MULTIPLE SOLUTIONS FOR A NULL MASS NEUMANN PROBLEM IN EXTERIOR DOMAINS

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Abstract. In this paper we study the existence and multiplicity of solutions for the semilinear elliptic equation $-\Delta u = Q(x)f'(u)$ in an exterior domain with Neumann boundary conditions. We prove the existence of a positive ground state as well as a sign-changing solution under a double power growth condition on the nonlinearity.

1. Introduction

In this paper we are concerned with the existence and multiplicity of solutions for the problem

\[\begin{cases}
-\Delta u = Q(x)f'(u) & \text{in } \Omega, \\
\frac{\partial u}{\partial \eta} = 0 & \text{on } \partial \Omega,
\end{cases}\]  

(P)

where $\Omega = \mathbb{R}^N \setminus \overline{A}$, for $N \geq 3$, $A$ is a smooth bounded set and an exterior domain with no bounded components and $f'(0) = f''(0) = 0$, known in the literature as the null (zero) mass case. When $\Omega = \mathbb{R}^N$ and $Q \equiv 1$, Berestycki and Lions in [7] proved the existence of a (positive) ground state solution for the problem with $f(s)$ behaving as $s^q$ for $s$ small and as $s^p$ for $s$ large ($s > 0$), with $2 < p < 2^* := 2N/(N-2) < q$. This situation arises in certain problems related to Yang-Mills equations. More recently, the work

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of Benci and Micheletti [6] caught our attention because they proved that the Dirichlet problem in an exterior domain \( \Omega \)

\[
\begin{align*}
\begin{cases}
-\Delta u &= f'(u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{align*}
\]

(1.1)

with \( f'(0) = f''(0) = 0 \), has no ground state solution, contrary to the case of the entire \( \mathbb{R}^N \).

A parallel with the scalar field equation \(-\Delta u + u = |u|^{p-2}u \) in \( H^1(\Omega) \), \( \Omega \) an exterior domain, was inspiring. Benci and Cerami [3] proved that under Dirichlet boundary conditions there is no ground state for this problem and, a little later, Esteban [13] proved existence of ground state solutions with Neumann boundary conditions instead. That led us to studying \( (P) \), the zero mass case, with Neumann boundary conditions.

Moreover, the work of Cao [9] for the subcritical pure power case in an exterior domain \( \Omega \)

\[
\begin{align*}
\begin{cases}
-\Delta u + u &= Q(x)|u|^{p-1}u \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \eta} &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{align*}
\]

(1.2)

with \( Q(x) \to Q_\infty > 0 \) as \( |x| \to \infty \), and in Alves, Carrião and Medeiros [1] for the \( m \)-Laplacian, which proves the existence of sign-changing solutions in \( H^1(\Omega) \), suggested we should also look for a nodal solution of problem \( (P) \).

The variational approach is similar but the main difficulty in our case is that the natural setting for this problem is in the space \( D^{1,2}(\Omega) \) and the Orlicz space \( L^p + L^q \) which is required by the growth of the nonlinearity \( f \) at 0 and at \( \infty \). In order to apply critical point theory to studying the problem we use some recent abstract results of Benci and Fortunato [4] and an embedding result due to Pan and Wang [17] to obtain the regularity of the associated energy functional. Moreover, since \( \Omega \) is an unbounded domain, which is not necessarily symmetric, the existence of solutions is more difficult in view of the lack of compactness from the embedding \( D^{1,2}(\Omega) \hookrightarrow L^p + L^q \).

In order to overcome this difficulty we use a version of the splitting lemma found in a recent paper by Benci and Micheletti [6] to show that a Palais-Smale sequence either converges strongly to its weak limit or it differs from it by one or more functions which, after a suitable translation, altogether converge to a solution of the limit problem

\[
-\Delta w = Q_\infty f'(w) \quad \text{in } \mathbb{R}^N.
\]

(1.3)
Multiple solutions for a null mass Neumann problem

It is worthwhile mentioning that, in order to compare the minimax level associated to problem \((P)\) and get compactness, it is crucial to take advantage of the asymptotic behavior of the solutions of the above problem. Since it has polynomial decay, the arguments are somehow different from that of the positive mass case. However, we are able to adapt the arguments of Cao [9] (see also [15]) for this setting and obtain a positive ground state solution for \((P)\). After noting that this ground state also has polynomial decay, we prove the existence of a nodal solution for the problem \((P)\).

For these purposes we are going to assume that the potential \(Q\) satisfies

\((Q_1)\) \(Q : \Omega \to (0, \infty)\) is continuous;
\((Q_2)\) there exists \(Q_\infty > 0\) such that \(\lim_{|x| \to \infty} Q(x) = Q_\infty\).

Concerning the nonlinearity \(f\), we start by assuming that

\((f_1)\) \(f : \mathbb{R} \to \mathbb{R}\) is of class \(C^2\);
\((f_2)\) \(f(0) = f'(0) = f''(0) = 0\) and there exist positive constants \(c_0, \hat{c}_0, p, q\) with \(2 < p < 2^* < 2^* + 1 < q\) such that, for each \(s \in \mathbb{R}\), there hold
\[c_0 \min\{|s|^p, |s|^q\} \leq f(s) \quad \text{and} \quad f''(s) \leq \hat{c}_0 \min\{|s|^{p-2}, |s|^{q-2}\};\]
\((f_3)\) there exists \(\theta > 2\) such that, for each \(s \neq 0,\)
\[0 < \theta f(s) \leq sf'(s) < f''(s)s^2.\]

Let \(D^{1,2}(\Omega) = \{u \in L^{2^*}(\Omega) : |\nabla u| \in L^2(\Omega)\}\) and consider the functional \(I : D^{1,2}(\Omega) \to \mathbb{R}\) given by
\[I(u) := \frac{1}{2} \int_{\Omega} |\nabla u(x)|^2 dx - \int_{\Omega} Q(x)f(u(x))dx.\]

In view of \((f_1)\) and \((f_2)\) we can check that \(I\) is well defined. Making use of the double power growth of the nonlinearity at 0 and \(\infty\) and the appropriate spaces, we obtain that \(I \in C^1\) and its critical points are weak solutions of \((P)\).

We recall that a solution \(u_1\) of \((P)\) is called a ground state solution if it possesses minimum energy among all solutions; that is,
\[I(u_1) = \min\{I(u) : u \neq 0\ \text{is a solution of } (P)\}.\]

The first result of this paper can be stated as follows.

