# Multiplicity of solutions for homogeneous elliptic systems with critical growth

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**Abstract**

In this paper we are concerned with the number of nonnegative solutions of the elliptic system

$$(P) \qquad \begin{cases} -\Delta u = Q_u(u,v) + \frac{1}{2^*}H_u(u,v), & \text{in } \Omega, \\ -\Delta v = Q_v(u,v) + \frac{1}{2^*}H_v(u,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 4$, $2^* := 2N/(N-2)$ and $Q_u, H_u$ and $Q_v$, $H_v$ are the partial derivatives of the homogeneous functions $Q$, $H \in C^1(\mathbb{R}^2_+, \mathbb{R})$, where $\mathbb{R}^2_+ := [0,\infty) \times [0,\infty)$. In the proofs we apply variational methods and Ljusternik-Schnirelmann theory.

*Key words:* Elliptic systems; critical problems; category of Ljusternik-Schnirelmann; homogeneous problems.

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## 1 Introduction

In this paper we are concerned with the number of nonnegative solutions of the elliptic system

$$(P) \qquad \begin{cases} -\Delta u = Q_u(u,v) + \frac{1}{2^*} H_u(u,v), & \text{in } \Omega, \\ -\Delta v = Q_v(u,v) + \frac{1}{2^*} H_v(u,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 4$, $2^* := 2N/(N-2)$ and $Q_u, H_u$ and $Q_v$, $H_v$ are the partial derivatives of the homogeneous functions $Q$, $H \in C^1(\mathbb{R}^2_+, \mathbb{R})$, where $\mathbb{R}^2_+ := [0,\infty) \times [0,\infty)$.

We are interested in the case that $H$ has critical growth. More specifically, the assumptions on $H = H(s,t)$ are the following.

$(H_0)$ $H$ is $2^*$-homogeneous, that is,

$$H(\theta s, \theta t) = \theta^{2^*} H(s,t) \quad \text{for each } \theta > 0,\ (s,t) \in \mathbb{R}^2_+;$$

$(H_1)$ $H_s(0,1) = 0$, $H_t(1,0) = 0$;
$(H_2)$ $H(s,t) > 0$ for each $s, t > 0$;
$(H_3)$ $H_s(s,t) \geq 0$, $H_t(s,t) \geq 0$ for each $(s,t) \in \mathbb{R}^2_+$;
$(H_4)$ the 1-homogeneous function $(s,t) \mapsto H(s^{1/2^*}, t^{1/2^*})$ is concave in $\mathbb{R}^2_+$;

The function $Q = Q(s,t)$ is a lower order perturbation term satisfying

$(Q_0)$ $Q$ is $q$-homogeneous for some $2 \leq q < 2^*$;
$(Q_1)$ $Q_s(0,1) = 0$, $Q_t(1,0) = 0$.

In order to present our results we introduce the following numbers

$$\mu := \min \{Q(s,t) \,:\, s^q + t^q = 1,\ s,\ t \geq 0\} \tag{1.1}$$

and

$$\lambda := \max \{Q(s,t) \,:\, s^q + t^q = 1,\ s,\ t \geq 0\}. \tag{1.2}$$

We say that a weak solution $z = (u,v) \in H^1_0(\Omega) \times H^1_0(\Omega)$ of problem $(P)$ is nonnegative if $u$, $v \geq 0$ in $\Omega$. If $Y$ is a closed set of a topological space $Z$, we denote by $\operatorname{cat}_Z(Y)$ the Ljusternik-Schnirelmann category of $Y$ in $Z$, namely the least number of closed and contractible sets in $Z$ which cover $Y$. We are now ready to state our first result.

**Theorem 1.1** *Suppose that $H$ satisfies $(H_0)-(H_4)$ and $Q$ satisfies $(Q_0)-(Q_1)$. Then there exists $\Lambda>0$ such that the problem $(P)$ has at least $\operatorname{cat}_\Omega(\Omega)$ nonzero nonnegative solutions provided $\lambda$, $\mu\in(0,\Lambda)$.*

In the proof we apply variational methods, Ljusternik-Schnirelmann theory and a technique introduced by Benci and Cerami [3]. It consists in making precise comparisons between the category of some sublevel sets of the associated functional with the category of the set $\Omega$. In order to overcame the lack of compactness due to the critical growth of $H$ we use the ideas of Brezis and Nirenberg [4], besides the paper of Morais Filho and Souto [15], where it is proved that the number

$$S_H := \inf\left\{\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)\mathrm{d}x \,:\, u,v\in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} H(u^+,v^+)\mathrm{d}x=1\right\} \tag{1.3}$$

plays an important role when dealing with critical systems like $(P)$. Actually, we use the above constant and adapt some calculations performed in Myiagaki [16] to localize the energy levels where the Palais-Smale condition fails. We would like to mention that, as a byproduct of our arguments, we extend the existence result of [15] for any subcritical degree of homogeneity of the perturbation $Q$ (see Theorems 2.4 and 2.5).

Notice that condition $(Q_1)$ discard examples like $Q(s,t)=s^q+t^q+st^{q-1}$ since, in this case, $Q_s(0,1)=1$. However, we can also consider this situation if the subcritical perturbation satisfies $q>2$. More specifically, the following holds

**Theorem 1.2** *Suppose that $H$ satisfies $(H_0)-(H_4)$, $Q$ satisfies $(Q_0)$ with $q>2$ and*

$(\widehat{Q_1})$ $Q_s(0,1)>0$ *and* $Q_t(1,0)>0$.

*If we set*

$$\widehat{\lambda} := \max\{Q_s(0,1),\ Q_t(1,0)\} \tag{1.4}$$

*then there exists $\Lambda>0$ such that the problem $(P)$ has at least $\operatorname{cat}_\Omega(\Omega)$ nonzero nonnegative solutions provided $\lambda$, $\mu$, $\widehat{\lambda}\in(0,\Lambda)$.*

The difference when dealing with $(Q_1)$ or $(\widehat{Q_1})$ is just in the way we extend the function $Q$ to the whole $\mathbb{R}^2$. Since we want to apply minimax methods this extension needs to be made in a smooth way. We refer to the beginning of the next section for more details about the possible extensions.

Concerning the class of nonlinearities we are considering, we present in Section 5 some examples of functions satisfying our hypothesis. There, we also make some comments about the possibility of proving that the solutions are positive in $\Omega$ and we state other settings in which our results hold, including the possibility of having a sum of subcritical terms with different degrees of homogeneity. As a final remark, we would like to mention that the theorems remain valid for $N = 3$ if the degree of homogeneity of $Q$ satisfies $4 < q < 6$ (see Remark 4.5).

The starting point on the study of the system $(P)$ is its scalar version

$$-\Delta u = \theta |u|^{q-2}u + |u|^{2^*-2}u \ \text{ in } \Omega, \ \ u \in H_0^1(\Omega), \tag{1.5}$$

with $2 \leq q < 2^*$. In a pioneer work Brezis and Nirenberg [4] showed that, for $q = 2$, the existence of positive solutions is related with the interaction between the parameter $\theta$ with the first eigenvalue $\theta_1(\Omega)$ of the operator $(-\Delta, H_0^1(\Omega))$. Among other results they showed that, if $q = 2$, the problem has at least one positive solution provided $N \geq 4$ and $0 < \theta < \theta_1(\Omega)$. They also obtained some results for the case $2 < q < 2^*$.

After the paper of Brezis and Nirenberg, a lot of works dealing with critical nonlinearities have been appeared. Concerning the question of multiplicity, we recall that Rey [17] and Lazzo [13] proved that, for $q = 2$, the problem (1.5) has at least $\text{cat}_\Omega(\Omega)$ positive solutions (see also the well known paper of Benci and Cerami [3] where the subcritical case was considered) provided $\theta > 0$ is small. This result was extended for the $p$-Laplacian operator and $p \leq q < p^*$ by Alves and Ding [1]. The results presented here can be viewed as versions of the papers [17,13,1] for the case of systems.

As far we know, the first results for homogeneous system like $(P)$ are due to Morais Filho and Souto [15] (see also [2]). After this work many results have been appeared (see [7–9,18,12,10,11] and references therein). Among then, the most related with our paper if the work of Han [9], where the author considered the case $Q(s,t) = \alpha_1 s^2 + \alpha_2 t^2$ and $H(s,t) = s^\alpha t^\beta$ with $\alpha + \beta = 2^*$. His results was complemented by Ishiwata in [11,12], with different classes of homogeneous nonlinearities being considered. Our paper extends and/or complements the results found in [15,2,9,11,12]. Although there are some multiplicity results for systems like $(P)$ via Ljusternik-Schnirelmann theory, we do not know any article that relates the topology of $\Omega$ with the number of solutions and contains a general class of nonlinearities such as those considered here.

The paper is organized as follows. In Section 2 we present the abstract framework of the problem, we prove a local compactness result and obtain the existence of one nonnegative solution for $(P)$. Section 3 is devoted to the proof of some technical results concerned the properties of sequences which minimize $S_H$ and the asymptotic behavior of the minimax levels associated to the problem. Theorems 1.1 and 1.2 are proved in Section 4 and we devote the last section for some further remarks about examples and possible extensions of the results.

