

**NODAL SOLUTIONS
FOR A NONHOMOGENEOUS ELLIPTIC
EQUATION WITH SYMMETRY**

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ABSTRACT. We consider the semilinear problem $-\Delta u + \lambda u = |u|^{p-2}u + f(u)$ in Ω , $u = 0$ on $\partial\Omega$ where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $2 < p < 2^* = 2N/(N-2)$ and $f(t)$ behaves like $t^{p-1-\varepsilon}$ at infinity. We show that if Ω is invariant by a nontrivial orthogonal involution then, for $\lambda > 0$ sufficiently large, the equivariant topology of Ω is related with the number of solutions which change sign exactly once. The results are proved by using equivariant Lusternik–Schnirelmann theory.

1. Introduction

Consider the problem

$$(1.1) \quad \begin{cases} -\Delta u + \lambda u = |u|^{p-2}u + f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\lambda \geq 0$, $2 < p < 2^* := 2N/(N-2)$ and the function $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies

$$(f_1) \quad \lim_{t \rightarrow \infty} f(t)/t^{p-1} = 0;$$

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(f₂) there exists $\gamma > 0$ such that

$$\frac{d}{dt} \left(\frac{f(t)}{t^{1+\gamma}} \right) \geq 0 \quad \text{for any } t > 0;$$

(f₃) $f(t) \geq 0$ for any $t > 0$.

We are interested in investigating the effect of the topology of Ω on the number of solutions of (1.1). The starting point of our study is the paper of Benci and Cerami [4], where the authors considered $f \equiv 0$ and proved that (1.1) possesses at least $\text{cat}(\Omega)$ positive solutions provided λ is large enough or p is close to 2^* . Here, $\text{cat}(\Omega)$ stands the usual Lusternik–Schnirelmann category of $\overline{\Omega}$ in itself. The result for λ large was extended for nonhomogeneous nonlinearities by the same authors in [5]. Since the work [4], multiplicity results for problems like (1.1) involving the category have been intensively studied (see [6], [7], [11] for subcritical, and [16], [14], [2], [1] for critical nonlinearities).

In the aforementioned works, the authors obtained positive solutions. Castro, Cossio and Neuberger considered in [10] a slightly different class of nonlinearities and proved that the problem possesses a solution which changes sign exactly once. This means that the solution u is such that $\Omega \setminus u^{-1}(0)$ has exactly two connected components, u is positive in one of them and negative in the other. In [3], Bartsch obtained infinite nodal solutions for (1.1). Motivated by these works and by a recent paper of Castro and Clapp [9], we are interested in relating the topology of Ω with the number of solutions which change sign exactly once.

We deal with the problem

$$(P_\lambda^\tau) \quad \begin{cases} -\Delta u + \lambda u = |u|^{p-2}u + f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(\tau x) = -u(x) & \text{for all } x \in \Omega, \end{cases}$$

where $\tau: \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a linear orthogonal transformation such that $\tau \neq \text{id}$, $\tau^2 = \text{id}$, and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain such that $\tau\Omega = \Omega$. Since we are looking for nodal solutions we suppose that f is odd, that is,

$$(f_4) \quad f(-t) = -f(t) \quad \text{for any } t \in \mathbb{R}.$$

Before to state our main results, we would like to quote the paper [8], where Cao, Li and Zhong proved that, under (f₁)–(f₄), the problem without symmetry (1.1) has at least $\text{cat}(\Omega)$ positive solutions. Quite recently, Furtado [13] considered the problem (P_λ^τ) for $f \equiv 0$ and proved that, if $\lambda \geq 0$ is fixed and p is sufficiently close to 2^* , then there exists an effect of the equivariant topology of Ω on the number of solutions which change sign exactly once. In view of this and the results of [4], [5], [8], it is natural to ask if the same kind of result holds for the nonhomogeneous symmetric problem (P_λ^τ) when p is fixed and λ

is large. In this paper we give an affirmative answer to this question by proving the following result.

