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On a class of nonlinear elliptic equations with fast increasing weight and critical growth ☆

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ABSTRACT

We are concerned with the existence of rapidly decaying solutions for the equation

$$-\mathrm{div}\big(K(x)\nabla u\big) = \lambda K(x)|x|^{\beta}|u|^{q-2}u + K(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N,$$

where $N \geqslant 3$, $2 \leqslant q < 2^* := 2N/(N-2)$, $\lambda > 0$ is a parameter, $K(x) := \exp(|x|^{\alpha}/4)$, $\alpha \geqslant 2$ and the number β is given by $\beta := (\alpha - 2)\frac{(2^*-q)}{(2^*-2)}$. We obtain a positive solution if $2 < q < 2^*$ and a sign changing solution if $q = 2$. The existence results depend on the values of the parameter λ. In the proofs we apply variational methods.

1. Introduction

We consider the nonlinear equation

$$-\mathrm{div}\big(K(x)\nabla u\big) = \lambda K(x)|x|^{\beta}|u|^{q-2}u + K(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \tag{P}$$

where $N \geqslant 3$, $2 \leqslant q < 2^* := 2N/(N-2)$, $\lambda > 0$ is a parameter, $K(x) := \exp(|x|^{\alpha}/4)$, $\alpha \geqslant 2$ and the number β is given by

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$$\beta := (\alpha - 2)\frac{(2^* - q)}{(2^* - 2)}.$$

As pointed out in [7], one of the motivations for studying the above equation relies on the fact that, for $\alpha = q = 2$ and $\lambda = (N-2)/(N+2)$, it appears when one tries to find self-similar solutions

$$v(x,t) = t^{\frac{2-N}{N+2}} u\bigl(xt^{\frac{-1}{2}}\bigr)$$

to the parabolic equation

$$v_t - \Delta v = |v|^{\frac{4}{N-2}} v, \quad \mathbb{R}^N \times (0, +\infty).$$

The radially symmetric case with $\alpha = q = 2$ was considered in [1]. As far as we know, the first variational approach was done by Escobedo and Kavian in [6]. In that article the authors have considered $\alpha = q = 2$ and $N \geqslant 3$, and have proved that the existence of positive solutions is related with the interaction of the parameter λ with the first positive eigenvalue of the associated linear problem

$$-\operatorname{div}\bigl(K(x)\nabla u\bigr) = \lambda K(x)|x|^{\alpha-2}u, \quad x \in \mathbb{R}^N. \tag{LP}$$

Among other results, they have noticed a dichotomy in the existence range of λ for $N = 3$, relative to space dimensions $N \geqslant 4$. More precisely, for $N \geqslant 4$, there is a solution if and only if $\lambda \in (N/4, N/2)$. If $N = 3$, there is a positive solution for $\lambda \in (1, 3/2)$, and there is no solution for $\lambda \leqslant 3/4$ and $\lambda \geqslant 3/2$.

Later on, many authors considered the case $\alpha = q = 2$ and addressed questions of existence, symmetry and asymptotic behavior of solutions of (P), of the associated parabolic equation and its variants (see [8,10,12,9] and references therein). We also quote the paper of Ohya [11], where some results for a p-Laplacian type operator can be found.

Recently, Catrina, Furtado and Montenegro [4] have obtained some results for $\alpha \geqslant 2$ and $q = 2$. After calculating the first eigenvalue of (LP) as $\lambda_1 = \alpha(N - 2 + \alpha)/4$, they have proved that, if $2 \leqslant \alpha \leqslant N - 2$, then the problem (P) has a positive solution if, and only if, $\lambda \in (\lambda_1/2, \lambda_1)$. If $\alpha > N - 2$ and $\lambda \in (\alpha^2/4, \lambda_1)$ then the problem (P) has a positive solution. Moreover, also in this last case, the problem has no solution if $\lambda \leqslant \lambda_1/2$ or $\lambda \geqslant \lambda_1$. So, if $\alpha > 2$, the critical dimension of the problem depends on the value of α.

Due to the presence of the critical Sobolev exponent in (P), it is natural to make a parallel with the Brezis and Nirenberg problem

$$-\Delta u = \lambda|u|^{q-2}u + |u|^{2^*-2}u, \quad u \in H_0^1(\Omega), \tag{BN}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $N \geqslant 3$. The aforementioned results of [6,4] can be viewed as versions of those ones presented in [2] for the above problem when $q = 2$. The nonexistence results for $\lambda \geqslant \lambda_1$ are a consequence of a Pohozaev type identity. In the case that $2 < q < 2^*$, this identity does not give any information. So, we can expect existence of solution for any $\lambda > 0$. A result in this direction to the problem (BN) was presented in [2, Section 2]. In our first result we give an answer for this question when we deal with the problem (P). More specifically, we shall prove the following result.

Theorem 1.1. *The problem (P) has a positive solution in each of the following cases*

(i) $N \geqslant \alpha + 2$, $2 < q < 2^*$, $\lambda > 0$;
(ii) $2 < N < \alpha + 2$, $2^* - \frac{4}{\alpha} < q < 2^*$, $\lambda > 0$;
(iii) $2 < N < \alpha + 2$, $2 < q \leqslant 2^* - \frac{4}{\alpha}$, $\lambda > 0$ *is sufficiently large.*

The restriction on $\lambda > 0$ in the last item above is of technical nature. However, we would like to emphasize that, for $\alpha = 2$, this item becomes $N = 3$, $2 < q \leqslant 4$ and λ large. A similar condition have already appeared in the paper of Brezis and Nirenberg (cf. [2, Example 2.4]) for the problem (*BN*). We do not know if the statement (iii) holds for arbitrary values of λ.

In order to present our second result let us recall that, according to [4, Theorem 1.1], the problem (P) has no positive solution if $q = 2$ and $\lambda \geqslant \lambda_1$. However, we can ask for the existence of sign changing solutions. Actually, Capozzi, Fortunato and Palmieri have proved in [3, Theorem 0.1] that (*BN*) has a sign changing solution whenever $q = 2$, $N \geqslant 4$ and λ is greater than or equal to the first eigenvalue of $(-\Delta, H_0^1(\Omega))$ (see also [5] for a previous weaker result). In our second theorem we present a version of this last result to the problem (P).