## 2 The PS condition and an existence result

We start this section fixing some notation. We denote $B_R(0) := \{x \in \mathbb{R}^N : \|x\| < R\}$ and by $C_0^\infty(A)$ the set of all functions $f : A \to \mathbb{R}$ of class $C^\infty$ with compact support contained in the open set $A \subset \mathbb{R}^N$. We denote by $\|f\|_p$ the $L^p$-norm of $f \in L^p(A)$. In order to simplify the notation, we write $\int_A f$ instead of $\int_A f(x)\mathrm{d}x$. We also omit the set $A$ whenever $A = \Omega$.

We remark for future reference that, if $p \geq 1$ and $F$ is a $p$-homogeneous $C^1$-function, then the following holds

(i) if we set $M_F := \max\{F(s,t) : s,\, t \in \mathbb{R},\, |s|^p + |t|^p = 1\}$ then, for each $(s,t) \in \mathbb{R}^2$, we have that

$$|F(s,t)| \leq M_F(|s|^p + |t|^p)\,; \tag{2.1}$$

(ii) $\nabla F$ is a $(p-1)$-homogeneous function and, for each $(s,t) \in \mathbb{R}^2$, we have that

$$sF_s(s,t) + tF_t(s,t) = pF(s,t). \tag{2.2}$$

Throughout the paper we suppose that $H$ satisfies $(H_0) - (H_4)$. In view of $(H_1)$, we can extend the function $H$ to the whole $\mathbb{R}^2$ by considering

$$\widetilde{H}(s,t) := H(s^+, t^+), \tag{2.3}$$

where $s^+ := \max\{s, 0\}$. It is easy to check that $\widetilde{H}$ is of class $C^1$ and its restriction to $[0,\infty)\times[0,\infty)$ coincides with $H$. In order to simplify the notating we shall write, from now on, only $H$ to denote the above extension.

The extension of the function $Q$ is more delicate. We first consider the case that $(Q_1)$ is assumed. In this setting we can extend as above, that is,

$$\widetilde{Q}(s,t) := Q(s^+, t^+). \tag{2.4}$$

However, if we suppose that $Q$ satisfies $(\widehat{Q_1})$ instead of $(Q_1)$, it can be proved that the above extension is not differentiable. Thus, with this other condition we extend $Q$ in the following way

$$\widetilde{Q}(s,t) := Q(s^+, t^+) - \nabla Q(s^+, t^+) \cdot (s^-, t^-), \tag{2.5}$$

where $s^- = \max\{-s, 0\}$. We can check that this extension is of classe $C^1$.

**Remark 2.1** *Since $\nabla Q$ is $(q-1)$-homogeneous we can use (2.4) to get*

$$-s^- Q_s(s^+, t^+) = \begin{cases} -sQ_s(0, t^+) = -s(t^+)^{q-1}Q_s(0,1), & \text{if } s < 0, \\ 0, & \text{if } s \geq 0, \end{cases}$$

*for each* $(s,t) \in \mathbb{R}^2$*. Hence*

$$|-s^-Q_s(s^+,t^+)| \le Q_s(0,1)(|s|^q + |t|^q).$$

*Analogously,*

$$|-t^-Q_t(s^+,t^+)| \le Q_t(1,0)(|s|^q + |t|^q)$$

*and therefore it follows from (2.4) that the extension* $\widetilde{Q}$ *satisfies*

$$\begin{aligned} |\widetilde{Q}(s,t)| &\le |Q(s^+,t^+)| + |(s^-,t^-)\cdot\nabla Q(s^+,t^+)| \\ &\le (\lambda + \widehat{\lambda})(|s|^q + |t|^q), \end{aligned} \tag{2.6}$$

*for each* $(s,t) \in \mathbb{R}^2$*, whenever* $Q$ *satisfies* $(\widehat{Q_1})$*.*

As before, we shall write only $Q$ to denote the $C^1$-extension $\widetilde{Q}$.

By using (2.1) and well know arguments, we see that the weak solutions of $(P)$ are precisely the critical points of the $C^1$-functional $I_{\lambda,\mu} : X \to \mathbb{R}$ given by

$$I_{\lambda,\mu}(z) := \frac{1}{2}\|z\|^2 - \int Q_{\lambda,\mu}(z) - \frac{1}{2^*}\int H(z), \ \ z \in X,$$

where $X$ is the Sobolev space $H_0^1(\Omega) \times H_0^1(\Omega)$ endowed with the norm

$$\|(u,v)\|^2 := \int \left(|\nabla u|^2 + |\nabla v|^2\right).$$

We notice that, in the definition of $I_{\lambda,\mu}$, we are denoting $Q_{\lambda,\mu}(z) := Q(z)$ for $z \in \mathbb{R}^2$. We shall write $Q_{\lambda,\mu}$ instead of $Q$ just to emphasize that the smallness condition in the statement of the main theorems depends on the value of the parameters $\mu$ and $\lambda$ defined in (1.1)-(1.2).

We introduce the Nehari manifold of $I_{\lambda,\mu}$ by setting

$$\mathcal{N}_{\lambda,\mu} := \left\{z \in X \setminus \{(0,0)\} : I'_{\lambda,\mu}(z)z = 0\right\}$$

and define the minimax $c_{\lambda,\mu}$ as

$$c_{\lambda,\mu} := \inf_{z\in\mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(z).$$

In what follows, we present some properties of $c_{\lambda,\mu}$ and $\mathcal{N}_{\lambda,\mu}$. Its proofs can be done as in [19, Chapter 4]. First of all, we note that there exists $r = r_{\lambda,\mu} > 0$, such that

$$\|z\| \ge r > 0 \quad \text{for each } z \in \mathcal{N}_{\lambda,\mu}. \tag{2.7}$$

It is standard to check that $I_{\lambda,\mu}$ satisfies Mountain Pass geometry. So, we can use the homogeneity of $Q$ and $H$ to prove that $c_{\lambda,\mu}$ can be alternatively characterized by

$$c_{\lambda,\mu} = \inf_{\gamma\in\Gamma_{\lambda,\mu}} \max_{t\in[0,1]} I_{\lambda,\mu}(\gamma(t)) = \inf_{z\in X\setminus\{0\}} \max_{t\geq 0} I_{\lambda,\mu}(tz) > 0, \tag{2.8}$$

where $\Gamma_{\lambda,\mu} := \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ I_{\lambda,\mu}(\gamma(1)) < 0\}$. Moreover, for each $z \in X \setminus \{0\}$, there exists a unique $t_z > 0$ such that $t_z z \in \mathcal{N}_{\lambda,\mu}$. The maximum of the function $t \mapsto I_{\lambda,\mu}(tz)$, for $t \geq 0$, is achieved at $t = t_z$.

Let $E$ be a Banach space and $J \in C^1(E, \mathbb{R})$. We say that $(z_n) \subset E$ is a Palais-Smale sequence at level $c$ ($(\mathrm{PS})_c$ sequence for short) if $J(z_n) \to c$ and $J'(z_n) \to 0$. We say that $J$ satisfies $(\mathrm{PS})_c$ if any $(\mathrm{PS})_c$ sequence possesses a convergent subsequence.

**Lemma 2.2** *If $Q$ satisfies $(Q_0)$ then the functional $I_{\lambda,\mu}$ satisfies the $(PS)_c$ condition for all $c < \frac{1}{N}S_H^{N/2}$, provide one of the conditions below holds*

*(i)* $2 < q < 2^*$ *and $Q$ satisfies $(Q_1)$ or $(\widehat{Q_1})$;*

*(ii)* $q = 2$*, $Q$ satisfies $(Q_1)$ and the parameter $\lambda$ defined in (1.2) belongs to $(0, \theta_1(\Omega)/2)$, where $\theta_1(\Omega) > 0$ denotes the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.*

*Proof.* Let $(z_n) = ((u_n, v_n)) \subset X$ be such that $I'_{\lambda,\mu}(z_n) \to 0$ and $I_{\lambda,\mu}(z_n) \to c < \frac{1}{N}S_H^{N/2}$. We start by proving that $(z_n)$ is bounded in $X$. If the item (i) above is true it suffices to use the definition of $I_{\lambda,\mu}$ to obtain $c_1 > 0$ such that

$$\begin{aligned} c + c_1\|z_n\| + o_n(1) &\geq I_{\lambda,\mu}(z_n) - \frac{1}{q}I'_{\lambda,\mu}(z_n)z_n \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\|z_n\|^2 + \left(\frac{2^* - q}{2^* q}\right)\int H(z_n) \\ &\geq \left(\frac{q-2}{2q}\right)\|z_n\|^2, \end{aligned}$$

where hereafter $o_n(1)$ denotes a quantity approaching zero as $n \to \infty$. The above expression implies that $(z_n) \subset X$ is bounded. In the case that (ii) occurs, it follows from (2.4) that

$$\int Q(z_n) = \int Q(u_n^+, v_n^+) \leq \lambda \int |z_n|^2 \leq \frac{\lambda}{\theta_1(\Omega)}\|z_n\|^2,$$

and therefore we get

$$\begin{aligned} c + c_1\|z_n\| + o_n(1) &\geq I_{\lambda,\mu}(z_n) - \frac{1}{2^*}I'_{\lambda,\mu}(z_n)z_n \\ &= \frac{1}{N}\|z_n\|^2 - \frac{2}{N}\int Q(z_n) \geq \frac{1}{N}\left(1 - \frac{2\lambda}{\theta_1(\Omega)}\right)\|z_n\|^2. \end{aligned}$$

Since $2\lambda < \theta_1(\Omega)$ the boundedness of $(z_n)$ follows as in the first case.

