset W_0^{1,p}(\Omega)^\tau$, we know that $\|w_n\|^p = 2\|w_n^+\|^p$ and $|w_n|_{p^*}^{p^*} = 2|w_n^+|_{p^*}^{p^*}$. Thus

$$S|w_n^+|_{p^*}^p \leqslant \|w_n^+\|^p = b/2 + o(1).$$

Taking the limit we conclude that $S(b/2)^{p/p^*} \leqslant b/2$. We have now two possibilities: $b=0$ or $b \geqslant 2S^{N/p}$. The second case cannot occur by (3.5). Thus $b=0$ and we infer from (3.4) that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$. The lemma is proved. □

We are now ready to present the proof of our existence result.

Proof of Theorem 1.1. Let $(u_n) \subset \mathcal{N}_\lambda^\tau$ be a minimizing sequence for m_λ^τ. By Ekeland's variational principle [8], we may assume that it is a Palais–Smale sequence. In view of Lemma 2.3, we have that $m_\lambda^\tau < (2/N)S^{N/p}$ and therefore, by Lemma 3.2, we obtain a minimum u of I_λ on $\mathcal{N}_\lambda^\tau$. Now, Lemma 3.1 and the principle of symmetric criticality [16] (see also [15, Proposition 1]) imply that u (and also $-u$) is a solution of (D_λ^τ) which changes sign exactly once. □

Before presenting the proof of Theorem 1.2, we recall some facts about equivariant Ljusternik–Schnirelmann theory. An involution on a topological space X is a continuous function $\tau_X : X \to X$ such that τ_X^2 is the identity map of X. A subset $A \subset X$ is called τ_X-invariant if $\tau_X(A) = A$. If X and Y are topological spaces equipped with involutions τ_X and τ_Y, respectively, then an equivariant map is a continuous function $f : X \to Y$ such that $f \circ \tau_X = \tau_Y \circ f$. Two equivariant maps $f_0, f_1 : X \to Y$ are equivariantly homotopic if there is a homotopy $\Theta : X \times [0,1] \to Y$ such that $\Theta(x,0) = f_0(x)$, $\Theta(x,1) = f_1(x)$ and $\Theta(\tau_X(x),t) = \tau_Y(\Theta(x,t))$, for all $x \in X$, $t \in [0,1]$.

Definition 3.3. The equivariant category of an equivariant map $f : X \to Y$, denoted by (τ_X, τ_Y)-cat(f), is the smallest number k of open τ_X-invariant subsets $X_1, \ldots, X_k$ of X which cover X and which have the property that, for each $i=1,\ldots,k$, there is a point $y_i \in Y$ and a homotopy $\Theta_i : X_i \times [0,1] \to Y$ such that $\Theta_i(x,0) = f(x)$, $\Theta_i(x,1) \in \{y_i, \tau_Y(y_i)\}$ and $\Theta_i(\tau_X(x),t) = \tau_Y(\Theta_i(x,t))$ for every $x \in X_i$, $t \in [0,1]$. If no such covering exists we define (τ_X, τ_Y)-cat$(f) = \infty$.

If A is a τ_X-invariant subset of X and $\imath : A \hookrightarrow X$ is the inclusion map we write

$$\tau_X\text{-cat}_X(A) = (\tau_X, \tau_X)\text{-cat}(\imath) \quad \text{and} \quad \tau_X\text{-cat}(X) = \tau_X\text{-cat}_X(X).$$

The following properties can be verified.

Lemma 3.4. (i) *If $f : X \to Y$ and $h : Y \to Z$ are equivariant maps, then*

$$(\tau_X, \tau_Z)\text{-cat}(h \circ f) \leqslant \tau_Y\text{-cat}(Y).$$

(ii) *If $f_0, f_1 : X \to Y$ are equivariantly homotopic, then* (τ_X, τ_Y)-cat$(f_0) = (\tau_X, \tau_Y)$-cat(f_1).

We denote by $\tau_a : V \to V$ the antipodal involution $\tau_a(u) = -u$ on the vector space V. A τ_a-invariant subset of V is usually called a symmetric subset. Equivariant Ljusternik–Schnirelmann category provides a lower bound for the number of pairs $\{u, -u\}$ of critical points of an even functional, as stated in the following abstract result (see [7, Theorem 1.1], [20, Theorem 5.7]).

Theorem 3.5. *Let $M \subset V$ be a complete symmetric $C^{1,1}$-submanifold of some Banach space V. Suppose $I \in C^1(V, \mathbb{R})$ is even, bounded below on M and satisfies $(\mathrm{PS})_c$ for all $c \leqslant d$. Then the functional I has at least τ_a-cat$(\{u \in M : I(u) \leqslant d\})$ antipodal pairs $\{u, -u\}$ of critical points with $I(\pm u) \leqslant d$.*

Coming back to our problem we set, for any given $r > 0$,

$$\Omega_r^- = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega \cup \Omega^\tau) \geqslant r\}.$$

From now on we fix $r > 0$ small in such a way that the inclusion maps $\Omega_r^- \hookrightarrow \Omega \backslash \Omega^\tau$ and $\Omega \hookrightarrow \Omega_r^+$ are equivariant homotopy equivalences. In order to simplify the notation we denote by I_λ^d the set $\{u \in \mathcal{N}_\lambda^\tau : I_\lambda(u) \leqslant d\}$.