Theorem 1.2. *If $q = 2$ and $N \geqslant \alpha + 2$, then the problem (P) has a sign changing solution for any $\lambda \geqslant \lambda_1$.*

Our problem is variational in nature. Indeed, for any $\alpha \geqslant 2$, let us denote by $H(\alpha)$ the Hilbert space obtained as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_K := \left(\int_{\mathbb{R}^N} K(x)|\nabla u|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$$

which is induced by the inner product

$$(u, v)_K := \int_{\mathbb{R}^N} K(x)(\nabla u \cdot \nabla v)\,\mathrm{d}x.$$

We shall look for solution belonging in $H(\alpha)$ and therefore our solution has a fast decay rate at infinity.

Since it might happen $q > 2$, the minimization argument employed in [6,4] does not work here. So, we shall use a different approach by considering the functional

$$I(u) := \int_{\mathbb{R}^N} K(x)\left(\frac{1}{2}|\nabla u|^2 - \frac{\lambda}{q}|x|^\beta |u|^q - \frac{1}{2^*}|u|^{2^*}\right)\mathrm{d}x$$

which is well defined, belongs to $C^1(H(\alpha), \mathbb{R})$ and whose critical points are precisely the weak solutions of Eq. (P). By using the fast decay rate of the elements of $H(\alpha)$ we are able to obtain compactness embeddings of this space in some weighted Lebesgue spaces. Hence, we can argue as in [2] to prove a local compactness result for the functional I.

In the proof of Theorem 1.1 we shall apply the Mountain Pass Theorem. The main difficult is to localize correctly the minimax level in the range where we have compactness. We achieve this objective by adapting some estimates performed in [6,4]. However, since we have many degrees of homogeneity in our equation, the calculations are more involved and the estimates will be done in several distinct cases depending on the relation between α and the dimension N. For Theorem 1.2 we use the Linking Theorem and an adaptation of the ideas contained in [3]. As in the first theorem, the calculations are more difficult. Moreover, since the domain is unbounded, some estimates presented in [3] do not apply here. We are able to overcome the difficulties by making fine estimates and calculating the precise decay rate of the solutions of (*LP*).

The paper is organized as follows. In the next section we present the variational setting of problem (P) and we prove the local compactness result for I. Section 3 is devoted to the proof of Theorem 1.1. The existence of sign changing solution is proved in the final Section 4.

2. The variational setting

In this section we present the variational framework to deal with problem (P). Throughout the paper we write $\int u$ instead of $\int_{\mathbb{R}^N} u(x)\,\mathrm{d}x$.

For each $q \in [2, 2^*]$ we denote by $L^q(\alpha)$ the following space

$$L^q(\alpha) := \left\{ u \text{ measurable in } \mathbb{R}^N\colon\ |u|_{q,K} := \left(\int K(x)|x|^\beta |u|^q \right)^{1/q} < \infty \right\}.$$

Let S be the best constant of the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. It is proved in [4, Proposition 2.1] that

$$\int K(x)|x|^{\alpha-2}|u|^2 \leqslant \frac{8}{\alpha(N-2+\alpha)} \int K(x)|\nabla u|^2 \tag{2.1}$$

and

$$\int K(x)|u|^{2^*} \leqslant S^{-2^*/2} \left(\int K(x)|\nabla u|^2 \right)^{2^*/2}. \tag{2.2}$$

It follows that the space $H^1(\alpha)$ is continuously embedded in the spaces $L^2(\alpha)$ and $L^{2^*}(\alpha)$. Moreover, the authors also proved that the first embedding is compact (see [4, Proposition 2.2]). By applying interpolation we can obtain similar results for $2 < q < 2^*$, namely

Proposition 2.1. *Let $\alpha \geqslant 2$ be fixed. Then the embedding $H(\alpha) \hookrightarrow L^q(\alpha)$ is continuous for all $q \in [2, 2^*]$ and it is compact for all $q \in [2, 2^*)$.*

Proof. Let $q \in (2, 2^*)$ be fixed and $\tau = 2(2^* - q)/q(2^* - 2) \in (0, 1)$. Hölder's inequality with exponents $p = (2^* - 2)/(2^* - q)$ and $p' = (2^* - 2)/(q - 2)$ provides

$$\begin{aligned}
\int K(x)|x|^\beta |u|^q &= \int K(x)|x|^{(\alpha-2)(2^*-q)/(2^*-2)}|u|^q \\
&= \int K(x)^{1/p}|x|^{(\alpha-2)(2^*-q)/(2^*-2)}|u|^{q\tau} K(x)^{1/p'}|u|^{(1-\tau)q} \\
&\leqslant \left(\int K(x)|x|^{\alpha-2}|u|^2 \right)^{1/p} \left(\int K(x)|u|^{2^*} \right)^{1/p'} \\
&\leqslant C_q \left(\int K(x)|\nabla u|^2 \right)^{1/p + 2^*/(2p')} = C_q \left(\int K(x)|\nabla u|^2 \right)^{q/2},
\end{aligned}$$

where $C_q = 8^{1/p}(\alpha(N-2+\alpha))^{-1/p} S^{-2^*/(2p')}$ and we have used (2.1), (2.2) and the definition of p. From the above inequality we obtain the first statement of the lemma. In order to prove the second one we take $q \in (2, 2^*)$, $\sigma \in (0, 1)$ such that $1/q = \sigma/2 + (1-\sigma)/2^*$ and argue as above to obtain

$$|u|_{q,K} \leqslant |u|_{2,K}^\sigma |u|_{2^*,K}^{1-\sigma}, \quad \text{for all } u \in L^2(\alpha) \cap L^{2^*}(\alpha).$$

Since the embedding $H^1(\alpha) \hookrightarrow L^2(\alpha)$ is compact, it follows from the above inequality and (2.2) that $H^1(\alpha)$ is compactly embedded in $L^q(\alpha)$. This finishes the proof. □

We consider in the sequel the linear problem

$$-\operatorname{div}\big(K(x)\nabla u\big) = \lambda K(x)|x|^{\alpha-2}u, \quad \text{in } \mathbb{R}^N. \tag{LP}$$

If we denote by $(x, y) := \sum_{i=1}^{N} x_i y_i$ the scalar product of $x, y \in \mathbb{R}^N$, we can easily check that (LP) is equivalent to

$$-\Delta u - \frac{\alpha}{4}(x, \nabla u)|x|^{\alpha-2} = \lambda u|x|^{\alpha-2}, \quad \text{in } \mathbb{R}^N. \tag{2.3}$$

The compactness of the embedding $H^1(\alpha) \hookrightarrow L^2(\alpha)$ and standard spectral theory for compact operators provide a sequence of positive eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \lambda_n = +\infty$.