THEOREM 1.1. *Suppose $p \in (2, 2^*)$ and f satisfies (f_1) – (f_4) . Then there exists $\bar{\lambda}(p)$ such that, for all $\lambda \geq \bar{\lambda}(p)$, the problem (P_λ^τ) has at least $\tau\text{-cat}_\Omega(\Omega \setminus \Omega^\tau)$ pairs of solutions which change sign exactly once.*

Here, $\Omega^\tau = \{x \in \Omega : \tau x = x\}$ and $\tau\text{-cat}$ is the G_τ -equivariant Lusternik–Schnirelmann category for the group $G_\tau = \{\text{id}, \tau\}$ (see Section 3). There are several situations where the equivariant category turns out to be larger than the nonequivariant one. The classical example is the unit sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ with $\tau = -\text{id}$. In this case $\text{cat}(\mathbb{S}^{N-1}) = 2$ whereas $\tau\text{-cat}(\mathbb{S}^{N-1}) = N$. Thus, as a consequence of Theorem 1.1 we have

COROLLARY 1.2. *Suppose $p \in (2, 2^*)$ and f satisfies (f_1) – (f_4) . Assume further that Ω is symmetric with respect to the origin, $0 \notin \Omega$ and there is an odd map $\varphi: \mathbb{S}^{N-1} \rightarrow \Omega$. Then there exists $\bar{\lambda}(p)$ such that, for all $\lambda \geq \bar{\lambda}(p)$, the problem (1.1) has at least N pairs of odd solutions which change sign exactly once.*

The above results complement those of [9] where the authors considered the critical semilinear problem

$$-\Delta u = \lambda u + |u|^{2^*-2}u, \quad u \in H_0^1(\Omega), \quad u(\tau x) = -u(x) \quad \text{in } \Omega,$$

and obtained the same results for $\lambda > 0$ small enough. They also complement the results of [8] since we obtain nodal solutions under the same hypothesis on f , as well the aforementioned works which deal only with positive solutions.

The paper is organized as follows. In Section 2 we present the abstract framework of the problem and some technical results. Section 3 is devoted to recalling some facts about equivariant Lusternik–Schnirelmann theory. The main results are proved in Section 4.

2. Functional setting and some technical results

Throughout this paper, we denote by H the Sobolev space $H_0^1(\Omega)$ endowed with the norm

$$\|u\| = \left\{ \int_\Omega |\nabla u|^2 dx \right\}^{1/2}$$

and by $|u|_s$ the $L^s(\Omega)$ -norm of a function $u \in L^s(\Omega)$. For simplicity of notation, we write only $\int_\Omega u$ instead of $\int_\Omega u(x) dx$.

We start by noting that the involution τ of Ω induces an involution of H , which we also denote by τ , in the following way: for each $u \in H$ we define $\tau u \in H$ by

$$(2.1) \quad (\tau u)(x) = -u(\tau x).$$

If we set $H^\tau = \{u \in H : \tau u = u\}$ as being the subspace of τ -invariant functions, it follows from the above expression that any function $u \in H^\tau$ satisfies the symmetry condition which appears in (P_λ^τ) .

It is well known that the nontrivial weak solutions of the problem (1.1) are precisely the nontrivial critical points of the C^2 -functional $E_\lambda: H \rightarrow \mathbb{R}$ given by

$$E_\lambda(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + \lambda u^2) - \frac{1}{p} \int_\Omega |u|^p - \int_\Omega F(u),$$

where $F(t) = \int_0^t f(s) ds$ is the primitive of f . All of them lie in the Nehari manifold of E_λ given by

$$\begin{aligned} \mathcal{N}_\lambda &= \{u \in H \setminus \{0\} : \langle E'_\lambda(u), u \rangle = 0\} \\ &= \left\{ u \in H \setminus \{0\} : \|u\|^2 + \lambda |u|_2^2 = |u|_p^p + \int_\Omega f(u)u \right\}. \end{aligned}$$