**Theorem 1.1.** Suppose that \(f\) satisfies \((f_1) - (f_3)\) and \(Q\) satisfies \((Q_1) - (Q_2)\). Then problem \((P)\) has a positive ground state solution provided \(Q\) satisfies
\[(Q_3)\] \[Q(x) \geq Q_\infty - C|x|^{-\gamma} \quad \text{for each } x \in \Omega,\]
for some \(C > 0\) and \(\gamma > 2(N - 1)\).
In our second result we are interested in the question of multiple solutions for \((P)\). In this case, we seek a nodal solution and for this purpose we need a condition stronger than \((f_3)\), namely
\[
(\hat{f}_3) \text{ there exist } \theta > 2 \text{ and } \bar{c}_0 > 0 \text{ such that, for each } s \neq 0, \text{ there hold}
0 < \theta f(s) \leq sf'(s) \text{ and } f''(s)s^2 - f'(s)s \geq \bar{c}_0 \min \{|s|^p, |s|^q\}, \text{ where}
p \text{ and } q \text{ are given in } (f_2).
\]

We shall prove the following multiplicity result.

**Theorem 1.2.** Suppose that \(f\) is even and satisfies \((f_1) - (f_2)\) and \((\hat{f}_3)\). Suppose also that \(Q\) satisfies \((Q_1) - (Q_2)\) and
\[
(\hat{Q}_3) \quad Q(x) \geq Q_\infty + C|x|^{-\gamma} \text{ for each } x \in \Omega,
\]
for some \(C > 0\) and \(\gamma < (N - 2)/2\). Then problem \((P)\) has in addition to a positive ground state solution a sign-changing solution.

An example of a function \(f\) satisfying our hypotheses can be obtained as follows. Let us consider the function \(\bar{f} : [0, \infty) \rightarrow \mathbb{R}\) defined as
\[
\bar{f}(s) = \begin{cases} 
s^q & \text{if } 0 \leq s \leq 1, \\
a + bs + cs^p & \text{if } s \geq 1,
\end{cases}
\]
with \(a, b, c \in \mathbb{R}\) chosen in such a way that \(\bar{f} \in C^2\) and define \(f(s) = \bar{f}(|s|)\), for \(s \in \mathbb{R}\). We refer the reader to [2] for more examples. We also would like to mention that there is no pure power \(f(s) = |s|^\sigma - 1 s\) which satisfies \((\hat{f}_3)\) with \(2 < p < 2^* < q\). Thus, we are considering here nonlinearities other than those included in [9, 11].

We believe that our results hold just assuming the natural condition \(q > 2^*\). However, the calculations presented here do not allow us to consider this case. In spite of this, if we consider \(\Omega = \mathbb{R}^N\) and remove the boundary condition on \((P)\), we are able to prove Theorem 1.2 getting two solutions for the null mass scalar field equation. More specifically, as a byproduct of our calculations, we have the following theorem, which complements some results of [7, 22, 5, 15].

**Theorem 1.3.** Suppose that \(f\) is even and satisfies \((f_1), (\hat{f}_3)\) and \((f_2)\) with \(2 < p < 2^* < q\). Suppose also that \(Q\) satisfies \((Q_1) - (Q_2)\) and \((\hat{Q}_3)\), for some \(C > 0\) and \(\gamma < (N - 2)/2\). Then the problem
\[
\begin{align*}
-\Delta u &= Q(x)f'(u) \text{ in } \mathbb{R}^N, \\
u &\in D^{1,2}(\mathbb{R}^N),
\end{align*}
\]
has a positive ground state solution and a sign-changing solution.
The paper is organized as follows. In Section 2 we present the variational framework and a version of the “splitting lemma” for these spaces which is going to be the main tool in posterior compactness arguments. We prove the existence of a (positive) ground state solution in Section 3. Finally, in Section 4 we present the proofs of the multiplicity results.

2. The variational framework

In this section we present the variational framework to deal with problem (\(P\)) and also give some preliminary results. The natural space to consider problem (\(P\)) in is

\[ D^{1,2}(\Omega) := \{ u \in L^{2^*} (\Omega) : |\nabla u| \in L^2(\Omega) \} \]

which has its natural norm given by \( \| u \|_{L^{2^*}(\Omega)} + \| \nabla u \|_{L^2(\Omega)} \). However, since we are dealing with the zero mass case, it is useful to consider only the \( L^2 \)-norm of the gradient. Since \( \Omega \) has no bounded components, it is proved in [17, Proposition 2.1] that

\[
\inf \left\{ \int_{\Omega} |\nabla u(x)|^2 dx : u \in D^{1,2}(\Omega), \| u \|_{L^{2^*}(\Omega)} = 1 \right\} > 0. \tag{2.1}
\]

Thus we can denote by \( E \) the Banach space \( D^{1,2}(\Omega) \) endowed with the norm

\[
\| u \| := \left( \int_{\Omega} |\nabla u(x)|^2 dx \right)^{1/2}.
\]

If \( u \in L^1(D) \), we write only \( \int_D u \) instead of \( \int_D u(x)dx \). Given a subset \( D \subset \mathbb{R}^N \), we denote by \( D^c \) the complement of \( D \); that is, \( D^c := \mathbb{R}^N \setminus D \).

As stated in the introduction, it is important to consider the limit problem associated to (\(P\)), namely the autonomous problem

\[-\Delta w = Q_\infty f'(w) \quad \text{in} \quad \mathbb{R}^N, \quad (P_\infty)\]

whose solutions are the critical points of the functional \( I_\infty : D^{1,2}(\mathbb{R}^N) \to \mathbb{R} \) given by

\[
I_\infty(w) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 - \int_{\mathbb{R}^N} Q_\infty f(w).
\]

Let \( \mathcal{N}_\infty \) be the Nehari manifold of \( I_\infty \); that is,

\[
\mathcal{N}_\infty := \{ w \in D^{1,2}(\mathbb{R}^N) \setminus \{0\} : I'_\infty(w)w = 0 \},
\]

and consider the following minimization problem:

\[
c_\infty := \inf_{w \in \mathcal{N}_\infty} I_\infty(w).
\]
In Berestycki and Lions [7] it is proved that \( c_\infty \) is attained. Moreover, by using a result due to Flucher and Müller [14], we get the polynomial decay of the solutions. More specifically, the following proposition holds.