Lemma 2.2. *If $u \in H^1(\alpha)$ satisfies (LP), then $u \in C^2(\mathbb{R}^N)$ and there exists $C > 0$ such that*

$$\big|u(x)\big| \leqslant Ce^{-\frac{1}{8}|x|^\alpha}, \quad \textit{for each } x \in \mathbb{R}^N.$$

Proof. Let $w := \exp(|x|^\alpha/8)u = K(x)^{1/2}u$. Since u is a solution of (LP) we can use the Brezis–Kato theorem and a standard boot-strap argument to conclude that $u \in C^2(\mathbb{R}^N)$. Hence, w is also regular and we can easily compute

$$\begin{aligned}\Delta w = K(x)^{1/2}\bigg(&\frac{\alpha^2}{64}|x|^{2\alpha-2}u + \frac{\alpha}{4}|x|^{\alpha-2}(x, \nabla u) + \frac{\alpha}{8}N|x|^{\alpha-2}u \\ &+ \frac{\alpha}{8}(\alpha-2)|x|^{\alpha-2}u + \Delta u\bigg).\end{aligned}$$

Recalling that u satisfies (2.3) and $\lambda_1 = \frac{\alpha}{4}(N+\alpha-2)$ we obtain

$$\Delta w = \left(\frac{\alpha^2}{64}|x|^\alpha - \lambda + \frac{\lambda_1}{2}\right)|x|^{\alpha-2}w. \tag{2.4}$$

Since $u \in H(\alpha)$ we have that $w \in L^2(\mathbb{R}^N)$. Moreover,

$$\int |\nabla w|^2 \leqslant c_1 \int K(x)|x|^{2\alpha-2}u^2 + c_1 \int K(x)|\nabla u|^2. \tag{2.5}$$

By setting $\theta(x) := \frac{1}{4}|x|^\alpha$, we can proceed as in [4, Proposition 2.1] to get

$$\int K(x)|\nabla u|^2 \geqslant \frac{1}{2}\int K(x)\left(\Delta\theta(x) + \frac{1}{2}\big|\nabla\theta(x)\big|^2\right)u^2 \geqslant \frac{\alpha^2}{64}\int K(x)|x|^{2\alpha-2}u^2.$$

This, (2.5) and $u \in H(\alpha)$ imply that $\int |\nabla w|^2$ is finite, and therefore we conclude that $w \in H^1(\mathbb{R}^N)$.

We now choose $R > 0$ such that

$$\frac{\alpha^2}{64}|x|^\alpha - \lambda + \frac{\lambda_1}{2} > 0, \quad \text{for each } |x| \geqslant R.$$

Let $M = M(R) := \sup\{w(x)\colon x \in B_R(0)\}$ and suppose that $M > 0$. If we take $\varphi = (w-M)^+$ as a test function in the weak formulation of (2.4) we get

$$\int_{B_R(0)^c} |\nabla\varphi|^2 = -\int_{B_R(0)^c} \left(\frac{\alpha^2}{64}|x|^\alpha - \lambda + \frac{\lambda_1}{2}\right)|x|^{\alpha-2} w\varphi \leqslant 0,$$

from which it follows that $\varphi \equiv 0$, or equivalently, $w(x) \leqslant M$ in $\mathbb{R}^N$. If $M \leqslant 0$ for any $R > 0$ then $w \leqslant 0$, and therefore we also obtain an upper bound to w.

Since $-w$ also satisfies (2.4) we can proceed as above to obtain an upper bound for $-w$. Thus

$$\left|w(x)\right| \leqslant C := \sup_{x\in B_R(0)} \left|w(x)\right|,$$

for some large ball $B_R(0)$. The result follows from the definition of w. $\square$

Remark 2.3. We can employ the same argument above to conclude that any solution of (P) is of class $C^2(\mathbb{R}^N)$ and decay as $e^{-\frac{1}{8}|x|^\alpha}$ at infinity. More specifically, if u solves (P), then u and its gradient ∇u have this decay property (see [6] for details).

Since problem (P) has a variational structure we shall consider the functional $I : H(\alpha) \to \mathbb{R}$ given by

$$I(u) := \frac{1}{2}\int K(x)|\nabla u|^2 - \frac{\lambda}{q}\int K(x)|x|^\beta |u|^q - \frac{1}{2^*}\int K(x)|u|^{2^*}.$$

Standard calculations and Proposition 2.1 show that $I \in C^1(H(\alpha), \mathbb{R})$ and the derivative of I at the point u is given by

$$I'(u)v = \int K(x)\nabla u \cdot \nabla v - \lambda\int K(x)|x|^\beta |u|^{q-2}uv - \int K(x)|u|^{2^*-2}uv,$$

for any $v \in H(\alpha)$. Hence, the critical points of I are precisely the weak solutions of Eq. (P).

Lemma 2.4. *Suppose that $(u_n) \subset H(\alpha)$ satisfies*

$$\lim_{n\to\infty} I(u_n) = d < \frac{1}{N}S^{N/2} \quad \textit{and} \quad \lim_{n\to\infty} I'(u_n) = 0. \tag{2.6}$$

Then (u_n) is bounded and, along a subsequence, (u_n) weakly converges to a nontrivial solution of the problem (P).

Proof. In view of (2.6) we have that

$$\begin{aligned} d + o(1) + o(1)\|u_n\|_K &\leqslant I(u_n) - \frac{1}{q}I'(u_n)u_n \\ &\leqslant \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|_K^2 + c_1\left(\frac{1}{q} - \frac{1}{2^*}\right)\|u_n\|_K^{2^*}, \end{aligned}$$

where $o(1)$ denotes a quantity approaching zero as $n \to \infty$. Hence (u_n) is bounded and, up to a subsequence, we have that $u_n \rightharpoonup u$ weakly in $H(\alpha)$. Since $K(x)|x|^\beta \in L^\infty_{loc}(\mathbb{R}^N)$, we can use the Lebesgue Theorem and standard arguments to conclude that $I'(u) = 0$.