In order to obtain τ -invariant solutions, we will look for critical points of E_λ restricted to the τ -invariant Nehari manifold

$$\mathcal{N}_\lambda^\tau = \{u \in \mathcal{N}_\lambda : \tau u = u\} = \mathcal{N}_\lambda \cap H^\tau.$$

By using conditions $(f_2) - (f_4)$ we can check that

$$(2.2) \quad 0 \leq (2 + \gamma)F(t) \leq f(t)t,$$

for any $t \in \mathbb{R}$. Thus, if $u \in \mathcal{N}_\lambda$, we have

$$\begin{aligned} E_\lambda(u) &= \left(\frac{1}{2} - \frac{1}{p} \right) \int_\Omega |u|^p + \frac{1}{2} \int_\Omega f(u)u - \int_\Omega F(u) \\ &\geq \left(\frac{1}{2} - \frac{1}{p} \right) \int_\Omega |u|^p + \left(\frac{1}{2} - \frac{1}{2 + \gamma} \right) \int_\Omega f(u)u > 0, \end{aligned}$$

and therefore the following minimization problems are well defined

$$m_\lambda = \inf_{u \in \mathcal{N}_\lambda} E_\lambda(u) \quad \text{and} \quad m_\lambda^\tau = \inf_{u \in \mathcal{N}_\lambda^\tau} E_\lambda(u).$$

By using the symmetry of the problem (P_λ^τ) we can obtain the following relation between the two minimizers defined above.

LEMMA 2.1. *For any $\lambda \geq 0$, we have that $2m_\lambda \leq m_\lambda^\tau$.*

PROOF. Let $u \in \mathcal{N}_\lambda^\tau$ and set $u^\pm = \max\{\pm u, 0\}$. Since $u \in H^\tau$, we can use (2.1) to conclude that u is negative in $\tau(A)$ whenever u is positive in some subset $A \subset \Omega$. We claim that

$$(2.3) \quad \int_\Omega f(u^\pm)u^\pm = \frac{1}{2} \int_\Omega f(u)u, \quad \|u^\pm\|^2 = \frac{1}{2} \|u\|^2 \quad \text{and} \quad |u^\pm|_s = \frac{1}{2} |u|_s,$$

for any $2 \leq s < 2^*$. Indeed, if we set $\Omega^+ = \{x \in \Omega : u(x) > 0\}$, we can use (2.1) to verify that $\Omega^- = \{x \in \Omega : u(x) < 0\} = \tau(\Omega^+)$. Recalling that $u = u^+ - u^-$ and f is an odd function, we obtain

$$(2.4) \quad \int_{\Omega} f(u)u = \int_{\Omega^+} f(u^+)u^+ - \int_{\Omega^-} f(-u^-)u^- = \int_{\Omega} f(u^+)u^+ + \int_{\Omega} f(u^-)u^-.$$

Moreover, since $\tau = \tau^{-1}$, we can use a change of variables to conclude that

$$\begin{aligned} \int_{\Omega} f(u^+)u^+ &= \int_{\Omega^+} f(u(x))u(x) dx = \int_{\tau^{-1}(\Omega^+)} f(u(\tau y))u(\tau y) dy \\ &= \int_{\Omega^-} f(-u(y))(-u(y)) dy = \int_{\Omega} f(u^-)u^-. \end{aligned}$$

This and (2.4) imply the first equality in (2.3). The other ones can be proved in a similar way.