**Proposition 2.1.** The problem \((P_\infty)\) has a positive and radially symmetrical solution \( \omega \in \mathcal{D}^{1,2}(\mathbb{R}^N) \) such that \( I_\infty(\omega) = c_\infty \). Moreover, there exist positive constants \( a_1, a_2 \) such that, for any \( |x| \) large, there hold
\[
a_1|x|^{2-N} \leq \omega(x) \leq a_2|x|^{2-N}, \quad a_1|x|^{1-N} \leq |\nabla \omega(x)| \leq a_2|x|^{1-N}. \tag{2.2}
\]

To get compactness, we shall use the following version of a result due to Struwe [18].

**Lemma 2.2** (splitting lemma). Let \( (u_n) \subset E \) be such that \( I(u_n) \to c, \ I'(u_n) \to 0 \) and \( u_n \rightharpoonup u_0 \) weakly in \( E \). Then \( I'(u_0) = 0 \) and we have either

(a) \( u_n \rightharpoonup u_0 \) strongly in \( E \), or

(b) there exist \( k \in \mathbb{N}, (y^j_n) \in \mathbb{R}^N \) with \( |y^j_n| \to \infty, \ j = 1, \ldots, k, \) and nontrivial solutions \( w^1, \ldots, w^k \) of the problem \((P_\infty)\), such that
\[
I(u_n) \to I(u_0) + \sum_{j=1}^k I_\infty(w^j) \tag{2.3}
\]

and
\[
\|u_n - u_0 - \sum_{j=1}^k w^j(\cdot - y^j_n)\| \to 0.
\]

**Proof.** The proof can be done by using the Orlicz space \( L^p + L^q \) and adapting the arguments presented in [6, Lemma 4.1]. We point out just the main differences in our case. First, we recall that the space \( L^p + L^q \) is that of functions \( v : \Omega \to \mathbb{R} \) such that \( v = v_1 + v_2 \) with \( v_1 \in L^p(\Omega) \) and \( v_2 \in L^q(\Omega) \). It is a Banach space with norm \( \|v\|_{L^p + L^q} := \inf\{\|v_1\|_{L^p} + \|v_2\|_{L^q} : v = v_1 + v_2 \} \). If \( 2 < p < 2^* < q \), by interpolation \( L^{2^*}(\Omega) \subset L^p(\Omega) \cap L^q(\Omega) \). By (2.1) we get the Sobolev embedding \( \mathcal{D}^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega) \). Recalling that \( L^p(\Omega) \cap L^q(\Omega) \subset L^p + L^q \) we obtain \( \mathcal{D}^{1,2}(\Omega) \hookrightarrow L^p + L^q \), with continuous embedding (see [4] for more details).

We start by noticing that, since \( (u_n) \) is bounded in \( L^{2^*}(\Omega) \), \( (f'(u_n)) \) is bounded in \( L^{2N/(N+2)}(\Omega) \). So \( f'(u_n) \rightharpoonup f'(u_0) \) weakly in \( L^{2N/(N+2)}(\Omega) \) (see [8]) and we can use the boundedness of \( Q \) to conclude that
\[
\int_{\Omega} Q(x)f'(u_n)v \to \int_{\Omega} Q(x)f'(u_0)v
\]
for each \( v \in E \). Hence \( I'(u_0) = 0 \). Moreover, using the mean value theorem, \((f_2)\) and an argument similar to the previous one, we obtain for \( \psi_n := u_n - u_0 \) the following:

(i) \( \|\psi_n\|^2 = \|u_n\|^2 - \|u_0\|^2 + o(1) \),

(ii) \( \int_\Omega Q(x)f'(\psi_n) = \int_\Omega Q(x)f'(u_n) - \int_\Omega Q(x)f'(u_0) + o(1) \),

(iii) \( \int_\Omega Q(x)f(\psi_n) = \int_\Omega Q(x)f(u_n) - \int_\Omega Q(x)f(u_0) + o(1) \),

where \( o(1) \) denotes a quantity approaching zero as \( n \to \infty \). Note that up to this point, assuming \((f_2)\), there is no need to step up to the space \( L^p + L^q \). Only now, in order to obtain a sequence \((y_n^1) \in \mathbb{R}^N \) with \( \|y_n^1\| \to \infty \), and \( \psi_n(x + y_n^1) \to w^1(x) \neq 0 \) we make use of the Orlicz space.

We decompose \( \mathbb{R}^N \) into \( N \)-dimensional hypercubes \( Q_i \) having edge length \( L \), to be chosen appropriately later. Then we set

\[
\Gamma_n := \{ x \in \Omega : |\psi_n(x)| > 1 \}
\]

\[
Q_{i,n}^+ := Q_i \cap \Gamma_n, \quad Q_{i,n}^- := Q_i \cap (\mathbb{R}^N \setminus \Gamma_n).
\]

From above, by item (i), the facts that \( I'(u_n)u_n \to 0 \) and \( I'(u_0) = 0 \) and \((f_2)\) we get

\[
\|\psi_n\|^2 = \|u_n\|^2 - \|u_0\|^2 + o(1)
\]

\[
= \int_\Omega Q(x)f'(u_n) - \int_\Omega Q(x)f'(u_0) + o(1) = \int_\Omega Q(x)f'(\psi_n) + o(1)
\]

\[
\leq c_1 \sum_i \left( \|\psi_n\|_{L^p(Q_{i,n}^+)}^p + \|\psi_n\|_{L^q(Q_{i,n}^-)}^q \right) \leq c_1 d_n \|\psi_n\|^2,
\]

(2.4)

where \( d_n = \sup_i \left\{ \max \left\{ \|\psi_n\|_{L^p(Q_{i,n}^+)}^{p-2}, \|\psi_n\|_{L^q(Q_{i,n}^-)}^{q-2} \right\} \right\} \).