In order to prove that $u \neq 0$ we suppose, by contradiction, that $u = 0$. Since the embedding $H^1(\alpha) \hookrightarrow L^q(\alpha)$ is compact we have that

$$\lim_{n\to\infty}\int K(x)|x|^{(\alpha-2)(2^*-q)/(2^*-2)}|u_n|^q = 0. \tag{2.7}$$

So, recalling that $I(u_n)\to d$, we obtain

$$\frac{1}{2}\int K(x)|\nabla u_n|^2 - \frac{1}{2^*}\int K(x)|u_n|^{2^*} = d + o(1). \tag{2.8}$$

Moreover, since (u_n) is bounded and $I'(u_n)\to 0$, we have that $I'(u_n)u_n \to 0$. This and (2.7) imply that

$$\int K(x)|\nabla u_n|^2 - \int K(x)|u_n|^{2^*} = o(1).$$

If $l \geqslant 0$ is such that $\int K(x)|\nabla u_n|^2 \to l$, it follows from the above equation that $\int K(x)|u_n|^{2^*}\to l$. Hence, we infer from (2.8) that

$$d = \left(\frac{1}{2}-\frac{1}{2^*}\right)l = \frac{l}{N}. \tag{2.9}$$

On the other hand, the inequality (2.2) implies that

$$S\left(\int K(x)|u_n|^{2^*}\right)^{2/2^*} \leqslant \int K(x)|\nabla u_n|^2.$$

Letting $n\to\infty$, we obtain $Sl^{2/2^*}\leqslant l$. By combining this inequality with (2.9) we conclude that $d \geqslant \frac{1}{N}S^{N/2}$, which is a contradiction with the hypothesis. So, $u\neq 0$ and the lemma is proved. □

3. Positive solution for $2 < q < 2^*$

In this section we use the classical Mountain Pass Theorem to obtain a positive solution for (P) when $2<q<2^*$. We start with an easy consequence of the embedding result of the previous section.

Lemma 3.1. *There exists $\rho,\sigma>0$ such that $I|_{\partial B_\rho(0)}\geqslant\sigma$. Moreover, there exists $e\in H^1(\alpha)$ such that $\|e\|_K\geqslant\rho$ and $I(e)<0$.*

Proof. By using Proposition 2.1 we get

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|_K^2 - \frac{\lambda}{q}|u|_{q,K}^q - \frac{1}{2^*}|u|_{2^*,K}^{2^*} \\ &\geqslant \|u\|_K^2\left(\frac{1}{2} - c_1\|u\|_K^{q-2} - c_2\|u\|_K^{2^*-2}\right) \geqslant \sigma > 0, \end{aligned}$$

for any $u\in H(\alpha)$ such that $\|u\|_K=\rho$, with $\rho>0$ sufficiently small. Moreover, for any $u\in H^1(\alpha)\setminus\{0\}$ there holds $\lim_{t\to\infty} I(tu) = -\infty$. Thus, it suffices to set $e := tu$, for $t>0$ large enough, to obtain $\|e\|_K\geqslant\rho$ and $I(e)<0$. □

We now define

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I\big(\gamma(t)\big),$$

where

$$\Gamma := \bigl\{\gamma \in C\bigl([0,1], H(\alpha)\bigr):\ \gamma(0) = 0 \text{ and } \gamma(1) = e\bigr\}.$$

It follows from the Mountain Pass Theorem that there exits a Palais–Smale sequence for I at level c. In view of Lemma 2.4 we can obtain a nontrivial solution of (P) provided

$$c < \frac{1}{N} S^{N/2}.$$

We devote the rest of this section to show that, under the hypothesis of Theorem 1.1, the above inequality is satisfied.

We first notice that, arguing as in [13, Lemma 4.1], we can obtain the following characterization for the minimax level

$$c = \inf_{u \in H(\alpha)\setminus\{0\}} \max_{t \geqslant 0} I(tu). \tag{3.1}$$

Hence, it suffices to prove the following.

Proposition 3.2. *Under the hypothesis of Theorem* 1.1 *there exists* $v \in H^1(\alpha)$ *such that*

$$\sup_{t \geqslant 0} I(tv) < \frac{1}{N} S^{N/2}.$$

Proof. We adapt the arguments and calculations performed in [4] (see also [2]). Let $\varphi \in C^\infty(\mathbb{R}^N, [0,1])$ be such that $\varphi \equiv 1$ on $B_1(0)$ and $\varphi \equiv 0$ outside $B_2(0)$. For any $\varepsilon > 0$, let us consider

$$u_\varepsilon := K(x)^{-1/2}\varphi(x)\left(\frac{1}{\varepsilon + |x|^2}\right)^{(N-2)/2}$$

and consider

$$v_\varepsilon := \frac{u_\varepsilon}{|u_\varepsilon|_{2^*,K}}.$$

The function $h(t) := I(tv_\varepsilon)$, $t \geqslant 0$, has a unique maximum point $t_\varepsilon > 0$. By using $h'(t_\varepsilon) = 0$ and $\lambda > 0$, we obtain

$$\|v_\varepsilon\|_K^2 = t_\varepsilon^{2^*-2} + \lambda t_\varepsilon^{q-2}|v_\varepsilon|_{q,K}^q \geqslant t_\varepsilon^{2^*-2},$$

or equivalently,

$$\hat{t} := \|v_\varepsilon\|_K^{2/(2^*-2)} \geqslant t_\varepsilon.$$

Since the function $g(t) := (1/2)t^2\hat{t}^{2^*-2} - (1/2^*)t^{2^*}$ is increasing on $[0,\hat{t}]$ and $|v_\varepsilon|_{2^*,K} = 1$ we get

$$\begin{aligned} I(t_\varepsilon v_\varepsilon) &= g(t_\varepsilon) - \frac{t_\varepsilon^q}{q}\lambda|v_\varepsilon|_{q,K}^q \\ &\leqslant g(\hat{t}) - \frac{t_\varepsilon^q}{q}\lambda|v_\varepsilon|_{q,K}^q = \frac{1}{N}\bigl(\|v_\varepsilon\|_K^2\bigr)^{N/2} - \frac{t_\varepsilon^q}{q}\lambda|v_\varepsilon|_{q,K}^q. \end{aligned} \tag{3.2}$$

In what follows we consider several cases depending on the values of N and α.

Case 1. $N > \alpha + 2$.