Since F is even and $F(0) = 0$, we can argue as above to conclude that

$$\int_{\Omega} F(u) = \int_{\Omega} F(u^+ - u^-) = \int_{\Omega} F(u^+) + \int_{\Omega} F(u^-).$$

Moreover, since $u \in \mathcal{N}_{\lambda}^{\tau}$, it follows from (2.3) that $u^{\pm} \in \mathcal{N}_{\lambda}$. Thus, we can use the above equation and (2.3) to get

$$E_{\lambda}(u) = E_{\lambda}(u^+) + E_{\lambda}(u^-) \geq 2m_{\lambda},$$

which concludes the proof of the lemma. \square

In what follows we denote by $\|E'_{\lambda}(u)\|_*$ the norm of the derivative of the restriction of E_{λ} to $\mathcal{N}_{\lambda}^{\tau}$ at u , which is defined by (see [18, Section 5.3])

$$\|E'_{\lambda}(u)\|_* = \min_{\theta \in \mathbb{R}} \|E'_{\lambda}(u) - \theta J'_{\lambda}(u)\|_{(H^{\tau})^*},$$

where $(H^{\tau})^*$ is the dual space of H^{τ} and $J_{\lambda}: H^{\tau} \rightarrow \mathbb{R}$ is given by

$$J_{\lambda}(u) = \|u\|^2 + \lambda|u|_2^2 - |u|_p^p - \int_{\Omega} f(u)u.$$

LEMMA 2.2. *If u is a critical point of E_{λ} restricted to $\mathcal{N}_{\lambda}^{\tau}$, then $E'_{\lambda}(u) = 0$ in the dual space of H .*

PROOF. By definition, there exists $\theta \in \mathbb{R}$ such that $\langle E'_{\lambda}(u) - \theta J'_{\lambda}(u), \phi \rangle = 0$, for all $\phi \in H^{\tau}$. Since $u \in \mathcal{N}_{\lambda}^{\tau}$, we can take $\phi = u$ to get $\theta \langle J'_{\lambda}(u), u \rangle = 0$.

By using (f₂)–(f₄) we can check that

$$f(t)t - f'(t)t^2 \leq -\gamma f(t)t \leq 0 \quad \text{for any } t \in \mathbb{R}.$$

This and the definition of J_λ imply that

$$\begin{aligned} \langle J'_\lambda(u), u \rangle &= 2\|u\|^2 + 2\lambda|u|_2^2 - p|u|_p^p - \int_\Omega \{f'(u)u^2 + f(u)u\} \\ &= (2-p)|u|_p^p + \int_\Omega \{f(u)u - f'(u)u^2\} < 0. \end{aligned}$$

Thus $\theta = 0$ and therefore $\langle E'_\lambda(u), \phi \rangle = 0$ for all $\phi \in H^\tau$. The result follows from the principle of symmetric criticality [15] (see also [18, Theorem 1.28]). \square

Let V be a Banach space, M be a C^1 -manifold of V and $I: V \rightarrow \mathbb{R}$ a C^1 -functional. We recall that I restricted to M satisfies de Palais–Smale condition at level c ((PS) $_c$ for short) if any sequence $(u_n) \subset M$ such that $I(u_n) \rightarrow c$ and $\|I'(u_n)\|_* \rightarrow 0$ contains a convergent subsequence. We end this section by stating the compactness property satisfied by E_λ .

LEMMA 2.3. *The functional E_λ restricted to \mathcal{N}_λ^τ satisfies the Palais–Smale condition at any level $c \in \mathbb{R}$.*

PROOF. Since we are dealing with a subcritical nonlinearity, the proof follows from the boundedness of Ω , the Ambrosetti–Rabinowitz condition in (2.2) and standard arguments (see [5]). We omit the details. \square

3. Equivariant Lusternik–Schnirelmann theory

We recall in this section some facts about equivariant Lusternik–Schnirelmann theory. An involution on a topological space X is a continuous function $\tau_X: X \rightarrow X$ such that τ_X^2 is the identity map of X . A subset A of X is called τ_X -invariant if $\tau_X(A) = A$. If X and Y are topological spaces equipped with involutions τ_X and τ_Y , respectively, then an equivariant map is a continuous function $f: X \rightarrow Y$ such that $f \circ \tau_X = \tau_Y \circ f$. Two equivariant maps $f_0, f_1: X \rightarrow Y$ are equivariantly homotopic if there is an homotopy $\Theta: X \times [0, 1] \rightarrow Y$ such that $\Theta(x, 0) = f_0(x)$, $\Theta(x, 1) = f_1(x)$ and $\Theta(\tau_X(x), t) = \tau_Y(\Theta(x, t))$, for all $x \in X, t \in [0, 1]$.