By (2.4) it follows that \( d_n \not\to 0 \) as \( n \to \infty \). So, there exists \( \tilde{d} > 0 \) and a sequence \((i_n) \subset \mathbb{N} \) such that for the hypercubes \( Q_{i,n} = Q_{i,n} \) we have the following inequality:

\[
0 < \tilde{d} < \max \left\{ \|\psi_n\|_{L^p(Q_{i,n}^+)}^{p-2}, \|\psi_n\|_{L^q(Q_{i,n}^-)}^{q-2} \right\}.
\]

(2.5)

Now we call \( y_{i,n}^1 \) the center of the hypercube \( Q_{i,n} \). If \((y_{i,n}^1)\) were bounded, by passing to a subsequence, \( y_{i,n}^1 \) would be in the same \( Q_j \), so they coincide. Since \( \|\psi_n\| \) is bounded, up to a subsequence \( \psi_n \) converges to \( \psi \)
strongly in \( L^p(Q_j) \) and weakly in \( D^{1,2}(Q_j) \). We claim that \( \psi \neq 0 \). Indeed, if \( \|\psi_n\|_{L^p(Q_j)} \to 0 \) as \( n \to \infty \), then

\[
\|\psi_n\|_{L^p(Q_j)} \to 0 \quad \text{and} \quad \int_{Q_j^-} |\psi_n|^q \, dx < \int_{Q_j^+} |\psi_n|^p \, dx \to 0,
\]

thus contradicting (2.5). It follows from the claim that \( \psi_n \rightharpoonup \psi \neq 0 \) in \( D^{1,2}(Q_j) \). But \( \psi_n = u_n - u_0 \to 0 \) in \( D^{1,2}(\Omega) \) leads again to a contradiction. This way we conclude that \( |y_1^n| \to \infty \). Now we call \( w^1 \) the weak limit in \( D^{1,2}(\mathbb{R}^N) \) of the sequence \( \psi_n(\cdot + y_1^n) \) and the rest of the proof follows as in [6, Lemma 4.1] by iterating this procedure. □

We now recall that \( I \) is said to satisfy the Palais-Smith condition at level \( c \in \mathbb{R} \) (\((PS)\) for short) if any sequence \((u_n) \subset E\) such that \( I(u_n) \to c \) and \( I'(u_n) \to 0 \) possesses a convergent subsequence.

**Corollary 2.3.** The functional \( I \) satisfies \((PS)\) for any \( c < c_\infty \).

**Proof.** Let \((u_n) \subset E\) be such that \( I(u_n) \to c < c_\infty \) and \( I'(u_n) \to 0 \). The second inequality in \((f_3)\) and standard arguments show that \((u_n) \subset E\) is bounded. Hence, up to a subsequence, \( u_n \rightharpoonup u_0 \) weakly in \( E \). By Lemma 2.2 we have \( I'(u_0) = 0 \). We conclude from \((f_3)\) that

\[
I(u_0) = \int_{\Omega} Q(x) \left( \frac{1}{2} f'(u_0) u_0 - f(u_0) \right) \geq 0.
\]

If \( u_n \not\to u_0 \) in \( E \), we can invoke Lemma 2.2 again to obtain \( k \in \mathbb{N} \) and nontrivial solutions \( w^1, \ldots, w^k \) of \((P_\infty)\) satisfying

\[
\lim_{n \to \infty} I(u_n) = c = I(u_0) + \sum_{j=1}^{k} I_\infty(w^j) \geq kc_\infty \geq c_\infty,
\]

contrary to the hypothesis. Hence \( u_n \to u_0 \) strongly in \( E \). □

We close this section by stating a technical result which will be useful in the sequel. The proof can be found in [1, Lemma 3.1].

**Lemma 2.4.** Let \( g \in C^2(\mathbb{R}, [0, \infty)) \) be a convex and even function such that \( g(0) = 0 \) and \( g'(s) \geq 0 \) for all \( s \in [0, \infty) \). Then, for each \( s, t \geq 0 \) there holds

\[
|g(s - t) - g(s) - g(t)| \leq 2(g'(s)t + g'(t)s).
\]
3. Positive solution

We devote this section to the proof of Theorem 1.1. We look for critical points of the functional \( I \) and we start by introducing the Nehari manifold of \( I \) defined as \( \mathcal{N} := \{ u \in E \setminus \{0\} : I'(u)u = 0 \} \). Let

\[
c_+ := \inf_{u \in \mathcal{N}} I(u).
\]

In what follows we present some properties of \( c_+ \) and \( \mathcal{N} \). For the proofs we refer to [21, Chapter 4]. First we note that, in view of \((f_3)\), the following holds:

the function \( f'(s)/s \) is increasing in \((0, \infty)\).  

Hence, for any \( u \in E \setminus \{0\} \), there exists a unique \( t_u > 0 \) such that \( t_u u \in \mathcal{N} \).

Since \( |f(s)| \leq c_1 |s|^{2^*} \), we can use Sobolev embedding to prove that the origin is a local minimum of \( I \). Moreover, condition \((f_3)\) provides \( r, c_2 > 0 \) such that \( f(s) \geq c_2 |s|^\theta \) for each \( |s| > r \).

Thus, if \( t > 1 \) and \( u \in C_0^\infty(\Omega) \) is a nonnegative function such that the set \( \{ x \in \Omega : u(x) \geq r \} \) has positive measure, we get

\[
\int_{\Omega} Q(x)f(tu) \geq c_2 \int_{\{ x : tu(x) \geq r \}} Q(x)f(tu) \geq c_2 t^\theta \int_{\{ x : u(x) \geq r \}} Q(x)|u|^\theta.
\]

Since \( \theta > 2 \) and \( \inf_{x \in \Omega} Q(x) > 0 \), we conclude that

\[
I(tu) \leq \frac{t^2}{2} \|u\|^2 - c_2 t^\theta \int_{\{ x : u(x) \geq r \}} Q(x)|u|^\theta \to -\infty
\]
as \( t \to \infty \). These observations show that \( I \) has the mountain pass geometry. By using \((f_1), (f_2)\) and the same arguments presented in the proof of [21, Theorem 4.2], we can prove that \( c_+ \) is positive, it coincides with the mountain pass level of \( I \) and has the following characterization:

\[
c_+ = \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)) = \inf_{u \in E \setminus \{0\}} \max_{t \geq 0} I(tu) > 0,
\]

where \( \Gamma := \{ g \in C([0,1], E) : g(0) = 0, \ I(g(1)) < 0 \} \).

In what follows we use the above remarks to obtain a relation between \( c_+ \) and \( c_\infty \).