In this case, according to [4], we have that

$$\|v_\varepsilon\|_K^2 = \begin{cases} S + O(\varepsilon^{\alpha/2}) + O(\varepsilon^{\alpha}) + O(\varepsilon^{N/2-1}), & \text{if } N > 2\alpha + 2, \\ S + O(\varepsilon^{\alpha/2}) + O(\varepsilon^{\alpha}|\log\varepsilon|) + O(\varepsilon^{N/2-1}), & \text{if } N = 2\alpha + 2, \\ S + O(\varepsilon^{\alpha/2}) + O(\varepsilon^{N/2-1}), & \text{if } \alpha + 2 < N < 2\alpha + 2, \end{cases}$$

where $f(\varepsilon) = O(\varepsilon^\delta)$ means that $\limsup_{\varepsilon\to 0^+} f(\varepsilon)/\varepsilon^\delta$ is finite. Since $N > \alpha + 2$ we have that $\alpha/2 < N/2 - 1$. Moreover, recalling that $\lim_{\varepsilon\to 0^+} \varepsilon^{\alpha/2}|\log\varepsilon| = 0$, we infer from the above estimates that

$$\|v_\varepsilon\|_K^2 = S + O\left(\varepsilon^{\alpha/2}\right). \tag{3.3}$$

We claim that $t_\varepsilon^q \geqslant qC_0 > 0$ for some $C_0 > 0$ and any $\varepsilon > 0$ small. Indeed, suppose by contradiction that for some sequence $\varepsilon_n \to 0^+$ we have that $t_{\varepsilon_n} \to 0$. Then it follows from (3.3) that $t_{\varepsilon_n} v_{\varepsilon_n} \to 0$ in $H^1(\alpha)$. Thus, we can use (3.1) and the continuity of I to obtain

$$0 < c \leqslant \sup_{t>0} I(t v_{\varepsilon_n}) = I(t_{\varepsilon_n} v_{\varepsilon_n}) \to I(0) = 0,$$

which does not make sense.

We can now use (3.2), (3.3), the above claim and the Mean Value Theorem to get

$$\begin{aligned} I(t_\varepsilon v_\varepsilon) &\leqslant \frac{1}{N}\left(S + O\left(\varepsilon^{\alpha/2}\right)\right)^{N/2} - \frac{t_\varepsilon^q}{q}\lambda|v_\varepsilon|_{q,K}^q \\ &\leqslant \frac{1}{N}S^{N/2} + O\left(\varepsilon^{\alpha/2}\right) - C_0\lambda|v_\varepsilon|_{q,K}^q \\ &= \frac{1}{N}S^{N/2} + \varepsilon^{\alpha/2}\left(O(1) - \lambda C_0\frac{1}{\varepsilon^{\alpha/2}}|v_\varepsilon|_{q,K}^q\right). \end{aligned} \tag{3.4}$$

In order to prove the lemma in this first case it suffices to show that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{\alpha/2}}|v_\varepsilon|_{q,K}^q = +\infty.$$

We are going to estimate $|v_\varepsilon|_{q,K}$. We first recall that

$$\begin{aligned} \int \frac{|x|^\beta}{(1+|x|^2)^{q(N-2)/2}} &= \left(\int_{B_1(0)} + \int_{B_1(0)^c}\right) \\ &\leqslant c_1 + \int_{B_1(0)^c} |x|^{\beta - q(N-2)}\,\mathrm{d}x < \infty \end{aligned} \tag{3.5}$$

whenever $q(N-2) - \beta - N < 0$. Let us consider the linear function

$$r(q) := q(N-2) - \beta - N.$$

Since $N - 2 - \alpha = r(2) < r(2^*)$ it follows that

$$\int \frac{|x|^{\beta}}{(1+|x|^2)^{q(N-2)/2}} < \infty \quad (\text{for } N \geqslant \alpha + 2). \tag{3.6}$$

Since $q > 2$, we have that $0 < c_2 < K(x)^{1-q/2}$ for each $|x| \leqslant 2$. Hence, recalling that $\varphi \equiv 0$ on $B_2(0)^c$, we get

$$|u_\varepsilon|_{q,K}^q = \int \frac{(K(x)^{1-q/2}\varphi^q)|x|^{\beta}}{(\varepsilon+|x|^2)^{q(N-2)/2}} \geqslant c_2 \int \frac{\varphi^q |x|^{\beta}}{(\varepsilon+|x|^2)^{q(N-2)/2}}. \tag{3.7}$$

On the other hand, by using the definition of φ again, we obtain

$$\begin{aligned}\int \frac{\varphi^q |x|^{\beta}}{(\varepsilon+|x|^2)^{q(N-2)/2}} &= \int\limits_{B_2(0)} \frac{|x|^{\beta}}{(\varepsilon+|x|^2)^{q(N-2)/2}} + \int\limits_{B_2(0)} \frac{(\varphi^q-1)|x|^{\beta}}{(\varepsilon+|x|^2)^{q(N-2)/2}} \\ &= \int\limits_{B_2(0)} \frac{|x|^{\beta}}{(\varepsilon+|x|^2)^{q(N-2)/2}} + O(1).\end{aligned}$$

Hence, we can use the change of variable $x \mapsto x/\sqrt{\varepsilon}$ to obtain

$$\int \frac{\varphi^q |x|^{\beta}}{(\varepsilon+|x|^2)^{q(N-2)/2}} = \varepsilon^{\beta/2+N/2-q(N-2)/2} \int\limits_{B_{2/\sqrt{\varepsilon}}(0)} \frac{|x|^{\beta}}{(1+|x|^2)^{q(N-2)/2}} \mathrm{d}x + O(1).$$

By (3.6) and the Lebesgue Theorem we have that the integral on the right-hand side above is convergent as $\varepsilon \to 0^+$. Thus, we can use (3.7) to deduce that

$$|u_\varepsilon|_{q,K}^q \geqslant O\left(\varepsilon^{\beta/2+N/2-q(N-2)/2}\right) + O(1). \tag{3.8}$$

According to [4, p. 1165] we also have

$$|u_\varepsilon|_{2^*,K}^{2^*} = \int K(x)|u_\varepsilon|^{2^*} = \varepsilon^{-N/2} A_0 + O(1) \quad (\text{for } N > 2), \tag{3.9}$$

with

$$A_0 := \int \frac{1}{(1+|x|^2)^N} \quad (\text{for } N > 2), \tag{3.10}$$

from which it follows that

$$|u_\varepsilon|_{2^*,K}^q = O\left(\varepsilon^{-q(N-2)/4}\right) + O(1) \quad (\text{for } N > 2).$$

This and (3.8) imply that

$$\begin{aligned}|v_\varepsilon|_{q,K}^q = \frac{|u_\varepsilon|_{q,K}^q}{|u_\varepsilon|_{2^*,K}^q} &\geqslant \frac{O(\varepsilon^{\beta/2+N/2-q(N-2)/2}) + O(1)}{O(\varepsilon^{-q(N-2)/4}) + O(1)} \\ &= \frac{O(\varepsilon^{\beta/2+N/2-q(N-2)/4}) + O(\varepsilon^{q(N-2)/4})}{O(1) + O(\varepsilon^{q(N-2)/4})}\end{aligned} \tag{3.11}$$

and therefore

$$\frac{1}{\varepsilon^{\alpha/2}}|v_\varepsilon|_{q,K}^q \geqslant \frac{O(\varepsilon^{s(q)})+O(\varepsilon^{t(q)})}{O(1)+O(\varepsilon^{q(N-2)/4})}, \tag{3.12}$$

where $s(q)$ and $t(q)$ are the liner functions

$$s(q) := \frac{\beta}{2}+\frac{N}{2}-\frac{q(N-2)}{4}-\frac{\alpha}{2}, \qquad t(q) := \frac{q(N-2)}{4}-\frac{\alpha}{2}. \tag{3.13}$$