DEFINITION 3.1. The equivariant category of an equivariant map $f: X \rightarrow Y$, denoted by (τ_X, τ_Y) -cat(f), is the smallest number k of open invariant subsets X_1, \dots, X_k of X which cover X and which have the property that, for each $i = 1, \dots, k$, there is a point $y_i \in Y$ and a homotopy $\Theta_i: X_i \times [0, 1] \rightarrow Y$ such that $\Theta_i(x, 0) = f(x)$, $\Theta_i(x, 1) \in \{y_i, \tau_Y(y_i)\}$ and $\Theta_i(\tau_X(x), t) = \tau_Y(\Theta_i(x, t))$ for every $x \in X_i, t \in [0, 1]$. If no such covering exists we define (τ_X, τ_Y) -cat(f) = ∞ .

If A is a τ_X -invariant subset of X and $\iota: A \hookrightarrow X$ is the inclusion map we write

$$\tau_X\text{-cat}_X(A) = (\tau_X, \tau_X)\text{-cat}(\iota) \quad \text{and} \quad \tau_X\text{-cat}(X) = \tau_X\text{-cat}_X(X).$$

The following properties can be verified.

LEMMA 3.2.

(a) If $f: X \rightarrow Y$ and $h: Y \rightarrow Z$ are equivariant maps then

$$(\tau_X, \tau_Z)\text{-cat}(h \circ f) \leq \tau_Y\text{-cat}(Y).$$

(b) If $f_0, f_1: X \rightarrow Y$ are equivariantly homotopic then

$$(\tau_X, \tau_Y)\text{-cat}(f_0) = (\tau_X, \tau_Y)\text{-cat}(f_1).$$

Let $\tau_a: V \rightarrow V$ be the antipodal involution $\tau_a(u) = -u$ on the vector space V . Equivariant Lusternik–Schnirelmann category provides a lower bound for the number of pairs $\{u, -u\}$ of critical points of an even functional, as stated in the following abstract result (see [12, Theorem 1.1], [17, Theorem 5.7]).

THEOREM 3.3. *Let $I: M \rightarrow \mathbb{R}$ be an even C^1 -functional on a complete symmetric $C^{1,1}$ -submanifold M of some Banach space V . Assume that I is bounded below and satisfies $(PS)_c$ for all $c \leq d$. Then, if $I^d = \{u \in M : I(u) \leq d\}$, the functional I has at least $\tau_a\text{-cat}_{I^d}(I^d)$ antipodal pairs $\{u, -u\}$ of critical points with $I(\pm u) \leq d$.*

4. Proofs of the main results

By standard regularity theory we know that if u is a solution of (P_λ^τ) , then it is of class C^1 . We say it changes sign k times if the set $\{x \in \Omega : u(x) \neq 0\}$ has $k + 1$ connected components. By (2.1), if u is a nontrivial solution of problem (P_λ^τ) then it changes sign an odd number of times. More specifically, we have the following relation between the number of nodal regions of a solution and its energy.