**Proposition 3.1.** Suppose that \( Q \) satisfies \((Q_1) - (Q_3)\). Then \( 0 < c_+ < c_\infty \).
**Proof.** Let \( \omega \) be a ground state solution of problem \((P_\infty)\) and define \( \omega_n(x) := \omega(x - x_n) \), where \( x_n := (0, \ldots, n) \). If \( \gamma_n \in (0, \infty) \) is such that

\[
I(\gamma_n \omega_n) = \max_{t \geq 0} I(t \omega_n),
\]

then

\[
\gamma_n^2 \int_\Omega |\nabla \omega_n|^2 = \int_\Omega Q(x) f'(\gamma_n \omega_n)(\gamma_n \omega_n).
\]  

(3.3)

It is easy to check that \((\gamma_n)\) is bounded and therefore, up to a subsequence, \( \gamma_n \to \gamma \). Since \( |x_n| \to \infty \), it follows from (3.3) that

\[
\int_{\mathbb{R}^N} |\nabla \omega|^2 = \int_{\mathbb{R}^N} Q_\infty \frac{f'(\gamma \omega)}{\gamma \omega} \omega^2.
\]  

(3.4)

Recalling that \( \omega \) is a ground state solution of \((P_\infty)\), we get

\[
\int_{\mathbb{R}^N} Q_\infty \frac{f'(\gamma \omega)}{\gamma \omega} \omega^2 = \int_{\mathbb{R}^N} Q_\infty \frac{f'(\omega)}{\omega} \omega^2.
\]

This and (3.1) implies that \( \gamma = 1 \).

We now notice that

\[
c_+ \leq I(\gamma_n \omega_n) = \frac{\gamma_n^2}{2} \int_\Omega |\nabla \omega_n|^2 - \int_\Omega Q(x) f(\gamma_n \omega_n)
\]

\[
= I_\infty(\gamma_n \omega_n) + t_n \left( \frac{s_n}{t_n} - \frac{\gamma_n^2}{2} \right)
\]  

(3.5)

where \( t_n \) and \( s_n \) are given by

\[
t_n := \int_{\Omega^c} |\nabla \omega_n|^2, \quad s_n := \int_{\Omega^c} Q_\infty f(\gamma_n \omega_n) + \int_\Omega (Q_\infty - Q(x)) f(\gamma_n \omega_n).
\]

Since \( \gamma_n \to 1 \), the above estimate shows that \( c_+ < c_\infty \) provided the following claim holds.

**Claim:** \( s_n/t_n \to 0 \) as \( n \to \infty \). For the proof of the claim, we first notice that the decay of \( \omega \) given by (2.2) enable us to estimate \( t_n \), for \( n \) large, as follows:

\[
t_n = \int_{\Omega^c} |\nabla \omega_n|^2 = \int_{\Omega^c} |\nabla \omega(x - x_n)|^2 \geq a_1 \int_{\Omega^c} \frac{1}{|x - x_n|^{2(N-1)}}.
\]

Since \( |x - x_n| \leq |x| + |n| \leq 2n \), we obtain

\[
t_n \geq a_1 \frac{|\Omega^c|}{(2n)^{2(N-1)}} = \frac{c_1}{n^{2(N-1)}}.
\]
We proceed now with the estimation of $s_n$. Let $R > 0$ be such that $\Omega^c \subset B_R(0)$. In view of (2.2), we have that

$$\gamma_n\omega_n(x) = \gamma_n\omega(x-x_n) \leq 1,$$

for $x \in B_R(0)$ and $n$ large enough. Hence, we can use $(f_2)$ and proceed as above to get

$$\int_{\Omega^c} Q\infty f(\gamma_n\omega_n) \leq c_2 \int_{B_R(0)} |\omega_n|^q \leq c_3 \int_{B_R(0)} \frac{1}{|x-x_n|^{(N-2)q}} = c_3 \int_{B_R(-x_n)} \frac{1}{|y|^{(N-2)q}} dy.$$

Since $B_R(-x_n) \subset \{y \in \mathbb{R}^N : n - R \leq |y| \leq n + R\}$, we obtain

$$\int_{\Omega^c} Q\infty f(\gamma_n\omega_n) \leq c_4 \int_{n-R}^{n+R} r^{N-1} r^{(N-2)q} dr \leq \frac{c_4}{(n+R)^{q-N}}, \quad (3.6)$$

for $n$ large enough.

Since $Q \in L^\infty(\Omega)$ we can similarly compute

$$\int_{\Omega \cap B_{n/2}(0)} (Q\infty - Q(x)) f(\gamma_n\omega_n) \leq c_5 \int_{B_{n/2}(0)} \frac{1}{|x-x_n|^{(N-2)q}} \leq \frac{c_6}{n^{(N-2)q-N}},$$

for large $n$. Moreover, by using $(Q_3)$, we get

$$\int_{\Omega \cap B_{n/2}(0)^c} (Q\infty - Q(x)) f(\gamma_n\omega_n) = \int_{B_{n/2}(0)^c} (Q\infty - Q(x)) f(\gamma_n\omega_n)$$

$$\leq c_7 \int_{B_{n/2}(0)^c} |x|^{-\gamma} f(\gamma_n\omega_n) \leq \frac{c_8}{n^\gamma} \int |\omega_n|^2 \leq \frac{c_9}{n^\gamma}.$$

All together, the above estimates provide

$$\frac{s_n}{t_n} \leq c_{10} \left( \frac{n^{2(N-1)}}{(n+R)^{q-N}} + \frac{n^{2(N-1)}}{n^{(N-2)q-N}} + \frac{n^{2(N-1)}}{n^\gamma} \right)$$

for $n$ large. Since $\gamma > 2(N-1)$ and $q > \frac{2N}{N-2} + 1$, we conclude that $s_n/t_n \to 0$. The proof is finished.

We are now ready to obtain the ground state solution of $(P)$.

**Proof of Theorem 1.1.** Since $I$ satisfies the geometry of the mountain pass theorem there exists a sequence $(u_n) \subset E$ such that $I(u_n) \to c_+$ and $I'(u_n) \to 0$. Proposition 3.1 and Corollary 2.3 imply that the sequence $(u_n)$ strongly converges to a function $u \in E$ such that $I(u) = c_+ > 0$ and $I'(u) = 0$. Clearly $u \neq 0$ and therefore $u$ is a ground state solution of $(P)$. Since we
are interested in positive solutions, we can suppose that \( f'(s) = 0 \) for any \( s \leq 0 \). Setting \( u^- := \max\{-u, 0\} \) and recalling that \( I'(u^-)u^- = 0 \), we have \( u^- \equiv 0 \). It follows from elliptic regularity and the strong maximum principle that \( u > 0 \) in \( \Omega \). The theorem is proved.