Since $0 = s(2) < s(2^*) = -\alpha/2$ and $2 < q < 2^*$ we conclude that $s(q) < 0$. Moreover, if $N \geqslant \alpha+2$, we have that $0 \leqslant (N-2-\alpha)/2 = t(2) < t(q)$, and therefore $t(q) > 0$ for any $2 < q < 2^*$. Thus, we can take the limit in (3.12) to conclude that

$$\lim_{\varepsilon\to 0} = \frac{1}{\varepsilon^{\alpha/2}}|v_\varepsilon|_{q,K}^q = +\infty \quad (\text{for } N \geqslant \alpha+2).$$

This establishes the statement of the lemma in Case 1.

Case 2. $N = \alpha + 2$.

In this case, we shall present a few more details on the estimate of $\|v_\varepsilon\|_K^2$ since it appears that its presentation in [4] contains a misprint. According to [4, p. 1167, Case 4] we have that

$$\|u_\varepsilon\|_K^2 = \int K(x)|\nabla u|^2 = \varepsilon^{1-N/2}A_1 + c_3|\log\varepsilon| + O(1),$$

with

$$A_1 := (N-2)^2\int \frac{|x|^2}{(1+|x|^2)^N} \quad (\text{for } N > 2). \tag{3.14}$$

Thus, we can use (3.9) and (3.10) to get

$$\begin{aligned}\|v_\varepsilon\|_K^2 = \frac{\|u_\varepsilon\|_K^2}{|u_\varepsilon|_{2^*,K}^2} &= \frac{\varepsilon^{1-N/2}A_1 + c_1|\log\varepsilon| + O(1)}{\varepsilon^{1-N/2}A_0^{1-2/N}+O(\varepsilon)} \\ &= \frac{A_1A_0^{-1+2/N} + c_1\varepsilon^{N/2-1}|\log\varepsilon| + O(\varepsilon^{-1+N/2})}{1+O(\varepsilon^{N/2})}.\end{aligned} \tag{3.15}$$

For any $0 < \delta < \alpha/2$ we have that $\varepsilon^{-1+N/2}|\log\varepsilon| = \varepsilon^{-1+N/2-\delta}\varepsilon^{\delta}|\log\varepsilon| = O(\varepsilon^{-1+N/2-\delta})$. Moreover, as proved in [2], $A_1A_0^{-1+2/N} = S$. Thus, recalling that $-1+N/2 = \alpha/2$, we get

$$\|v_\varepsilon\|_K^2 = S + O\big(\varepsilon^{-1+N/2-\delta}\big) = S + O\big(\varepsilon^{\alpha/2-\delta}\big).$$

This provides a positive lower bound for the values t_ε and therefore we can argue as in (3.4) to obtain

$$I(t_\varepsilon v_\varepsilon) \leqslant \frac{1}{N}S^{N/2} + \varepsilon^{\alpha/2-\delta}\left(O(1) - \lambda C_0\frac{1}{\varepsilon^{\alpha/2-\delta}}|v_\varepsilon|_{q,K}^q\right).$$

As in Case 1, we get

$$\frac{1}{\varepsilon^{\alpha/2-\delta}}|v_\varepsilon|_{q,K}^q \geqslant \frac{O(\varepsilon^{s(q)+\delta})+O(\varepsilon^{t(q)+\delta})}{O(1)+O(\varepsilon^{q(N-2)/4})},$$

where $s(q)$ and $t(q)$ are given in (3.13). Since $s(q)<0$ we can choose $\delta>0$ small enough in such a way that $s(q)+\delta<0$. So, recalling that $0=t(2)<t(q)$, we obtain

$$\lim_{\varepsilon\to 0} = \frac{1}{\varepsilon^{\alpha/2-\delta}}|v_\varepsilon|_{q,K}^q = +\infty \quad (\text{for } N=\alpha+2),$$

from which it follows that $I(t_\varepsilon v_\varepsilon) < \frac{1}{N}S^{N/2}$ for $\varepsilon>0$ sufficiently small.

Case 3. $2<N<\alpha+2$ and $2^*-4/\alpha<q<2^*$.

In this case, by using the calculations of [4, p. 1164] again, we have that

$$\begin{aligned}\|u_\varepsilon\|_K^2 &= \varepsilon^{1-N/2}A_1 + c_4\int\frac{\varphi^2|x|^\alpha}{(\varepsilon+|x|^2)^{N-1}} + c_5\int\frac{\varphi^2|x|^{2(\alpha-1)}}{(\varepsilon+|x|^2)^{N-2}} + O(1)\\ &= \varepsilon^{1-N/2}A_1 + O(1),\end{aligned}$$

where A_1 was defined in (3.14). Thus, we can proceed as in (3.15) to get

$$\|v_\varepsilon\|_K^2 = \frac{\|u_\varepsilon\|_K^2}{|u_\varepsilon|_{2^*,K}^2} = \frac{\varepsilon^{1-N/2}A_1+O(1)}{\varepsilon^{1-N/2}A_0^{1-2/N}+O(\varepsilon)} = S + O\big(\varepsilon^{-1+N/2}\big).$$

So, t_ε has a positive lower bound and we can argue as in (3.4) to obtain

$$I(t_\varepsilon v_\varepsilon) \leqslant \frac{1}{N}S^{N/2} + \varepsilon^{-1+N/2}\left(O(1) - \lambda C_0\frac{1}{\varepsilon^{-1+N/2}}|v_\varepsilon|_{q,K}^q\right).$$

We now recall that (3.5) holds provided $q(N-2)-\beta-N<0$. By replacing the value of β in this last inequality we can see that it occurs if, and only if,

$$q > q_1 := \frac{2^*(\alpha-2)+N(2^*-2)}{(N-2)(2^*-2)+(\alpha-2)} = \frac{2^*\alpha}{\alpha+2}.$$