LEMMA 4.1. *If u is a solution of problem (P_λ^τ) which changes sign $2k - 1$ times, then $E_\lambda(u) \geq km_\lambda^\tau$. In particular, if u is a nontrivial solution of (P_λ^τ) such that $E_\lambda(u) < 2m_\lambda^\tau$, then u changes sign exactly once.*

PROOF. The set $\{x \in \Omega : u(x) > 0\}$ has k connected components A_1, \dots, A_k . Let $u_i(x) = u(x)$ if $x \in A_i \cup \tau A_i$ and $u_i(x) = 0$, otherwise. We have that

$$\begin{aligned} 0 &= \langle E'_\lambda(u), u_i \rangle = \int_\Omega (\nabla u \cdot \nabla u_i + \lambda u u_i - |u|^{p-2} u u_i - f(u) u_i) \\ &= \|u_i\|^2 + \lambda |u_i|_2^2 - |u_i|_p^p - \int_\Omega f(u_i) u_i. \end{aligned}$$

Thus, $u_i \in \mathcal{N}_\lambda^\tau$ for all $i = 1, \dots, k$, and $E_\lambda(u) = E_\lambda(u_1) + \dots + E_\lambda(u_k) \geq km_\lambda^\tau$, as desired. \square

Given $r > 0$, we define the sets

$$\Omega_r^+ = \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) < r\} \quad \text{and} \quad \Omega_r^- = \{x \in \Omega : \text{dist}(x, \partial\Omega \cup \Omega^\tau) \geq r\}.$$

From now on we fix $r > 0$ sufficiently small in such way that the inclusion maps $\Omega_r^- \hookrightarrow \Omega \setminus \Omega^\tau$ and $\Omega \hookrightarrow \Omega_r^+$ are equivariant homotopy equivalences. We also define the barycenter map $\beta : H \setminus \{0\} \rightarrow \mathbb{R}^N$ by setting

$$\beta(u) = \frac{\int_{\Omega} x \cdot |\nabla u(x)|^2 dx}{\int_{\Omega} |\nabla u(x)|^2 dx}.$$

Let $E_{\lambda,r} : H_0^1(B_r(0)) \rightarrow \mathbb{R}$ be defined as

$$E_{\lambda,r}(u) = \frac{1}{2} \int_{B_r(0)} (|\nabla u|^2 + \lambda u^2) - \frac{1}{p} \int_{B_r(0)} |u|^p - \int_{B_r(0)} F(u),$$

and set

$$m_{\lambda,r} = \inf_{u \in \mathcal{N}_{\lambda,r}} E_{\lambda,r}(u),$$

where $\mathcal{N}_{\lambda,r}$ stands the Nehari manifold of $E_{\lambda,r}$. The following lemma is an important tool for the proof of Theorem 1.1.

LEMMA 4.2. *For any fixed $p \in (2, 2^*)$ there exists $\bar{\lambda}(p)$ such that, for any $\lambda \geq \bar{\lambda}(p)$, there hold*

- (a) $m_{\lambda,r} < 2m_{\lambda}$,
- (b) if $u \in \mathcal{N}_{\lambda}$ and $E_{\lambda}(u) \leq m_{\lambda,r}$, then $\beta(u) \in \Omega_r^+$.

PROOF. See [8, Corollary 3.20 and Lemma 3.24]. \square

For any given $d > 0$ we set $E_{\lambda}^d = \{u \in \mathcal{N}_{\lambda}^{\tau} : E_{\lambda}(u) \leq d\}$. By using the second statement of the above lemma we are able to prove the following result.

LEMMA 4.3. *For any fixed $p \in (2, 2^*)$, let $\bar{\lambda}(p)$ be given by Lemma 4.2. Then*

$$\tau_a\text{-cat}(E_{\lambda}^{2m_{\lambda,r}}) \geq \tau\text{-cat}_{\Omega}(\Omega \setminus \Omega^{\tau}), \quad \text{for any } \lambda \geq \bar{\lambda}(p).$$