In view of the above remarks we may suppose that $z_n \rightharpoonup z := (u,v)$ weakly in $X$ and $z_n \to z$ strongly in $L^q(\Omega) \times L^q(\Omega)$. Moreover, a standard argument shows that $I'_{\lambda,\mu}(z) = 0$.

By setting $\widetilde{z}_n := (\widetilde{u}_n, \widetilde{v}_n) = (u_n - u, v_n - v)$ we can use the strong convergence in $L^q(\Omega) \times L^q(\Omega)$ and [15, Lemma 5] to conclude that

$$\int Q_{\lambda,\mu}(z_n) = \int Q_{\lambda,\mu}(z) + o_n(1), \quad \int H(z_n) = \int H(z) + \int H(\widetilde{z}_n) + o_n(1). \tag{2.9}$$

This and the weak convergence of $(z_n)$ provide

$$c + o_n(1) = I_{\lambda,\mu}(z) + \frac{1}{2}\|\widetilde{z}_n\|^2 - \frac{1}{2^*}\int H(\widetilde{z}_n) \geq \frac{1}{2}\|\widetilde{z}_n\|^2 - \frac{1}{2^*}\int H(\widetilde{z}_n), \tag{2.10}$$

where we have used $I_{\lambda,\mu}(z) \geq 0$.

By using $I'_{\lambda,\mu}(z_n) \to 0$ and (2.9) again, we get

$$\begin{aligned} o_n(1) = I'_{\lambda,\mu}(z_n)z_n &= \|z_n\|^2 - q\int Q_{\lambda,\mu}(z_n) - \int H(z_n) \\ &= I'_{\lambda,\mu}(z)z + \|\widetilde{z}_n\|^2 - \int H(\widetilde{z}_n). \end{aligned}$$

Recalling that $I'_{\lambda,\mu}(z) = 0$, we can use the above equality and (2.10) to obtain

$$\lim_{n\to\infty} \|\widetilde{z}_n\|^2 = b = \lim_{n\ \to\infty} \int H(\widetilde{z}_n), \quad \frac{1}{N}b = \left(\frac{1}{2} - \frac{1}{2^*}\right) b \leq c,$$

for some $b \geq 0$.

In view of the definition of $S_H$, we have that

$$\|\widetilde{z}_n\|^2 \geq S_H \left(\int H(\widetilde{z}_n)\right)^{2/2^*}.$$

Taking the limit we get $b \geq S_H b^{2/2^*}$. So, if $b > 0$, we conclude that $b \geq S_H^{N/2}$ and therefore

$$\frac{1}{N}S_H^{N/2} \leq \frac{1}{N}b \leq c < \frac{1}{N}S_H^{N/2},$$

which is a contradiction. Hence $b = 0$ and therefore $z_n \to z$ strongly in $X$. □

Before presenting our next result we recall that, for each $\varepsilon > 0$, the function

$$\Phi_\varepsilon(x) := \frac{C_N \varepsilon^{(N-2)/4}}{(\varepsilon + |x|^2)^{(N-2)/2}}, \quad x \in \mathbb{R}^N, \tag{2.11}$$

where $C_N := N(N-2)^{(N-2)/4}$, satisfies $\|\nabla\Phi_\varepsilon\|_2^2 = \|\Phi_\varepsilon\|_{2^*}^{2^*} = S^{N/2}$, where $S$ is the best constant of the Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. Thus, using [15, Lemma 1] and the homogeneity of $H$, we obtain $A$, $B > 0$ such that

$$S_H = \frac{||(A\Phi_\varepsilon, B\Phi_\varepsilon)||^2}{\left(\int_{\mathbb{R}^N} H(A\Phi_\varepsilon, B\Phi_\varepsilon)\right)^{2/2*}} = \frac{(A^2+B^2)}{H(A,B)^{2/2^*}} \frac{S^{N/2}}{\|\Phi_\varepsilon\|_{2^*}^2},$$

from which it follows that

$$S_H = \frac{(A^2+B^2)}{H(A,B)^{2/2^*}} S. \tag{2.12}$$

The above equality and the ideas introduced by Brezis and Nirenberg [4] are the keystone of the following result.

**Lemma 2.3** *Suppose that $Q$ satisfies $(Q_0)$, with $2 < q < 2^*$, and $\lambda$, $\mu$ defined in (1.1)-(1.2) are positive. Then,*

$$c_{\lambda,\mu} < \frac{1}{N} S_H^{N/2}.$$

*The same result holds if $q = 2$ and and $\lambda$, $\mu \in (0, \theta_1(\Omega)/2)$.*

*Proof.* We consider a nonnegative function $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\phi \equiv 1$ in $B_R(0) \subset \Omega$, $\phi \equiv 0$ in $\mathbb{R}^N \setminus B_{2R}(0)$ and define

$$w_\varepsilon(x) := \frac{\phi(x)\Phi_\varepsilon(x)}{\|\phi\Phi_\varepsilon\|_{2^*}}.$$

where $\Phi_\varepsilon$ was defined in (2.11). Since $\|w_\varepsilon\|_{2^*} = 1$, we can use the homogeneity of $Q$ and $H$ to get, for any $t \geq 0$,

$$I_{\lambda,\mu}(tAw_\varepsilon, tBw_\varepsilon) = \frac{t^2}{2}(A^2+B^2)\|w_\varepsilon\|^2 - t^q Q_{\lambda,\mu}(A,B)\|w_\varepsilon\|_q^q - \frac{t^{2^*}}{2^*} H(A,B).$$

We shall denote by $h_\varepsilon(t)$ the right-hand side of the above equality and consider two distinct cases.

**Case 1**. $2 < q < 2^*$.

In this case there exists $t_\varepsilon > 0$ such that

$$h_\varepsilon(t_\varepsilon) = \max_{t \geq 0} h_\varepsilon(t). \tag{2.13}$$

Let

$$g_\varepsilon(t) := \frac{t^2}{2}(A^2+B^2)\|w_\varepsilon\|^2 - \frac{t^{2^*}}{2^*} H(A,B), \quad t \geq 0,$$

and notice that the maximum value of $g_\varepsilon$ occurs at the point

$$\tilde{t}_\varepsilon := \left\{ \frac{(A^2+B^2)\|w_\varepsilon\|^2}{H(A,B)} \right\}^{1/(2^*-2)}.$$

So, for each $t \geq 0$,

$$g_\varepsilon(t) \leq g_\varepsilon(\tilde{t}_\varepsilon) = \frac{1}{N}\left(\frac{(A^2+B^2)\|w_\varepsilon\|^2}{H(A,B)^{2/2^*}}\right)^{N/2},$$

and therefore

$$h_\varepsilon(t_\varepsilon) \leq \frac{1}{N}\left(\frac{(A^2+B^2)\|w_\varepsilon\|^2}{H(A,B)^{2/2^*}}\right)^{N/2} - t_\varepsilon^q Q_{\lambda,\mu}(A,B)\|w_\varepsilon\|_q^q. \tag{2.14}$$

We claim that, for some $c_2 > 0$, there holds

$$t_\varepsilon^q Q_{\lambda,\mu}(A,B) \geq c_2.$$

Indeed, if this is not the case, we have that $t_{\varepsilon_n} \to 0$ for some sequence $\varepsilon_n \to 0^+$. But it is proved in [4, (1.11) and (1.12)] that

$$\|w_\varepsilon\|^2 = S + O(\varepsilon^{(N-2)/2}). \tag{2.15}$$

Thus,

$$0 < c_{\lambda,\mu} \leq \sup_{t\geq 0} I_{\lambda,\mu}(tAw_{\varepsilon_n}, tBw_{\varepsilon_n}) = I_{\lambda,\mu}(t_{\varepsilon_n}Aw_{\varepsilon_n}, t_{\varepsilon_n}Bw_{\varepsilon_n}) \to 0,$$

which is a contradiction. So, the claim holds and we infer from (2.14) and (2.15) that

$$\begin{aligned} h_\varepsilon(t_\varepsilon) &\leq \frac{1}{N}\left(\frac{(A^2+B^2)}{H(A,B)^{2/2^*}}S + O(\varepsilon^{(N-2)/2})\right)^{N/2} - c_2\|w_\varepsilon\|_q^q \\ &\leq \frac{1}{N}S_H^{N/2} + O(\varepsilon^{(N-2)/2}) - c_2\|w_\varepsilon\|_q^q. \end{aligned}$$