Straightforward calculations show that $2<q_1<2^*$ if, and only if, $N<\alpha+2$. Hence, we are in the setting of Case 3 and the expression in (3.11) also holds. Thus,

$$\frac{1}{\varepsilon^{-1+N/2}}|v_\varepsilon|_{q,K}^q \geqslant \frac{O(\varepsilon^{\hat{s}(q)})+O(\varepsilon^{\hat{t}(q)})}{O(1)+O(\varepsilon^{q(N-2)/4})}$$

where $\hat{s}(q)$ and $\hat{t}(q)$ are the liner functions

$$\hat{s}(q) := \frac{\beta}{2} - \frac{q(N-2)}{4} + 1, \qquad \hat{t}(q) := \frac{q(N-2)}{4} + 1 - \frac{N}{2}.$$

Since $\hat{t}(2)=0$ we have that $\hat{t}(q)>0$ for any $q>2$. A direct calculation shows that $\hat{s}(q)<0$ if, and only if,

$$q > q_2 := \frac{2^*\alpha-4}{\alpha} = 2^* - \frac{4}{\alpha}.$$

Since $N < \alpha + 2$ we have that $q_1 < q_2$. Thus, for any $2^* - 4/\alpha < q < 2^*$, we have that

$$\lim_{\varepsilon \to 0} = \frac{1}{\varepsilon^{1+N/2}} |v_\varepsilon|_{q,K}^q = +\infty$$

and the proposition follows as in Case 1.

Case 4. $2 < N < \alpha + 2$, $2 < q \leqslant 2^* - 4/\alpha$ and λ large.

In this case the proof is easier because we can take large values for λ. More specifically, let $v \in H(\alpha)$ satisfying $|v|_{2^*,K} = 1$ and consider $t_\lambda > 0$ such that $I_\lambda(t_\lambda v) = \sup_{t \geqslant 0} I_\lambda(tv)$, that is,

$$\|v\|_K^2 - \lambda t_\lambda^{q-2} |v|_{q,K}^q = t_\lambda^{2^*-2}.$$

The above expression implies that $(t_\lambda)_{\lambda \in \mathbb{R}^+}$ is bounded. If $\limsup_{\lambda \to \infty} = t_0 > 0$, the above equality would imply that $v = 0$, which does not make sense. Hence, $\lim_{\lambda \to \infty} t_\lambda = 0$. Thus,

$$\sup_{t>0} I_\lambda(tv) = \left(\frac{1}{2} - \frac{1}{q}\right) t_\lambda^2 \|v\|_K^2 + \left(\frac{1}{q} - \frac{1}{2^*}\right) t_\lambda^{2^*},$$

and therefore,

$$\lim_{\lambda \to \infty} \sup_{t>0} I_\lambda(tv) = 0.$$

Hence, there exists $\lambda^* > 0$ such that, for any $\lambda \geqslant \lambda^*$, the statement of the proposition holds. This finishes the proof. □

4. Sign-changing solution for $q = 2$

In this section we prove Theorem 1.2 by applying the following variant of the Mountain Pass Theorem.

Theorem 4.1. *Let X be a real Banach space with $X = Y \oplus Z$ and $\dim Y < \infty$. Suppose $I \in C^1(X\mathbb{R})$ satisfies*

(I_1) there exist $\rho, \sigma > 0$ such that $I|_{\partial B_\rho(0) \cap Z} \geqslant \sigma$;
(I_2) there exist $e \in \partial B_1(0) \cap Z$ and $R > \rho$ such that

$$I|_{\partial Q} \leqslant 0,$$

with

$$Q := \left(\overline{B_R(0)} \cap Y\right) \oplus \{te\colon\ 0 < t < R\}.$$

If we define

$$c := \inf_{\gamma \in \Gamma} \max_{u \in Q} I\big(\gamma(u)\big)$$

where

$$\Gamma := \left\{\gamma \in C(\overline{Q}, X)\colon\ \gamma \equiv \mathrm{Id} \text{ on } \partial Q\right\},$$

then there exists a sequence $(u_n) \subset X$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$.

Let $\lambda > 0$ be such that $\lambda_n \leqslant \lambda < \lambda_{n+1}$ and set

$$Y = \text{span}\{\varphi_1, \ldots, \varphi_n\}, \quad Z = Y^{\perp},$$

in such a way that $H^1(\alpha) = Y \oplus Z$. The following result is the key stone in the proof of Theorem 1.2.

Proposition 4.2. *If $N \geqslant \alpha + 2$ and $\lambda_n \leqslant \lambda < \lambda_{n+1}$, then there exists $z \in Z \setminus \{0\}$ such that*

$$\max_{u \in Y + \mathbb{R}z} I(u) < \frac{1}{N} S^{N/2}.$$

The proof of this proposition is rather long and technique. Before presenting it, let us see how we can use it to obtain a nodal solution for (P).

Proof of Theorem 1.2. Let Y and Z be defined as above. For any $u \in Z$ we have that

$$I(u) \geqslant \left(\frac{\lambda_{n+1} - \lambda}{2}\right)\|u\|_K^2 - \frac{1}{2^*}|u|_{2^*,K}^{2^*},$$

and therefore we can argue as in the proof of Lemma 3.1 to conclude that I satisfies the condition (I_1) of the previous abstract result.

We also have that

$$I(u) \leqslant \left(\frac{\lambda_n - \lambda}{2}\right)\|u\|_K^2 - \frac{1}{2^*}|u|_{2^*,K}^{2^*} \leqslant 0, \quad \forall u \in Y.$$

Moreover, if $z \in Z \setminus \{0\}$ is given by Proposition 4.2, we can use the equivalence of norms in the finite dimensional space $Y \oplus \mathbb{R}z$, to get

$$I(u) \to -\infty \quad \text{as } \|u\|_K \to \infty,\ u \in Y \oplus \mathbb{R}z.$$

Thus, condition (I_2) is satisfied for $R > 0$ sufficiently large.

By applying Theorem 4.1 and Proposition 4.2 we obtain $(u_n) \subset H^1(\alpha)$ satisfying

$$\lim_{n\to\infty} I(u_n) = c < \frac{1}{N} S^{N/2} \quad \text{and} \quad \lim_{n\to\infty} I'(u_n) = 0.$$

It follows from Lemma 2.4 that, along a subsequence, (u_n) weakly converges to a nontrivial solution of (P). Since this problem has no positive solution for $\lambda \geqslant \lambda_1$, we conclude that this solution changes sign in $\mathbb{R}^N$. □

It remains to prove Proposition 4.2. We first introduce some notations which will be useful in the sequel. For any $u, v \in H^1(\alpha)$ we denote by

$$(u, v)_K := \int K(x)\nabla u \cdot \nabla v, \qquad (u, v)_{2,K} := \int K(x)|x|^{\alpha-2} u v,$$

the inner product in $H^1(\alpha)$ and $L^2(\alpha)$, respectively.