\( \square \)

**Remark 3.2.** Let \( u \in E \) be the positive solution of \((P)\) given by Theorem 1.1. Since \( Q(x) \) is bounded and \( f(u) \leq c_1 |u|^{2^*} \), we can use a decay result due to Egnell [12, Theorem 2] to conclude that, for some constant \( a_3 > 0 \), there holds
\[
  u(x) \leq a_3 |x|^{2-N} \quad \text{for each} \quad x \in \Omega.
\]

**4. Sign-changing solution**

We start by introducing the closed sets
\[
  \mathcal{N}_\pm := \{u \in E : u^+ \not\equiv 0, u^- \not\equiv 0, I'(u^+)u^+ = 0 = I'(u^-)u^-\}.
\]

Note that any solution of \((P)\) which belongs to \( \mathcal{N}_\pm \) changes sign. Moreover, it is easy to check that \( I \) is bounded from below in \( \mathcal{N}_\pm \). Thus we can consider the following minimization problem:
\[
  c_\pm := \inf_{u \in \mathcal{N}_\pm} I(u).
\]

As in the last section, this new minimizer is related to the previous one.

**Proposition 4.1.** Suppose that \( Q \) satisfies \((Q_1)\), \((Q_2)\) and \((\hat{Q}_3)\). Then
\[
  0 < c_\pm < c_+ + c_\infty.
\]

**Proof.** Let \( \omega \) be given by Proposition 2.1 and define \( \omega_n(x) := \omega(x - x_n) \), where \( x_n := (0, ..., 0, n) \). From now on we denote by \( u \) a positive ground state solution of \((P)\) given by Theorem 1.1. For any \( \alpha, \beta > 0 \) we consider the functions
\[
  h^\pm(\alpha, \beta, n) := \int_\Omega |\nabla(\alpha u - \beta \omega_n)^\pm|^2 - \int_\Omega Q(x)f'(\alpha u - \beta \omega_n)\pm(\alpha u - \beta \omega_n)^\pm.
\]

Recalling that \( I'(u)u = 0 \) and using (3.1) we get
\[
  \int_\Omega (|\nabla(u/2)|^2 - Q(x)f'(u/2)(u/2)) = \int_\Omega Q(x)\left(\frac{f'(u)}{u} - \frac{f'(u/2)}{u/2}\right)(\frac{u}{2})^2 > 0
\]

and
\[
  \int_\Omega |\nabla(2u)|^2 - \int_\Omega Q(x)f'(2u)(2u) < 0.
\]
Claim 1. For \( n \) sufficiently large there holds
\[
\int_{\Omega} |\nabla (\omega_n/2)|^2 - \int_{\Omega} Q(x)f'(\omega_n/2) (\omega_n/2) > 0, \tag{4.4}
\]
\[
\int_{\Omega} |\nabla (2\omega_n)|^2 - \int_{\Omega} Q(x)f'(2\omega_n)(2\omega_n) < 0. \tag{4.5}
\]
We only prove (4.4), since the other inequality can be proved in the same way. First, notice that
\[
\int_{\Omega} |\nabla (\omega_n/2)|^2 - \int_{\Omega} Q(x)f'(\omega_n/2) (\omega_n/2) = \delta + A_n + C_n, \tag{4.6}
\]
where
\[
\delta := \int_{\mathbb{R}^N} |\nabla (\omega_n/2)|^2 - \int_{\mathbb{R}^N} Q_\infty f'(\omega_n/2)(\omega_n/2),
\]
\[
A_n := \int_{\Omega} (Q_\infty - Q(x)) f'(\omega_n/2)(\omega_n/2),
\]
and
\[
C_n := \int_{\Omega_c - x_n} Q_\infty f'(\omega/2)(\omega/2) - \int_{\Omega_c - x_n} |\nabla (\omega/2)|^2.
\]
It follows from (3.1) that \( \delta > 0 \). Thus, it suffices to check that \( A_n, C_n \to 0 \) as \( n \to \infty \). In order to do this we take \( \epsilon > 0 \) and use \((Q_2)\) to choose \( R > 0 \) in such a way that \( \Omega_c \subset B_R(0) \) and \( |Q(x) - Q_\infty| < \epsilon \) on \( B_R(0)^c \). This and \((f_2)\) provide
\[
\int_{B_R(0)^c} (Q_\infty - Q(x)) f'(\omega_n/2)(\omega_n/2) \leq c_1 \epsilon \int_{B_R(0)^c} |\omega_n|^2^* \leq c_2 \epsilon. \tag{4.7}
\]
On the other hand, since \( Q \) is bounded and \( \omega \in L^{2^*}(\mathbb{R}^N) \), we can use \((f_2)\) again and Lebesgue’s theorem to get
\[
\int_{\Omega \cap B_R(0)} (Q_\infty - Q(x)) f'(\omega_n/2)(\omega_n/2) \leq c_3 \int_{(\Omega \cap B_R(0)) - x_n} |\omega|^2^* = o(1).
\]
This and (4.7) show that \( A_n = o(1) \). Proceeding as in the above equation we can also check that \( C_n = o(1) \). This proves (4.4) and therefore the claim holds.

Since \( u(x) \to 0 \) as \( |x| \to \infty \), it follows from (4.2)-(4.5) that there exists \( n_0 > 0 \) such that
\[
h^+(1/2, \beta, n) > 0, \quad h^+(2, \beta, n) < 0,
\]
for \( n \geq n_0 \) and \( \beta \in [1/2, 2] \). Now, for each \( \alpha \in [1/2, 2] \), we have
\[
h^-(\alpha, 1/2, n) > 0, \quad h^-(\alpha, 2, n) < 0.
\]
Hence, we can apply a variant of the mean value theorem due to Miranda [16] 
(see also [20, Theorem 1]), to obtain \(a^*, \beta^* \in [1/2, 2]\) such that 
\(h^\pm(a^*, \beta^*, n) = 0\), for any \(n \geq n_0\). Thus, \(a^*u - \beta^*\omega_n \in N_\pm\) for \(n \geq n_0\). In view of 
the definition of \(c_\pm\), it suffices to show that 
\[
\sup_{\frac{1}{2} \leq \alpha, \beta \leq 2} I(\alpha u - \beta \omega_n) < c_+ + c_\infty,
\]
for some \(n \geq n_0\).