It is proved in [16, Claim 2, p. 778] that $\lim_{\varepsilon\to 0^+} \varepsilon^{(2-N)/2}\|w_\varepsilon\|_q^q = +\infty$. Thus, we conclude from the above inequality that, for each $\varepsilon > 0$ small, there holds

$$c_{\lambda,\mu} \leq \sup_{t\geq 0} I_{\lambda,\mu}(tAw_\varepsilon, BAw_\varepsilon) = h_\varepsilon(t_\varepsilon) < \frac{1}{N}S_H^{N/2}.$$

**Case 2**. $q = 2$.

In this case we have that $h'_{\varepsilon}(t) = 0$ if, and only if,

$$(A^2 + B^2)\|w_\varepsilon\|^2 - 2Q_{\lambda,\mu}(A,B)\|w_\varepsilon\|_2^2 = t^{2^*-2}H(A,B).$$

Since we are supposing $\lambda < \theta_1(\Omega)/2$, we can use Poincaré's Inequality to obtain

$$\begin{aligned}2Q_{\lambda,\mu}(A,B)\|w_\varepsilon\|_2^2 &\leq 2\lambda(A^2+B^2)\|w_\varepsilon\|_2^2 \\ &< \theta_1(\Omega)(A^2+B^2)\|w_\varepsilon\|_2^2 \leq (A^2+B^2)\|w_\varepsilon\|^2.\end{aligned}$$

Thus, there exists $t_\varepsilon > 0$ satisfying (2.13). By using the definition of $w_\varepsilon$ and [4, (1.12) and (1.13)] we get

$$\|w_\varepsilon\|_2^2 = \begin{cases} \varepsilon^{(N-2)/4} + O(\varepsilon^{(N-2)/2}) & \text{if } N \geq 5, \\ \varepsilon^{(N-2)/2}|\log\varepsilon| + O(\varepsilon^{(N-2)/2}) & \text{if } N = 4. \end{cases} \tag{2.16}$$

Arguing as in the first case we conclude that, for $\varepsilon > 0$ small, there holds

$$h_\varepsilon(t_\varepsilon) \leq \frac{1}{N}S_H^{N/2} + O(\varepsilon^{(N-2)/2}) - c_2\|w_\varepsilon\|_2^2 < \frac{1}{N}S_H^{N/2},$$

where we have used (2.16) in the last inequality. This concludes the proof. □

As a byproduct of Lemmas 2.2 and 2.3 we obtain the following generalizations of [15, Theorem 1].

**Theorem 2.4** *Suppose $H$ satisfies $(H_0)-(H_4)$ and $Q$ satisfies $(Q_0)-(Q_1)$. Then the problem $(P)$ possesses a nonzero nonnegative solution whenever $2 < q < 2^*$ and $\lambda$, $\mu > 0$, or $q = 2$ and $\lambda$, $\mu \in (0, \theta_1(\Omega)/2)$.*

*Proof.* Since $I_{\lambda,\mu}$ satisfies the geometric conditions of the Mountain Pass Theorem, there exists $(z_n) \subset X$ such that

$$I_{\lambda,\mu}(z_n) \to c_{\lambda,\mu}, \quad I'_{\lambda,\mu}(z_n) \to 0.$$

It follows from Lemma 2.2 and Lemma 2.3 (with Remark 4.5 in the case $N = 3$) that $(z_n)$ converges, along a subsequence, to a nonzero critical point $z = (u,v) \in X$ of $I_{\lambda,\mu}$. According to (2.3), (2.4) and (2.2), we have that

$$I'_{\lambda,\mu}(z)z^- = -\|z^-\|^2 - \int \left(\nabla Q(u^+v^+)\cdot(u^-v^-) + \frac{1}{2^*}\nabla H(u^+v^+)\cdot(u^-v^-)\right).$$

Since $z$ is a critical point and the integral above vanishes, it follows that $z^- = 0$. Hence, $u$, $v \geq 0$ in $\Omega$ and the theorem is proved. □

**Theorem 2.5** *Suppose $H$ satisfies $(H_0)-(H_4)$ and $Q$ satisfies $(Q_0)$ and $(\widehat{Q_1})$. Then the problem $(P)$ possesses a nonzero nonnegative solution whenever $\lambda,\ \mu > 0$.*

*Proof.* As before, we obtain a nonzero critical point $z$ of $I_{\lambda,\mu}$. A simple calculation shows that the extension given in (2.5) is such that $Q_s(s,t) \geq 0$ for $s \leq 0$, and $Q_t(s,t) \geq 0$ for $t \leq 0$. Hence, using the extension of $H$ and arguing as in the previous theorem we obtain

$$0 = I'_{\lambda,\mu}(z)z^- = -\|z^-\|^2 - \int \Big(Q_u(u,v)u^- + Q_v(u,v)v^-\Big) \leq -\|z^-\|^2,$$

and the result follows. □

**Remark 2.6** *The two above theorems remains valid if we suppose that $N = 3$ and $4 < q < 6$. Indeed, it suffices to notice that in this case, according to [16, p. 779], the function $w_\varepsilon$ defined at the beginning of the proof of Lemma 2.3 satisfies $\lim_{\varepsilon\to 0^+} \varepsilon^{(2-N)/2}\|w_\varepsilon\|_q^q = +\infty$. So, the same arguments of Case 1 in that lemma hold.*

## 3 Some technical results

In this section we denote by $\mathcal{M}(\mathbb{R}^N)$ the Banach space of finite Radon measures over $\mathbb{R}^N$ equipped with the norm

$$|\sigma| = \sup_{\varphi\in C_0(\mathbb{R}^N),\|\varphi\|_\infty\leq 1} |\sigma(\varphi)|.$$

A sequence $(\sigma_n) \subset \mathcal{M}(\mathbb{R}^N)$ is said to converge weakly to $\sigma \in \mathcal{M}(\mathbb{R}^N)$ provided $\sigma_n(\varphi) \to \sigma(\varphi)$ for all $\varphi \in C_0(\mathbb{R}^N)$. By the Banach-Alaoglu theorem, every bounded sequence $(\sigma_n) \subset \mathcal{M}(\mathbb{R}^N)$ contains a weakly convergent subsequence.

The next result is a version of the Second Concentration-Compactness Lemma of P.L.Lions [14, Lemma I.1]. It is also inspired by some previous lemmas due to Chabrowski [5] and Bianchi, Chabrowski and Szulkin [6], where the terms which measure the loss of mass of weakly convergente subsequence have firstly appeared.

**Lemma 3.1** *Suppose that the sequence* $(w_n) \subset \mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ *satisfies*

$$\begin{aligned} & w_n \rightharpoonup w && \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N), \\ & w_n(x) \to w(x) && \text{for a.e. } x \in \mathbb{R}^N, \\ & |\nabla(w_n - w)|^2 \rightharpoonup \sigma && \text{weakly in } \mathcal{M}(\mathbb{R}^N), \\ & H(w_n - w) \rightharpoonup \nu && \text{weakly in } \mathcal{M}(\mathbb{R}^N) \end{aligned}$$

*and define*

$$\sigma_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |\nabla w_n|^2 \, dx, \qquad \nu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} H(w_n) \, dx. \tag{3.1}$$

*Then*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx = \|\sigma\| + \sigma_\infty + \int_{\mathbb{R}^N} |\nabla w|^2 \, dx, \tag{3.2}$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} H(w_n) \, dx = \|\nu\| + \nu_\infty + \int_{\mathbb{R}^N} H(w) \, dx, \tag{3.3}$$

$$\|\nu\|^{2/2^*} \le S_H^{-1} \|\sigma\| \quad \text{and} \quad \nu_\infty^{2/2^*} \le S_H^{-1} \sigma_\infty. \tag{3.4}$$

*Moreover, if* $w = 0$ *and* $\|\nu\|^{2/2^*} = S_H^{-1}\|\sigma\|$, *then there exists* $x_0$, $x_1 \in \mathbb{R}^N$ *such that* $\nu = \delta_{x_0}$ *and* $\sigma = \delta_{x_1}$.