PROOF. We claim that, for any $\lambda \geq \bar{\lambda}(p)$ fixed, there exist two maps

$$\Omega_r^- \xrightarrow{\alpha_{\lambda}} E_{\lambda}^{2m_{\lambda,r}} \xrightarrow{\gamma_{\lambda}} \Omega_r^+$$

such that $\alpha_{\lambda}(\tau x) = -\alpha_{\lambda}(x)$, $\gamma_{\lambda}(-u) = \tau\gamma_{\lambda}(u)$, and $\gamma_{\lambda} \circ \alpha_{\lambda}$ is equivariantly homotopic to the inclusion map $\Omega_r^- \hookrightarrow \Omega_r^+$. Assuming the claim and recalling that the maps $\Omega_r^- \hookrightarrow \Omega \setminus \Omega^{\tau}$ and $\Omega \hookrightarrow \Omega_r^+$ are equivariant homotopy equivalences, we can use Lemma 3.2 to get

$$\tau_a\text{-cat}(E_{\lambda}^{2m_{\lambda,r}}) \geq \tau\text{-cat}_{\Omega_r^+}(\Omega_r^-) = \tau\text{-cat}_{\Omega}(\Omega \setminus \Omega^{\tau}).$$

In order to prove the claim we follow [9]. Let $v_{\lambda} \in \mathcal{N}_{\lambda,r}$ be a positive radial function such that $E_{\lambda,r}(v_{\lambda}) = m_{\lambda,r}$. We define $\alpha_{\lambda} : \Omega_r^- \rightarrow E_{\lambda}^{2m_{\lambda,r}}$ by

$$(4.1) \quad \alpha_{\lambda}(x) = v_{\lambda}(\cdot - x) - v_{\lambda}(\cdot - \tau x).$$

It is clear that $\alpha_{\lambda}(\tau x) = -\alpha_{\lambda}(x)$. Furthermore, since v_{λ} is radial and τ is an isometry, we have that $\alpha_{\lambda}(x) \in H^{\tau}$. The definition of Ω_r^- implies that

$|x - \tau x| \geq 2r$ for any $x \in \Omega_r^-$. Thus, we can check that $E_\lambda(\alpha_\lambda(x)) = 2m_{\lambda,r}$ and $\alpha_\lambda(x) \in \mathcal{N}_\lambda^\tau$. All this considerations show that α_λ is well defined.

Given $u \in E_\lambda^{2m_{\lambda,r}}$ we can proceed as in the proof of Lemma 2.1 to conclude that $u^+ \in \mathcal{N}_\lambda$ and $2E_\lambda(u^+) = E_\lambda(u) \leq 2m_{\lambda,r}$. It follows from Lemma 4.2(b) that $\gamma_\lambda: E_\lambda^{2m_{\lambda,r}} \rightarrow \Omega_r^+$ given by $\gamma_\lambda(u) = \beta(u^+)$ is well defined. A simple calculation shows that $\gamma_\lambda(-u) = \tau\gamma_\lambda(u)$. Moreover, using (4.1) and the fact that v_λ is radial we get

$$\gamma_\lambda(\alpha_\lambda(x)) = \frac{\int_{B_r(x)} y \cdot |\nabla v_\lambda(y-x)|^2 dy}{\int_{B_r(x)} |\nabla v_\lambda(y-x)|^2 dy} = \frac{\int_{B_r(0)} (y+x) \cdot |\nabla v_\lambda(y)|^2 dy}{\int_{B_r(0)} |\nabla v_\lambda(y)|^2 dy} = x,$$

for any $x \in \Omega_r^-$. This concludes the proof. \square

We are now ready to present the proof of our main results.