So, recalling that \(u\) is a positive solution of \((P)\), making some calculations 
and using (3.6), we get 
\[
I(\alpha u - \beta \omega_n) = \frac{1}{2} \int_\Omega \left( |\nabla(\alpha u)|^2 + |\nabla(\beta \omega_n)|^2 \right) - \alpha \beta \int_\Omega \nabla u \cdot \nabla \omega_n 
- \int_\Omega Q(x)f(\alpha u - \beta \omega_n) 
\leq I(\alpha u) + I_\infty(\beta \omega_n) + \int_{\Omega^c} Q_\infty f(\beta \omega_n) 
- \frac{1}{2} \int_{\Omega^c} |\nabla(\beta \omega_n)|^2 - s_n - t_n 
\leq c_+ + c_\infty + \frac{c_4}{(n + R)(N - 2)q - N} - s_n - t_n,
\]
where \(R > 0\) is such that \(\Omega^c \subset B_R(0)\), \(c_4 > 0\) and \(s_n\) and \(t_n\) are given by 
\[
s_n := \int_\Omega (Q(x) - Q_\infty)f(\beta \omega_n)
\]
and 
\[
t_n := \int_\Omega Q(x)(f(\alpha u - \beta \omega_n) - f(\alpha u) - f(\beta \omega_n)).
\]
By using \((\hat{Q}_3)\) and \((f_2)\), we obtain 
\[
s_n \geq C \int_\Omega |x|^{-\gamma}f(\beta \omega_n) \geq C \int_{\Omega - x_n} |x + x_n|^{-\gamma}f(\beta \omega) 
\geq C \int_{(\Omega - x_n) \cap \{x : \omega(x) > 1/\beta\}} |x - x_n|^{-\gamma}|\beta \omega|^p 
= C \int_{\{x : \omega(x) > 1/\beta\}} |x - x_n|^{-\gamma}|\beta \omega|^p,
\]
for \(n\) large, where we have used the boundedness of \(\{x : \omega(x) > 1/\beta\}\). If 
\(R_1 > 0\) is such that \(\{x : \omega(x) > 1/\beta\} \subset B_{R_1}(0)\) then, for each \(x \in \{x : \omega(x) > 1/\beta\}\),
\(\omega(x) > 1/\beta\) there holds \(|x + x_n|^{-\gamma} \geq (n + R_1)^{-\gamma}\). Thus, for \(n\) large, we have that
\[
s_n \geq c_5 \frac{c_6}{(n + R_1)} \int_{\{x: \omega(x) > 1/\beta\}} |\omega|^p = c_6 \frac{(n + R_1)^{-\gamma}}{(n + R_1)}.
\]
(4.9)

In order to estimate \(t_n\) we apply Lemma 2.4 and \((f_2)\) to obtain
\[
|t_n| \leq 2\|Q\|_{L^\infty(\Omega)} \int_\Omega (f'(\alpha u)\beta \omega_n + f'(\beta \omega_n)\alpha u)
\leq c_7 \int_\Omega (u^{2* - 1}\omega_n + u_n^{2* - 1}u).
\]
(4.10)

By using Holder’s inequality and (2.2), we obtain
\[
\int_{\Omega \cap B_n/2(0)} u^{2* - 1}\omega_n \leq \|u\|^{2* - 1}_{L^{2*}(\Omega)} \left( \int_{B_n/2(0)} \omega(x - x_n)^2 \right)^{1/2*}
\leq c_8 \left( \int_{B_n/2(0)} \frac{1}{|x - x_n|^{2N}} \right)^{1/2*}.
\]
(4.11)

But on \(B_n/2(0)\) we have that \(|x - x_n| \geq n/2\), and therefore
\[
\int_{\Omega \cap B_n/2(0)} u^{2* - 1}\omega_n \leq c_8 \left( \int_{B_n/2(0)} \frac{2^{2N}}{n^{2N}} \right)^{1/2*} = \frac{c_9}{n^{(N-2)/2}}.
\]
On the other hand (3.7) provides, for \(n\) large,
\[
\int_{\Omega \cap B_n/2(0)} u^{2* - 1}\omega_n \leq \|\omega\|_{L^{2*}(\mathbb{R}^N)} \left( \int_{B_n/2(0)} u^{2} \right)^{(2* - 1)/2*}
\leq c_{10} \left( \int_{n/2}^{\infty} \frac{r^{N-1}}{r^{2N}} \right)^{(N+2)/(2N)} = \frac{c_{11}}{n^{(N+2)/2}}.
\]
The above expression and (4.11) imply that, for \(n\) large,
\[
\int_{\Omega} u^{2* - 1}\omega_n \leq \frac{c_{12}}{n^{(N-2)/2}}.
\]
Arguing as above, we can check that the same kind of estimate holds for \(\int_{\Omega} \omega_n^{2* - 1}u\), and therefore it follows from (4.10) that, for \(n\) large,
\[
|t_n| \leq \frac{c_{13}}{n^{(N-2)/2}}.
\]

The above estimate, (4.9) and (4.8) provide, for \(n\) large,
\[
I(\alpha u - \beta \omega_n) \leq c_+ + c_\infty + c_{14} \left( \frac{1}{(n + R)(N-2)q-N} - \frac{1}{(n + R_1)^\gamma} + \frac{1}{n^{(N-2)/2}} \right).
\]
Since \( \gamma < \frac{N-2}{2} < (N-2)q - N \) we have that, for some \( n \geq n_0 \) large enough,
\[
\sup_{\frac{1}{2} \leq \alpha, \beta \leq 2} I(\alpha u - \beta \omega_n) < c_+ + c\infty.
\]
This concludes the proof. \( \square \)

The next result is a version of a result of [10] (see also [22, 9, 15]).

**Proposition 4.2.** There exists a sequence \( (u_n) \) in \( \mathcal{N}_\pm \) satisfying
\[
I(u_n) \to c_\pm \quad \text{and} \quad I'(u_n) \to 0.
\] (4.12)

Before making some comments about the proof of this proposition we need some remarks and a technical lemma. We start by noticing that, since
\[
|f'(s)| \leq c_1 |s|^{2^* - 1},
\]
we can use the embedding \( D^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega) \) to obtain \( \rho > 0 \) such that
\[
\|u_\pm\| \geq \rho \quad \text{for each} \quad u \in \mathcal{N}_\pm.
\] (4.13)
Moreover, the following holds.