*Proof.* We first recall that, in view of the definition of $S_H$, for each nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^N)$ we have that

$$\left( \int_{\mathbb{R}^N} \varphi^{2^*}(x) H(w_n) \mathrm{d}x \right)^{2/2^*} = \left( \int_{\mathbb{R}^N} H(\varphi(x) w_n) \mathrm{d}x \right)^{2/2^*} \le S_H^{-1} \|\varphi(x) w_n\|^2.$$

Moreover, arguing as in [15, Lemma 5], we have that

$$\int_{\mathbb{R}^N} \psi(x) H(w_n - w) \mathrm{d}x = \int_{\mathbb{R}^N} \psi(x) H(w_n) \mathrm{d}x - \int_{\mathbb{R}^N} \psi(x) H(w) \mathrm{d}x + o_n(1),$$

for each $\psi \in C_0^\infty(\mathbb{R}^N)$. Since $H$ is $2^*$-homogeneous, we can use the two above expressions and argue along the same lines of the proof of [19, Lemma 1.40] (see also [9, Lemma 2.2]) to conclude that (3.2)-(3.4) hold. If $w = 0$ and $\|\nu\|^{2/2^*} = S_H^{-1}\|\sigma\|$ the same argument of [19, step 3 of the proof of Lemma 1.40] implies that the measures $\nu$ and $\sigma$ are concentrated at single points $x_0$, $x_1 \in \mathbb{R}^N$, respectively. □

**Remark 3.2** *For future reference we notice that the last conclusion of the above result holds even if* $w \not\equiv 0$. *Indeed, in this case we can define* $\widetilde{w}_n :=$

*$w_n - w$ and notice that*

$$\begin{aligned}
&\widetilde{w}_n \rightharpoonup \widetilde{w} = 0 && \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N),\\
&\widetilde{w}_n(x) \to 0 && \text{for a.e. } x \in \mathbb{R}^N,\\
&|\nabla(\widetilde{w}_n - \widetilde{w})|^2 \rightharpoonup \widetilde{\sigma} && \text{weakly in } \mathcal{M}(\mathbb{R}^N),\\
&H(\widetilde{w}_n - \widetilde{w}) \rightharpoonup \widetilde{\nu} && \text{weakly in } \mathcal{M}(\mathbb{R}^N).
\end{aligned}$$

*But $\widetilde{w}_n - \widetilde{w} = w_n - w$ and therefore $\widetilde{\sigma} = \sigma$ and $\widetilde{\nu} = \nu$, where $\sigma$ and $\nu$ are as in Lemma 3.1. Thus, if $\|\nu\|^{2/2^*} = S_H^{-1}\|\sigma\|$ we also have that $\|\widetilde{\nu}\|^{2/2^*} = S_H^{-1}\|\widetilde{\sigma}\|$ and the result follows from the last part of Lemma 3.1.*

Before stating one of the main results of this section we introduce the following notation. Given $r > 0$, $y \in \mathbb{R}^N$ and a function $z \in X$, we extend $z$ to the whole $\mathbb{R}^N$ by setting $z(x) := 0$ if $x \in \mathbb{R}^N \setminus \Omega$ and define $z^{y,r} \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ as

$$z^{y,r}(x) := r^{(N-2)/2} z(rx + y), \quad x \in \mathbb{R}^N.$$

**Proposition 3.3** *Suppose $(z_n) \subset X$ is such that*

$$\int H(z_n) = 1 \quad \text{and} \quad \lim_{n\to\infty} \|z_n\|^2 = S_H.$$

*Then there exist $(r_n) \subset (0, \infty)$ and $(y_n) \subset \mathbb{R}^N$ such that the sequence $(z_n^{y_n,r_n})$ strongly converges to $z \neq 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$. Moreover, as $n \to \infty$, we have that $r_n \to 0$ and $y_n \to \overline{y} \in \overline{\Omega}$.*

*Proof.* We first extend $z_n$ by setting $z_n(x) := 0$ if $x \in \mathbb{R}^N \setminus \Omega$. For each $r > 0$ we consider

$$F_n(r) := \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} H(z_n).$$

Since $\lim_{r\to 0} F_n(r) = 0$ and $\lim_{r\to\infty} F_n(r) = 1$, there exist $r_n > 0$ and a sequence $(y_n^k)_{k\in\mathbb{N}} \subset \mathbb{R}^N$ satisfying

$$\frac{1}{2} = F_n(r_n) = \lim_{k\to\infty} \int_{B_{r_n}(y_n^k)} H(z_n).$$

Recalling that $\lim_{|y|\to\infty} \int_{B_{r_n}(y)} H(z_n) = 0$ we conclude that $(y_n^k)$ is bounded. Hence, up to a subsequence, $\lim_{k\to\infty} y_n^k = y_n \in \mathbb{R}^N$ and we obtain

$$\frac{1}{2} = \int_{B_{r_n}(y_n)} H(z_n).$$

We shall prove that the sequences $(r_n)$ and $(y_n)$ above satisfy the statements of the lemma. First notice that

$$\frac{1}{2} = \int_{B_{r_n}(y_n)} H(z_n) = \int_{B_1(0)} H(z_n^{y_n,r_n}) = \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} H(z_n^{y_n,r_n}). \tag{3.5}$$

If we denote $w_n := z_n^{y_n,r_n}$, a straightforward calculation provides

$$\lim_{n\to\infty} \|w_n\|^2 = \lim_{n\to\infty} \|z_n\|^2 = S_H, \qquad \int_{\mathbb{R}^N} H(w_n) = 1.$$

Hence, we can apply Lemma 3.1 to obtain $w \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ satisfying

$$S_H = \|\sigma\| + \sigma_\infty + \|w\|^2, \quad 1 = \|\nu\| + \nu_\infty + \int_{\mathbb{R}^N} H(w), \tag{3.6}$$

$$\|\nu\|^{2/2^*} \le S_H^{-1}\|\sigma\| \quad \text{and} \quad \nu_\infty^{2/2^*} \le S_H^{-1}\sigma_\infty. \tag{3.7}$$

The second equality above implies that $\int_{\mathbb{R}^N} H(w)$, $\|\nu\|$, $\nu_\infty \in [0,1]$. If one of these values belongs to the open interval $(0,1)$, we can use (3.6), $2/2^* < 1$, $(\int_{\mathbb{R}^N} H(w))^{2/2^*} \le S_H^{-1}\|w\|^2$ and (3.7) to get

$$\begin{aligned} S_H &= S_H\left(\|\nu\| + \nu_\infty + \int_{\mathbb{R}^N} H(w)\right) \\ &< S_H\left(\|\nu\|^{2/2^*} + \nu_\infty^{2/2^*} + \left(\int_{\mathbb{R}^N} H(w)\right)^{2/2^*}\right) \le S_H, \end{aligned}$$

which is a contradiction. Thus $\int_{\mathbb{R}^N} H(w)$, $\|\nu\|$, $\nu_\infty \in \{0,1\}$. Actually, it follows from (3.5) that $\int_{|x|>R} H(w_n) \le 1/2$ for any $R > 1$. Thus, we conclude that $\nu_\infty = 0$.

Let us prove that $\|\nu\| = 0$. Suppose, by contradiction, that $\|\nu\| = 1$. It follows from the first equality in (3.7) that $S_H \le \|\sigma\|$. On the other hand, the first inequality in (3.6) provides $\|\sigma\| \le S_H$. Hence, we conclude that $\|\sigma\| = S_H$. Since we are supposing that $\|\nu\| = 1$ we obtain $\|\nu\|^{2/2^*} = S_H^{-1}\|\sigma\|$. It follows from Remark 3.2 that $\nu = \delta_{x_0}$ for some $x_0 \in \mathbb{R}^N$. Thus, from (3.5), we get

$$\frac{1}{2} \ge \lim_{n\to\infty} \int_{B_1(x_0)} H(w_n) = \int_{B_1(x_0)} \mathrm{d}\nu = \|\nu\| = 1.$$

This contradiction proves that $\|\nu\| = 0$.

Since $\|\nu\| = \nu_\infty = 0$ we have that $\int_{\mathbb{R}^N} H(w) = 1$. This and (3.6) provide

$$\lim_{n\to\infty} \|w_n\|^2 = S_H \ge \|w\|^2 \ge S_H \left(\int_{\mathbb{R}^N} H(w)\right)^{2/2^*} = S_H.$$

So, $\|w\|^2 = S_H$ and therefore $w_n \to w \not\equiv 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ and $w_n(x) \to w(x)$ for a.e. $x \in \mathbb{R}^N$.

In order to conclude the proof we notice that

$$\|w_n\|_{L^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N)} = \frac{1}{r_n^2}\|z_n\|_{L^2(\Omega)\times L^2(\Omega)}.$$

Since $(z_n)$ is bounded and $w \not\equiv 0$, we infer from the above equality that, up to a subsequence, $r_n \to r_0 \geq 0$. If $|y_n| \to \infty$ we have that, for each fixed $x \in \mathbb{R}^N$, there exists $n_x \in \mathbb{N}$ such that $r_n x + y_n \notin \Omega$ for $n \geq n_x$. For such values of $n$ we have that $w_n(x) = 0$. Taking the limit and recalling that $x \in \mathbb{R}$ is arbitrary, we conclude that $w \equiv 0$, which is absurd. So, along a subsequence, $y_n \to y \in \mathbb{R}^N$.

We claim that $r_0 = 0$. Indeed, suppose by contradiction that $r_0 > 0$. Then, as $n$ becomes large, the set $\Omega_n := (\Omega - y_n)/r_n$ approaches $\Omega_0 := (\Omega - y)/r_0 \neq \mathbb{R}^N$. This implies that $w$ has compact support in $\mathbb{R}^N$. On the other hand, since $w$ achieves the infimum in (1.3) and $H$ is homogeneous, we can use the Lagrange Multiplier Theorem to conclude that $w = (u, v)$ satisfies

$$-\Delta u = \lambda H_u(u, v), \quad -\Delta v = \lambda H_v(u, v), \quad x \in \mathbb{R}^N,$$

for $\lambda = S_H/2^* > 0$. It follows from $(H_3)$ and the Maximum Principle that at least one the functions $u, v$ is positive in $\mathbb{R}^N$. But this contradicts $\operatorname{supp} w \subset \Omega_0$. Hence, we conclude that $r_0 = 0$. Finally, if $y \notin \overline{\Omega}$ we obtain $r_n x + y_n \notin \Omega$ for large values of $n$, and therefore we should have $w \equiv 0$ again. Thus, $y \in \overline{\Omega}$ and the proof is finished. □

We finalize this section with the study of the asymptotic behavior of the minimax level $c_{\lambda,\mu}$ as both the parameters approach zero.