PROOF OF THEOREM 1.1. Let $p \in (2, 2^*)$ and $\bar{\lambda}(p)$ be given by the Lemma 4.2. For any $\lambda \geq \bar{\lambda}(p)$, we can apply Theorem 3.3 and Lemma 4.3 to obtain τ -cat $_\Omega(\Omega \setminus \Omega^\tau)$ pairs $\pm u_i$ of critical points of the even functional E_λ restricted to \mathcal{N}_λ^τ verifying

$$E_\lambda(\pm u_i) \leq 2m_{\lambda,r} < 4m_\lambda \leq 2m_\lambda^\tau,$$

where we have used Lemma 4.2(a) and Lemma 2.1. It follows from Lemmas 2.2 and 4.1 that $\pm u_i$ are solutions of (P_λ^τ) which change sign exactly once. \square

PROOF OF COROLLARY 1.2. Let $\tau: \mathbb{R}^N \rightarrow \mathbb{R}^N$ be given by $\tau(x) = -x$. It is proved in [9, Corollary 3] that our assumptions imply τ -cat $(\Omega) \geq N$. Since $0 \notin \Omega$, $\Omega^\tau = \emptyset$. It suffices now to apply Theorem 1.1. \square

REFERENCES

- [1] C. O. ALVES, *Existence and multiplicity of solution for a class of quasilinear equations*, Adv. Nonlinear Studies **5** (2005), 73–87.
- [2] C. O. ALVES AND Y. H. DING, *Multiplicity of positive solutions to a p -Laplacian equation involving critical nonlinearity*, J. Math. Anal. Appl. **279** (2003), 508–521.
- [3] T. BARTSCH, *Critical point theory in partially ordered Hilbert spaces*, J. Funct. Anal. **186** (2001), 117–152.
- [4] V. BENCI AND G. CERAMI, *The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems*, Arch. Rational Mech. Anal. **114** (1991), 79–93.
- [5] ———, *Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology*, Calc. Var. Partial Differential Equations **2** (1994), 29–48.
- [6] V. BENCI, G. CERAMI AND D. PASSASEO, *On the number of positive solutions of some nonlinear elliptic problems*, Nonlinear Analysis. A tribute in honour of G. Prodi, Quaderno Scuola Norm. Sup., Pisa, 1991, pp. 93–107.
- [7] A. M. CANDELA, *Remarks on the number of positive solutions for a class of nonlinear elliptic problems*, Differential Integral Equations **5** (1992), 553–560.
- [8] D. CAO, G. LI AND X. ZHONG, *A note on the number of positive solutions of some nonlinear elliptic problems*, Nonlinear Anal. **27** (1996), 1095–1108.

- [9] A. CASTRO AND M. CLAPP, *The effect of the domain topology on the number of minimal nodal solutions of an elliptic equation at critical growth in a symmetric domain*, *Nonlinearity* **16** (2003), 579–590.
- [10] A. CASTRO, J. COSSIO AND M. NEUBERGER, *A sign-changing solution for a superlinear Dirichlet problem*, *Rocky Mountain J. Math.* **27** (1997), 1041–1053.
- [11] M. CLAPP, *On the number of positive symmetric solutions of a nonautonomous semilinear elliptic problem*, *Nonlinear Anal.* **42** (2000), 405–422.
- [12] M. CLAPP AND D. PUPPE, *Critical point theory with symmetries*, *J. Reine Angew. Math.* **418** (1991), 1–29.
- [13] M. F. FURTADO, *A relation between the domain topology and the number of minimal nodal solutions for a quasilinear elliptic problem*, *Nonlinear Anal.* **62** (2005), 615–628.
- [14] M. LAZZO, *Solutions positives multiples pour une équation elliptique non linéaire avec l'exposant critique de Sobolev*, *C. R. Acad. Sci. Paris Sér.* **314** (1992), 61–64.
- [15] R. S. PALAIS, *The principle of symmetric criticality*, *Comm. Math. Phys.* **69** (1979), 19–30.
- [16] O. REY, *A multiplicity result for a variational problem with lack of compactness*, *Nonlinear Anal.* **13** (1989), 1241–1249.
- [17] M. STRUWE, *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer, Berlin, 1990.
- [18] M. WILLEM, *Minimax Theorems*, Birkhäuser, Boston, 1996.

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