**Lemma 4.3.** There exists a constant \( c_1 > 0 \) such that
\[
\int_{\Omega} Q(x)(f''(u_\pm)(u_\pm)^2 - f'(u_\pm)u_\pm) \geq c_1 \quad \text{for each} \quad u \in \mathcal{N}_\pm.
\]

**Proof.** In view of \((\hat{f}_3)\) we have, for each \( u \in \mathcal{N}_\pm, \)
\[
\int_{\Omega} Q(x)(f''(u_\pm)(u_\pm)^2 - f'(u_\pm)u_\pm) \geq C \left( \int_{\{x: u_\pm(x) \geq 1\}} (u_\pm)^p + \int_{\{x: u_\pm(x) < 1\}} (u_\pm)^q \right).
\]
Thus, it suffices to show that the right-hand side of the above expression is bounded from below by a positive constant, uniformly for \( u \in \mathcal{N}_\pm \). Suppose, by contradiction, that this is false. Then there exists \( (u_n) \subset \mathcal{N}_\pm \) such that
\[
\lim_{n \to \infty} \left( \int_{\{x: u_n^+(x) \geq 1\}} (u_n^+)^p + \int_{\{x: u_n^+(x) < 1\}} (u_n^+)^q \right) = 0.
\]
However, in view of the definition of \( \mathcal{N}_\pm \) and \((f_2)\) we get
\[
\|u_n^+\|^2 = \int_{\Omega} Q(x)f'(u_n^+)u_n^+ \leq \|Q\|_{L^\infty(\Omega)} \left( \int_{\{x: u_n^+(x) \geq 1\}} (u_n^+)^p + \int_{\{x: u_n^+(x) < 1\}} (u_n^+)^q \right) \to 0,
\]
which contradicts (4.13). The proof for \( u^- \) is analogous. \( \square \)

We are now ready to prove Proposition 4.2. Since the proof follows some known arguments, we only present a sketch of it. The basic idea refers to the work of Tarantello [19].
**Proof of Proposition 4.2.** By using the Ekeland variational principle we obtain a sequence \((u_n) \subset N_{\pm}\) such that \(I(u_n) \to c_{\pm}\) as \(n \to \infty\) and
\[
I(v) \geq I(u_n) - \frac{1}{n} \|v - u_n\| \quad \text{for each} \quad v \in N_{\pm}.
\] (4.14)

For each \(\varphi \in E, n \in \mathbb{N},\) we introduce the \(C^1\)-functions \(h^{\pm}_n : \mathbb{R}^3 \to \mathbb{R}\) given by
\[
h^{\pm}_n(t, s, l) := \int_{\Omega} |\nabla (u_n + t\varphi + su^+_n + lu^-_n)\|^2
- \int_{\Omega} Q(x)f'(u_n + t\varphi + su^+_n + lu^-_n)^2(u_n + t\varphi + su^+_n + lu^-_n)^\pm.
\]

Note that \(h^{\pm}_n(0, 0, 0) = 0, (\partial h^{\pm}_n/\partial t)(0, 0, 0) = 0\) and \((\partial h^{-}_n/\partial s)(0, 0, 0) = 0\).

Since \(I'(u^+_n)u^-_n = 0,\) we can use \((\hat{f}_3)\) and Lemma 4.3 to get
\[
\frac{\partial h^{+}_n}{\partial s}(0, 0, 0) = 2 \int_{\Omega} |\nabla u^+_n|^2 - \int_{\Omega} Q(x)(f''(u^+_n)u^+_n)^2 + f'(u^+_n)u^+_n)
= \int_{\Omega} Q(x)(f'(u^+_n)u^+_n - f''(u^+_n)(u^+_n)^2 \leq -c_1 < 0.
\]

Similarly, we can check that
\[
\frac{\partial h^{-}_n}{\partial l}(0, 0, 0) < 0.
\]

Hence, we can apply the implicit function theorem to obtain \(\delta_n > 0\) and two \(C^1\)-functions \(s_n, l_n : (-\delta_n, \delta_n) \to \mathbb{R}\) such that \(s_n(0) = l_n(0) = 0\) and
\[
h^{\pm}_n(t, s_n(t), l_n(t)) = 0.
\] (4.15)

We now notice that
\[
s'_n(0) = \frac{(\partial h^{+}_n/\partial t)(0, 0, 0)}{(\partial h^{+}_n/\partial s)(0, 0, 0)}
= -2 \int_{\Omega} \nabla u^+_n \nabla \varphi - \int_{\Omega} Q(x)(f'(u^+_n)u^+_n + f(u^+_n))\varphi
\int_{\Omega} Q(x)(f'(u^+_n)u^+_n)^2 - f(u^+_n)u^+_n).
\]

The above inequality, the boundedness of \((u_n), (f_2)\) and Lemma 4.3 imply that \((s'_n(0))\) is bounded. In the same way it can be proved that \((l'_n(0))\) is bounded. Thus, we can follow the same lines of the proof of [15, Proposition 4.2] to obtain
\[
\|I'(u_n)\| \leq \frac{c_2}{n},
\]
for some $c_2 > 0$. This concludes the proof. □

We prove below our multiplicity results.

**Proof of Theorem 1.2.** Let $(u_n) \subset \mathcal{N}_\pm$ be the sequence given by the above proposition. We can easily check that $(u_n)$ is bounded in $E$. Hence, up to a subsequence, $u_n \rightharpoonup u_0$ weakly in $E$ with $I'(u_0) = 0$. In view of Lemma 2.2 we have either $u_n \to u_0$ strongly in $E$ or there exists $w^1, \ldots, w^k$ nontrivial solutions of $(P_\infty)$ satisfying the alternative (b) of Lemma 2.2. Since $c_+ < c_\infty$ it follows from (2.3) that $k \leq 1$.

Suppose that $u_0 \equiv 0$. In this case, since $c_\pm > 0$, we have that $k = 1$ and therefore

$$\|u_n - w^1(\cdot - y^1_n)\| \to 0.$$  

Since $|y^1_n| \to \infty$ and $(u_n) \subset \mathcal{N}_\pm$, the convergence above and (4.13) imply that $(w^1)^\pm \in \mathcal{N}_\infty$. Hence,

$$c_+ + c_\infty > c_\pm = I_\infty(w^1) = I_\infty((w^1)^+) + I_\infty((w^1)^-) \geq 2c_\infty,$$

contradicting $c_+ < c_\infty$. Thus $u_0 \not\equiv 0$ and we can use $c_\pm < c_+ + c_\infty$ again to conclude that $k = 0$, that is, $u_n \to u_0$ strongly in $E$. It follows from (4.13) that $u_0 \in \mathcal{N}_\pm$ is a sign-changing solution of $(P)$. □

**Proof of Theorem 1.3.** Since $\Omega = \mathbb{R}^N$ the integrals of the functionals $I$ and $I_\infty$ are taken over the whole space. Hence, by $(Q_3)$, we have that $c_1 < c_\infty$. The proof now follows the same lines of the calculations performed in this section only noticing that they hold just with the (weak) hypothesis that $2 < p < 2^* < q$. We omit the details. □

**References**