**Lemma 3.4** *We have that*

$$\lim_{\lambda,\,\mu \to 0^+} c_{\lambda,\mu} = c_{0,0} = \frac{1}{N} S_H^{N/2}.$$

*Proof.* We first prove the second equality. It follows from $\lambda = \mu = 0$ that $Q_{0,0} \equiv 0$. If $A$, $B$, $w_\varepsilon$, $g_\varepsilon$ and $t_\varepsilon$ are as in the proof of Lemma 2.3, we have that $(t_\varepsilon A w_\varepsilon, t_\varepsilon B w_\varepsilon) \in \mathcal{N}_{0,0}$. Thus

$$\begin{aligned} c_{0,0} \leq I_{0,0}(t_\varepsilon A w_\varepsilon, t_\varepsilon B w_\varepsilon) &= \frac{1}{N}\left\{\frac{(A^2+B^2)}{H(A,B)^{2/2^*}}\|w_\varepsilon\|^2\right\}^{N/2} \\ &= \frac{1}{N}\left\{\frac{(A^2+B^2)}{H(A,B)^{2/2^*}}(S + O(\varepsilon^{(N-2)/2}))\right\}^{N/2}. \end{aligned}$$

Taking the limit as $\varepsilon \to 0^+$ and using (2.12), we conclude that $c_{0,0} \leq \frac{1}{N} S_H^{N/2}$.

In order to obtain the reverse inequality we consider $(z_n) \subset X$ such that $I_{0,0}(z_n) \to c_{0,0}$ and $I'_{0,0}(z_n) \to 0$. The sequence $(z_n)$ is bounded and therefore

$I'_{0,0}(z_n)z_n = \|z_n\|^2 - \int H(z_n) = o_n(1)$. It follows that

$$\lim_{n\to\infty} \|z_n\|^2 = b = \lim_{n\ \to\infty} \int H(z_n).$$

Taking the limit in the inequality $S_H \left(\int H(z_n)\right)^{2/2^*} \le \|z_n\|^2$ we conclude, as in the proof of Lemma 2.2, that $Nc_{0,0} = b \ge S_H^{N/2}$. Hence,

$$c_{0,0} = \lim_{n\to\infty} I_{0,0}(z_n) = \lim_{n\to\infty} \left(\frac{1}{2}\|z_n\|^2 - \frac{1}{2^*}\int H(z_n)\right) = \frac{1}{N}b \ge \frac{1}{N}S_H^{N/2},$$

and therefore $c_{0,0} = \frac{1}{N}S_H^{N/2}$.

We proceed now with the calculation of $\lim_{\lambda,\,\mu\to 0^+} c_{\lambda,\mu}$. Let $(\lambda_n)$, $(\mu_n) \subset \mathbb{R}^+$ be such that $\lambda_n$, $\mu_n \to 0^+$. Since $\mu_n$ defined in (1.1) is positive, we have that $Q_{\lambda_n,\mu_n}(z) \ge 0$ whenever $z$ is nonnegative. Thus, for this kind of function, we have that $I_{\lambda_n,\mu_n}(z) \le I_{0,0}(z)$. It follows that

$$\begin{aligned} c_{\lambda_n,\mu_n} &= \inf_{z\ne(0,0)} \max_{t\ge 0} I_{\lambda_n,\mu_n}(tz) \\ &\le \inf_{z\ne(0,0),\, z\ge 0} \max_{t\ge 0} I_{\lambda_n,\mu_n}(tz) \\ &\le \inf_{z\ne(0,0),\, z\ge 0} \max_{t\ge 0} I_{0,0}(tz) = c_{0,0}, \end{aligned}$$

where we have used, in the last equality, that the infimum $c_{0,0}$ is attained at a nonnegative solution. The above inequality implies that

$$\limsup_{n\to\infty} c_{\lambda_n,\mu_n} \le c_{0,0}. \tag{3.8}$$

On the other hand, it follows from Theorem 2.4 that there exists $(z_n) = (u_n, v_n) \subset X$ such that

$$I_{\lambda_n,\mu_n}(z_n) = c_{\lambda_n,\mu_n}, \quad I'_{\lambda_n,\mu_n}(z_n) = 0.$$

Since $c_{\lambda_n,\mu_n}$ is bounded, the same argument performed in the proof of Lemma 2.2 implies that $(z_n)$ is bounded in $X$. Since $z_n \ge 0$ we obtain $0 \le \int Q_{\lambda_n,\mu_n}(z_n) \le \lambda_n \int(|u_n|^q + |v_n|^q)$, from which it follows that

$$\lim_{n\to\infty} \int Q_{\lambda_n,\mu_n}(z_n) = 0. \tag{3.9}$$

Let $t_n > 0$ be such that $t_n z_n \in \mathcal{N}_{0,0}$. Since $z_n \in \mathcal{N}_{\lambda_n,\mu_n}$, we have that

$$\begin{aligned} c_{0,0} &\leq I_{0,0}(t_n z_n) = I_{\lambda_n,\mu_n}(t_n z_n) + t_n^q \int Q_{\lambda_n,\mu_n}(z_n) \\ &\leq I_{\lambda_n,\mu_n}(z_n) + t_n^q \int Q_{\lambda_n,\mu_n}(z_n) \\ &= c_{\lambda_n,\mu_n} + t_n^q \int Q_{\lambda_n,\mu_n}(z_n). \end{aligned}$$

If $(t_n)$ is bounded, we can use the above estimate and (3.9) to get

$$c_{0,0} \leq \liminf_{n\to\infty} c_{\lambda_n,\mu_n}.$$

This and (3.8) proves the lemma.

It remains to check that $(t_n)$ is bounded. A straightforward calculation shows that

$$t_n = \left( \frac{\|z_n\|^2}{\int H(z_n)} \right)^{1/(2^*-2)}. \tag{3.10}$$

Since $z_n \in \mathcal{N}_{\lambda_n,\mu_n}$ we obtain

$$\|z_n\|^2 = q \int Q_{\lambda_n,\mu_n}(z_n) + \int H(z_n) \leq o_n(1) + S_H^{-2^*/2} \|z_n\|^{2^*}.$$

Hence $\|z_n\|^2 \geq c_1 > 0$, and therefore it follows from the above expression that $\int H(z_n) \geq c_2 > 0$. This, the boundedness of $(z_n)$ and (3.10) imply that $(t_n)$ is bounded. The lemma is proved. □

## 4 Proof of the main theorems

From now on we fix $r > 0$ such that the sets

$$\Omega_r^+ := \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) < r\}, \quad \Omega_r^- := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r\}$$

are homotopic equivalents to $\Omega$. We define the functional

$$J_{\lambda,\mu}(z) := \frac{1}{2}\|z\|^2 - \int Q_{\lambda,\mu}(z) - \frac{1}{2^*} \int H(z), \quad z \in X_{r,rad},$$

where $X_{r,rad} := \{(u, v) : u,\ v \in H_0^1(B_r(0))$ and $u,\ v$ are radial functions$\}$.

We denote by $\mathcal{M}_{\lambda,\mu}$ its associated Nehari manifold and set

$$m_{\lambda,\mu} := \inf_{z \in \mathcal{M}_{\lambda,\mu}} J_{\lambda,\mu}(z).$$

According to [15, Lemma 1] the infimum $S_H$ can be attained by functions belonging to $\mathcal{D}^{1,2}_{rad}(\mathbb{R}^N) \times \mathcal{D}^{1,2}_{rad}(\mathbb{R}^N)$. So, arguing as in the proof of Lemma 3.4 and Theorems 2.4 and 2.5 , we obtain the following result.

**Lemma 4.1** *Suppose $H$ satisfies $(H_0)-(H_4)$ and $Q$ satisfies $(Q_0)-(Q_1)$. Then the infimum $m_{\lambda,\mu}$ is attained by a positive radial function $z_{\lambda,\mu} \in X_{r,rad}$ whenever $2 < q < 2^*$ and $\lambda,\ \mu > 0$, or $q = 2$ and $\lambda,\ \mu \in (0, \theta_{1,rad}/2)$, where $\theta_{1,rad} > 0$ is the first eigenvalue of the operator $(-\Delta, H^1_{0,rad}(B_r(0)))$. Moreover*

$$m_{\lambda,\mu} < \frac{1}{N} S_H^{N/2} \quad \textit{and} \quad \lim_{\lambda,\, \mu \to 0^+} m_{\lambda,\mu} = \frac{1}{N} S_H^{N/2}.$$

*The same result hold if $Q$ satisfies $\widehat{(Q_1)}$ instead of $(Q_1)$ and $\lambda,\ \mu > 0$.*

We introduce the barycenter map $\beta_{\lambda,\mu} : \mathcal{N}_{\lambda,\mu} \to \mathbb{R}^N$ as follows

$$\beta_{\lambda,\mu}(z) := \frac{1}{S_H^{N/2}} \int H(z) x \,\mathrm{d}x.$$

This maps has the following property.