Lemma 3.6. *Let $\lambda_0 > 0$ be given by Lemma 2.6. Then, for any $\lambda \in (0, \lambda_0)$, we have*

$$\tau_a\text{-cat}(I_\lambda^{2m_{\lambda,r}}) \geqslant \tau\text{-cat}_\Omega(\Omega \backslash \Omega^\tau).$$

Proof. Let $\lambda \in (0, \lambda_0)$ be fixed. We claim that there exist two maps

$$\Omega_r^- \xrightarrow{\alpha_\lambda} I_\lambda^{2m_{\lambda,r}} \xrightarrow{\gamma_\lambda} \Omega_r^+$$

such that $\alpha_\lambda(\tau y) = -\alpha_\lambda(y)$, $\gamma_\lambda(-u) = \tau\gamma_\lambda(u)$, and $\gamma_\lambda \circ \alpha_\lambda$ is equivariantly homotopic to the inclusion map $\Omega_r^- \hookrightarrow \Omega_r^+$. If the claim is true, it follows from Lemma 3.4 and the choice of r that

$$\tau_a\text{-cat}(I_\lambda^{2m_{\lambda,r}}) \geqslant \tau\text{-cat}_{\Omega_r^+}(\Omega_r^-) = \tau\text{-cat}_\Omega(\Omega \backslash \Omega^\tau).$$

In order to prove the claim we take $v_\lambda \in \mathcal{N}_{\lambda, B_r(0)}$ a positive radial function such that $I_{\lambda, B_r(0)} = m_{\lambda,r}$ and define, for $y \in \Omega_r^-$,

$$\alpha_\lambda(y) = v_\lambda(\cdot - y) - v_\lambda(\cdot - \tau y).$$

It is clear that $\alpha_\lambda(\tau y) = -\alpha_\lambda(y)$. Furthermore, since v_λ is radial and τ is an isometry, we have that $\alpha_\lambda(y) \in W_0^{1,p}(\Omega)^\tau$. Note that, for every $y \in \Omega_r^-$, we have $|y - \tau y| \geqslant 2r$ (if this is not true, then $\overline{y} = (y + \tau y)/2$ satisfies $|y - \overline{y}| < r$ and $\tau\overline{y} = \overline{y}$, contradicting the definition of Ω_r^-). Thus, we can check that $\alpha_\lambda(y) \in \mathcal{N}_\lambda$ and $I_\lambda(\alpha_\lambda(y)) = 2m_{\lambda,r}$, and therefore $\alpha_\lambda(y) \in I_\lambda^{2m_{\lambda,r}}$.

If $u \in I_\lambda^{2m_{\lambda,r}}$, we can use (2.2) to conclude that $u^+ \in \mathcal{N}_\lambda$ and $I_\lambda(u^+) = I_\lambda(u)/2 \leqslant m_{\lambda,r}$. Hence, by Lemma 2.6, we conclude that $\gamma_\lambda(u) = \beta_\lambda(u^+) \in \Omega_r^+$. A simple calculation shows that $\gamma_\lambda(-u) = \tau\gamma_\lambda(u)$ and $\gamma_\lambda(\alpha_\lambda(y)) = y$, for every $y \in \Omega_r^-$. The lemma is proved. $\square$

We are now ready to present the proof of Theorem 1.2.

Proof of Theorem 1.2. Since m_λ and $m_{\lambda,r}$ have the same limit as $\lambda \to 0^+$, there exists $\widetilde{\lambda}_0 > 0$ such that

$$m_{\lambda,r} < 2m_\lambda \tag{3.6}$$

for all $0 < \lambda < \widetilde{\lambda}_0$. We will prove that the theorem holds for $\lambda_* = \min\{\lambda_0, \widetilde{\lambda}_0\}$, where λ_0 is given by Lemma 2.6.

Let $0<\lambda<\lambda_*$ be fixed. By Lemma 2.3 we have that $2m_{\lambda,r}<(2/N)S^{N/p}$ and therefore I_λ restricted to $\mathcal{N}_\lambda^\tau$ satisfies $(\mathrm{PS})_c$ for any $c\in[m_\lambda^\tau, 2m_{\lambda,r}]$. It follows from Theorem 3.5, Lemma 3.6 and the principle of symmetric criticality that I_λ has at least τ-$\mathrm{cat}_\Omega(\Omega\backslash\Omega^\tau)$ pairs $\pm u_i$ of critical points such that $I_\lambda(\pm u_i)\leqslant 2m_{\lambda,r}$. By using (3.6) and Lemma 2.3, we get

$$I_\lambda(\pm u_i)\leqslant 2m_{\lambda,r}<4m_\lambda\leqslant 2m_\lambda^\tau.$$

Hence, we conclude from Lemma 3.1 that the solutions $\pm u_i$ change sign exactly once. The theorem is proved. □

Proof of Corollary 1.3. Let $\tau:\mathbb{R}^N\to\mathbb{R}^N$ be given by $\tau(x)=-x$. It is proved in [5, Corollary 3] that our assumptions imply τ-$\mathrm{cat}(\Omega)\geqslant N$. Since $0\notin\Omega$, $\Omega^\tau=\emptyset$. It suffices now to apply Theorem 1.2. □

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