**Lemma 4.2** *If $Q$ satisfies $(Q_0)$ and $(Q_1)$ then there exists $\lambda^* > 0$ such that $\beta_{\lambda,\mu}(z) \in \Omega^+_{r/2}$ whenever $z \in \mathcal{N}_{\lambda,\mu}$, $\lambda,\ \mu \in (0, \lambda^*)$ and $I_{\lambda,\mu}(z) \le m_{\lambda,\mu}$. The same result holds if we replace $(Q_1)$ by $(\widehat{Q_1})$ and the parameter $\lambda^*$ defined in (1.4) also belongs to $(0, \lambda^*)$.*

*Proof.* We first assume that $Q$ satisfies $(Q_0)$ and $(Q_1)$. Arguing by contradiction, we suppose that there exist $(\lambda_n)$, $(\mu_n) \subset \mathbb{R}^+$ and $(w_n) \subset \mathcal{N}_{\lambda_n,\mu_n}$ such that $\lambda_n,\ \mu_n \to 0^+$ as $n \to \infty$, $I_{\lambda_n,\mu_n}(w_n) \le m_{\lambda_n,\mu_n}$ but $\beta_{\lambda_m,\mu_n}(w_n) \notin \Omega^+_{r/2}$.

Standard calculations show that $(w_n) = (u_n, v_n)$ is bounded in $X$. Moreover

$$0 = I'_{\lambda_n,\mu_n}(w_n) w_n = \|w_n\|^2 - q \int Q_{\lambda_n,\mu_n}(w_n) - \int H(w_n).$$

Since $\lambda_n \to 0$, we can use the boundedness of $(w_n)$ to get

$$0 \le \int Q_{\lambda_n,\mu_n}(w_n) \le \lambda_n \int (|u_n|^q + |v_n|^q) \to 0,$$

from which it follows that $\lim_{n\to\infty} \|w_n\|^2 = \lim_{n\to\infty} \int H(w_n) = b \ge 0$. Notice that

$$c_{\lambda_n,\mu_n} \le I_{\lambda_n,\mu_n}(w_n) = \frac{1}{2}\|w_n\|^2 - \int Q_{\lambda_n,\mu_n}(w_n) - \frac{1}{2^*} \int H(w_n) \le m_{\lambda_n,\mu_n}.$$

Recalling that $c_{\lambda_n,\mu_n}$ and $m_{\lambda_n,\mu_n}$ both converge to $\frac{1}{N} S_H^{N/2}$, we can use the above expression and $\int Q_{\lambda_n,\mu_n}(w_n) \to 0$ again to conclude that $b = S_H^{N/2}$, that is,

$$\lim_{n\to\infty} \|w_n\|^2 = S_H^{N/2} = \lim_{n\to\infty} \int H(w_n). \tag{4.1}$$

Let $t_n := (\int H(w_n))^{-1/2^*} > 0$ and notice that $z_n := t_n w_n$ satisfies the hypotheses of Proposition 3.3. Thus, for some sequences $(r_n) \subset (0,\infty)$ and $(y_n) \subset \mathbb{R}^N$ satisfying $r_n \to 0$, $y_n \to \overline{y} \in \overline{\Omega}$ we have that $z_n^{y_n,r_n} \to z$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)\times\mathcal{D}^{1,2}(\mathbb{R}^N)$.

The definition of $z_n$, (4.1), the strong convergence of $(z_n^{y_n,r_n})$ and the Lebesgue's Theorem provide

$$\begin{aligned}
\beta_{\lambda_n,\mu_n}(w_n) &= \frac{t_n^{-2^*}}{S_H^{N/2}} \int H(z_n)x\,\mathrm{d}x = (1+o_n(1)) \int H(z_n)x\,\mathrm{d}x \\
&= (1+o_n(1)) \int H(z_n^{y_n,r_n})(r_n x + y_n)\,\mathrm{d}x \\
&= (1+o_n(1)) \left( \int H(z)\overline{y}\,\mathrm{d}x + o_n(1) \right).
\end{aligned}$$

Since $\overline{y} \in \overline{\Omega}$ and $\int H(z) = 1$, the above expression implies that

$$\lim_{n\to\infty} \operatorname{dist}(\beta_{\lambda_n,\mu_n}(w_n), \overline{\Omega}) = 0,$$

which contradicts $\beta_{\lambda_n,\mu_n}(w_n) \notin \Omega^+_{r/2}$.

We now suppose that $Q$ satisfies $(\widehat{Q_1})$. Arguing by contradiction again we suppose that there exist $(\lambda_n)$, $(\mu_n)$, $(\widehat{\lambda}_n) \subset \mathbb{R}^+$ and $(w_n) \subset \mathcal{N}_{\lambda_n,\mu_n}$ such that $\lambda_n$, $\mu_n$, $\widehat{\lambda}_n \to 0^+$ as $n \to \infty$, $I_{\lambda_n,\mu_n}(w_n) \le m_{\lambda_n,\mu_n}$ but $\beta_{\lambda_m,\mu_n}(w_n) \notin \Omega^+_{r/2}$. The same argument of the first case holds provide we can prove that $\int Q_{\lambda_n,\mu_n}(w_n) \to 0$. Notice that, in this new setting, we do not know that the extension of $Q$ is nonnegative. However, we can use (2.6) to get

$$\left| \int Q_{\lambda_n,\mu_n}(w_n) \right| \le \int |Q_{\lambda_n,\mu_n}(w_n)| \le (\lambda_n + \widehat{\lambda}_n) \int (|u_n|^q + |v_n|^q) \to 0,$$

and the lemma is proved. □

According to Lemma 4.1, for each $\lambda$, $\mu > 0$ small the infimum $m_{\lambda,\mu}$ is attained by a nonnegative radial function $z_{\lambda,\mu}$. We consider

$$I_{\lambda,\mu}^{m_{\lambda,\mu}} := \{z \in X : I_{\lambda,\mu}(z) \le m_{\lambda,\mu}\}$$

and define the function $\gamma_{\lambda,\mu} : \Omega_r^- \to I_{\lambda,\mu}^{m_{\lambda,\mu}}$ by setting, for each $y \in \Omega_r^-$,

$$\gamma_{\lambda,\mu}(y)(x) := \begin{cases} z_{\lambda,\mu}(x-y) & \text{if } x \in B_r(y), \\ 0 & \text{otherwise.} \end{cases}$$

A change of variables and straightforward calculations show that the map $\gamma_{\lambda,\mu}$ is well defined. Since $z_{\lambda,\mu}$ is radial, we have that $\int_{B_r(0)} H(z_{\lambda,\mu})x \,\mathrm{d}x = 0$. Hence, for each $y \in \Omega_r^-$, we obtain

$$\beta_{\lambda,\mu}(\gamma_{\lambda,\mu}(y)) = \alpha(\lambda,\mu)y,$$

where

$$\alpha(\lambda,\mu) := \frac{1}{S_H^{N/2}} \int H(z_{\lambda,\mu}).$$

If we define $F_{\lambda,\mu} : [0,1] \times (\mathcal{N}_{\lambda,\mu} \cap I_{\lambda,\mu}^{m_{\lambda,\mu}}) \to \mathbb{R}^N$ by

$$F_{\lambda,\mu}(t,z) := \left( t + \frac{1-t}{\alpha(\lambda,\mu)} \right) \beta_{\lambda,\mu}(z),$$

we have the following.

**Lemma 4.3** *If $Q$ satisfies $(Q_0)$ and $(Q_1)$ then there exists $\lambda^{**} > 0$ such that,*

$$F_{\lambda,\mu}\left([0,1] \times \left(\mathcal{N}_{\lambda,\mu} \cap I_{\lambda,\mu}^{m_{\lambda,\mu}}\right)\right) \subset \Omega_r^+,$$

*whenever $\lambda, \mu \in (0, \lambda^{**})$. The same result holds if we replace $(Q_1)$ by $(\widehat{Q_1})$ and suppose that $\widehat{\lambda}$ also belongs to $(0,\lambda^{**})$.*

*Proof.* Arguing by contradiction, we suppose that there exist sequences $(\lambda_n)$, $(\mu_n) \subset \mathbb{R}^+$ and $(t_n, z_n) \in [0,1] \times (\mathcal{N}_{\lambda_n,\mu_n} \cap I_{\lambda_n,\mu_n}^{m_{\lambda_n,\mu_n}})$ such that $\lambda_n$, $\mu_n \to 0^+$, as $n \to \infty$, and $F_{\lambda_n,\mu_n}(t_n, z_n) \notin \Omega_r^+$. Up to a subsequence $t_n \to t_0 \in [0,1]$. Moreover, the compactness of $\overline{\Omega}$ and Lemma 4.2 imply that, up to a subsequence, $\beta_{\lambda_n,\mu_n}(z_n) \to y \in \overline{\Omega_{r/2}^+} \subset \Omega_r^+$. We claim that $\alpha(\lambda_n,\mu_n) \to 1$. If this is true, we can use the definition of $F$ to conclude that $F_{\lambda_n,\mu_n}(t_n,z_n) \to y \in \Omega_r^+$, which is a contradiction.

It remains to check the above claim. It follows from Lemma 4.1 that

$$m_{\lambda_n,\mu_n} = \frac{1}{2}\|z_{\lambda_n,\mu_n}\|^2 - \int_{B_r(0)} Q_{\lambda_n,\mu_n}(z_{\lambda_n,\mu_n}) - \frac{1}{2^*}\int_{B_r(0)} H(z_{\lambda_n,\mu_n}) < \frac{1}{N} S_H^{N/2}.$$

As before $\int_{B_r(0)} Q_{\lambda_n,\mu_n}(z_{\lambda_n,\mu_n}) \to 0$. This, $J'_{\lambda_n,\mu_n}(z_{\lambda_n,\mu_n}) = 0$, the above expression and the same arguments used in the proof of Lemma 4.1 imply that

$$\lim_{n\to\infty} \int H(z_{\lambda_n,\mu_n}) = S_H^{N/2}.$$

The equality above and the definition of $\alpha(\lambda,\mu)$ imply that $\alpha(\lambda_n,\mu_n) \to 1$. The lemma is proved. □

**Corollary 4.4** *Let $\Lambda := \min\{\lambda^*, \lambda^{**}\} > 0$, with $\lambda^*$ and $\lambda^{**}$ given by Lemmas*

*4.2 and 4.3, respectively. If $Q$ satisfies $(Q_0)$ and $(Q_1)$ with $\lambda, \mu \in (0, \Lambda)$ then*

$$\operatorname{cat}_{I_{\lambda,\mu}^{m_{\lambda,\mu}}}(I_{\lambda,\mu}^{m_{\lambda,\mu}}) \geq \operatorname{cat}_\Omega(\Omega).$$

*The same result holds if we replace $(Q_1)$ by $(\widehat{Q_1})$ and suppose that $\widehat{\lambda}$ also belongs to $(0, \Lambda)$.*

*Proof.* It suffices to use Lemmas 4.2 and 4.3 and argue as in [1, Lemma 4.3]. We omit the details. □

We are now ready to prove our main results.

*Proof of Theorems 1.1 and 1.2.* Let $\Lambda > 0$ be given by Corollary 4.4 and suppose that $Q$ satisfies $(Q_1)$ with $\lambda$, $\mu \in (0, \Lambda)$, or it satisfies $(\widehat{Q_1})$ with $\lambda$, $\mu$, $\widehat{\lambda} \in (0, \Lambda)$. Using Lemma 2.2 and arguing as in [1, Lemma 4.2] we can prove that the functional $I_{\lambda,\mu}$ restricted to $\mathcal{N}_{\lambda,\mu}$ satisfies the $(PS)_c$ condition for all $c < \frac{1}{N} S_H^{N/2}$. Since $m_{\lambda,\mu} < \frac{1}{N} S_H^{N/2}$, standard Ljusternik-Schnirelmann theory provides $\operatorname{cat}_{I_{\lambda,\mu}^{m_{\lambda,\mu}}}(I_{\lambda,\mu}^{m_{\lambda,\mu}})$ critical points of the constrained functional. If $z \in \mathcal{N}_{\lambda,\mu}$ is one of these critical points, the same argument of [1, Lemma 4.1] shows that $z$ is also a critical point of the unconstrained functional, and therefore a nontrivial solution of $(P)$. As before, the obtained solutions are nonnegative in $\Omega$. The results follow from Corollary 4.4. □

**Remark 4.5** *Theorems 1.1 and 1.2 remain valid if we suppose that $N = 3$ and $4 < q < 6$. Indeed, it suffices to argue as in the case $N \geq 4$ and recall that, by Remark 2.6, the existence results of Theorems 2.4 and 2.5 hold in this case.*

## 5 Some further remarks

We start this last section presenting some functions which satisfy our hypotheses. We have the following example from [15]. Let $2 \leq q < 2^*$ and

$$P_q(s,t) := a_1 s^q + a_2 t^q + \sum_{i=1}^{k} b_i s^{\alpha_i} t^{\beta_i}, \quad s,\, t \geq 0,$$

where $a_1$, $a_2$, $b_i \in \mathbb{R}$, $\alpha_i + \beta_i = q$, and $\alpha_i, \beta_i > 1$ if $q > 2$ and $\alpha_i = \beta_i = 1$ otherwise. The following functions and its possible combinations, with appropriated choices of the coefficients $a_1$, $a_2$, $b_i$, satisfy our hypotheses on $Q$

$$Q(s,t) = P_q(s,t), \quad Q(s,t) = \sqrt[r]{P_{rq}(s,t)} \;\text{ and }\; Q(s,t) = \frac{P_{q+l}(s,t)}{P_l(s,t)},$$

with $l > 0$. Hence, we see that our subcritical term is more general than those of [9,11,12].

The form of $H$ is more restricted due to $(H_4)$. This technical condition has already appeared in [15] and it is important to guarantee that the constant $S_H$ defined in (1.3) does not depend on $\Omega$. As quoted in [15], the concavity condition $(H_4)$ is satisfied if $H \in C^2(\mathbb{R}^2_+, \mathbb{R})$ is such that $H_{st}(s,t) \geq 0$ for each $(s,t) \in \mathbb{R}^2_+$.

Although we have more restrictions on the shape of $H$, it can have the polynomial form

$$H(s,t) = P_{2^*}(s,t).$$

Thus, differently from [9,11,12], we can deal here with functions $H$ which possesses coupled and no coupled terms. For example, the function

$$H(s,t) = a_1 s^{2^*} + a_2 t^{2^*} + a_3 s^{\alpha} t^{\beta},$$

with $a_i \in \mathbb{R}$, $\alpha, \beta > 1$, $\alpha + \beta = 2^*$ satisfies the hypotheses $(H_0) - (H_4)$ for appropriated choices of the coefficients $a_i$. We also mention that the positivity condition in $(H_2)$ can holds even if some of the coefficients $a_i$ are negative. As a simple example, suppose that $H$ is as above with $a_1$, $a_2 \geq 0$ and $a_3 < 0$. Since $s^{\alpha} v^{\beta} \leq s^{2^*} + t^{2^*}$, the condition $(H_2)$ holds for $a_3 > \max\{-a_1, -a_2\}$.

Another interesting remark is that we can obtain versions of our theorems by interchanging conditions like $(Q_1)$ and $(\widehat{Q_1})$ for both the functions $Q$ and $H$. More specifically, let us consider the following assumption

$(\widehat{H_1})$ $H_s(0,1) > 0$ and $H_t(1,0) > 0$.

A simple inspection of our proofs shows that Theorem 1.1 is valid if we suppose $(\widehat{H_1})$ and $(Q_1)$. The same is true for Theorem 1.2. This last theorem is also true if we suppose $(\widehat{H_1})$ and $(\widehat{Q_1})$. The difference among these various settings relies in the form of the possible coupled terms.

A simple inspection of our proofs show that, instead of just one subcritical term, we can consider in $(P)$ a subcritical nonlinear term of the form

$$\widetilde{Q}(s,t) = \sum_{i=1}^{k} Q_i(s,t),$$

with each function $Q_i$ being $q_i$-homogeneous, $2 \leq q_i < 2^*$, and satisfying the same kind of hypotheses of $Q$. In this case, for each $i = 1, \ldots, k$, we define the numbers $\mu_i$, $\lambda_i$ as in (1.1)-(1.2), and the results hold if $\max_{i=1,\ldots,k}\{\mu_i, \lambda_i\}$ is small enough.

With some additional conditions we can assure that the solutions obtained in this paper are positive. Indeed, if we suppose that

($Q_2$) $Q_s(s,t) \geq 0$, $Q_t(s,t) \geq 0$ for each $(s,t) \in \mathbb{R}^2_+$,

we can apply the Maximum Principle in each equation of $(P)$. Thus, if $(u,v)$ is a nonnegative solution, then $u \equiv 0$ or $u > 0$ in $\Omega$, the same holding for $v$. We need only to discard solutions of the type $(u,0)$ or $(0,v)$. This can be done if we guarantee some kind of strongly coupling for the system. In what follows, we present some situations where this can be done.

If we are under the conditions of Theorem 1.1 we assume a stronger form of $(Q_1)$ and $(H_1)$, namely that $\nabla Q(1,0) = \nabla Q(0,1) = \nabla H(1,0) = \nabla H(0,1) = (0,0)$. In this way, if $(u,0)$ is a solution then

$$0 = I'_{\lambda,\mu}(u,0)(u,0) = -\|u\|^2 - \int \Big( Q_u(u,0)u + \frac{1}{2^*} H_u(u,0)u \Big) = -\|u\|^2.$$

and therefore $u \equiv 0$. Analogously, if $(0,v)$ is a solution then $v \equiv 0$. In the setting of Theorem 1.2 and considering the solution $(u,0)$ we obtain, from the second equation, that

$$0 = Q_v(u,0) + H_v(u,0) = u^{q-1} Q_v(1,0).$$

Since from $(\widehat{Q_1})$ we have that $Q_v(1,0) > 0$, it follows that $u \equiv 0$. The argument for $(0,v)$ is analogous.


## Acknowledgements

The authors would like to thank the referees for their useful suggestions and comments.


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