We divide the proof in two distinct cases related with λ be or not to be an eigenvalue. We start with the case $\lambda_n < \lambda < \lambda_{n+1}$. For any $\varepsilon > 0$ we set

$$w_\varepsilon := \varepsilon^{\frac{N-2}{4}} u_\varepsilon = \varepsilon^{\frac{N-2}{4}} K(x)^{-1/2} \varphi(x) \left(\frac{1}{\varepsilon + |x|^2} \right)^{(N-2)/2},$$

and

$$z_\varepsilon := w_\varepsilon - \sum_{i=1}^{n} (w_\varepsilon, \varphi_i)_K \varphi_i,$$

where $\{\varphi_i\}_{i \in \mathbb{N}}$ is the sequence of eigenfunctions of the linearized problem (*LP*).

We shall prove that Proposition 4.2 holds for $z = z_\varepsilon$, with $\varepsilon > 0$ small enough. Since $Y \oplus \mathbb{R} z_\varepsilon = Y \oplus \mathbb{R} w_\varepsilon$ and

$$\max_{t \geqslant 0} I(tu) = \frac{1}{N} \left(\frac{\|u\|_K^2 - \lambda |u|_{2,K}^2}{|u|_{2^*,K}^2} \right)^{N/2}, \quad \forall u \in H^1(\alpha) \setminus \{0\},$$

it suffices to verify that

$$m_\varepsilon := \max_{u \in \Sigma_\varepsilon} \left(\|u\|_K^2 - \lambda |u|_{2,K}^2 \right) < S, \tag{4.1}$$

where

$$\Sigma_\varepsilon := \left\{ u = y + t w_\varepsilon \colon\ y \in Y,\ t \in \mathbb{R},\ |u|_{2^*,K} = 1 \right\}.$$

Lemma 4.3. *As $\varepsilon \to 0^+$, the following estimates hold*

$$\int K(x) |w_\varepsilon|^{2^*-1} = O\left(\varepsilon^{\frac{N-2}{4}}\right), \quad |w_\varepsilon|_{L^1(\mathbb{R}^N)} = O\left(\varepsilon^{\frac{N-2}{4}}\right), \tag{4.2}$$

$$\max\left\{ (y, w_\varepsilon)_K, (y, w_\varepsilon)_{2,K} \right\} = |y|_{2,K}\, O\left(\varepsilon^{\frac{N-2}{4}}\right), \quad y \in Y. \tag{4.3}$$

Proof. Since $\varphi \equiv 0$ outside $B_2(0)$, the definition of w_ε provides

$$\begin{aligned}
\int K(x) |w_\varepsilon|^{2^*-1} &= \int K(x) \left(\varepsilon^{(N-2)/4} \frac{K(x)^{-1/2} \varphi}{(\varepsilon + |x|^2)^{(N-2)/2}} \right)^{(N+2)/(N-2)} \\
&\leqslant c_1 \varepsilon^{(N+2)/4} \int_{B_2(0)} \frac{1}{(\varepsilon + |x|^2)^{(N+2)/2}} \\
&= c_1 \varepsilon^{(N+2)/4} \int_{B_{2/\sqrt{\varepsilon}}(0)} \frac{\varepsilon^{-1}}{(1 + |y|^2)^{(N+2)/2}} \, dy \\
&= c_1 \varepsilon^{(N-2)/4} \int \frac{1}{(1 + |y|^2)^{(N+2)/2}} \, dy = O\left(\varepsilon^{\frac{N-2}{4}}\right),
\end{aligned}$$

and therefore the first statement in (4.2) holds. For the second one it suffices to compute

$$|w_\varepsilon|_{L^1(\mathbb{R}^N)} \leqslant c_2\varepsilon^{(N-2)/4} \int\limits_{B_2(0)} \frac{1}{(\varepsilon+|x|^2)^{(N-2)/2}} \leqslant c_2\varepsilon^{(N-2)/4} \int\limits_{B_2(0)} |x|^{2-N} = O\big(\varepsilon^{(N-2)/4}\big).$$

If we take $y=\sum_{i=1}^n \beta_i\varphi_i \in Y$ and recall that φ_i solves (LP), we obtain

$$\big|(y,w_\varepsilon)_K\big| = \left|\sum_{i=1}^n \lambda_i\beta_i(\varphi_i,w_\varepsilon)_{2,K}\right| \leqslant c_3\left(\sum_{i=1}^n|\beta_i|\right)\int |x|^{\alpha-2}w_\varepsilon, \tag{4.4}$$

with $c_3 := \lambda_n \max\{|\varphi_1|_\infty,\dots,|\varphi_n|_\infty\}$. Notice that

$$\int |x|^{\alpha-2}w_\varepsilon \leqslant c_4\varepsilon^{(N-2)/4}\int\limits_{B_2(0)} \frac{|x|^{\alpha-2}}{(\varepsilon+|x|^2)^{(N-2)/2}} \leqslant c_4\varepsilon^{(N-2)/4}\int\limits_{B_2(0)} |x|^{\alpha-N} = O\big(\varepsilon^{(N-2)/4}\big).$$

Moreover, the equivalence of norms in Y, implies that $\sum_{i=1}^n|\beta_i| \leqslant c_4|y|_{2,K}$. By replacing this and the above inequality in (4.4), we obtain

$$\big|(y,w_\varepsilon)_K\big| = |y|_{2,K}\,O\big(\varepsilon^{\frac{N-2}{4}}\big).$$

The argument for $(y,w_\varepsilon)_{2,K}$ is analogous, and therefore (4.3) holds. The lemma is proved. □

Our next estimation is more involved.

Lemma 4.4. *For any $u=y+tw_\varepsilon\in\Sigma_\varepsilon$ we have that $t=O(1)$ as $\varepsilon\to0^+$.*

Proof. Given $u=y+tw_\varepsilon\in\Sigma_\varepsilon$, we set

$$A(u) := |u|_{2^*,K}^{2^*} - |y|_{2^*,K}^{2^*} - |tw_\varepsilon|_{2^*,K}^{2^*}.$$

Since $\dim V<\infty$ and the eigenfunctions of (LP) are regular, we conclude that $u\in C^2(\mathbb{R}^N)$. Hence,

$$\begin{aligned} A(u) &= \int\limits_{\mathbb{R}^N} K(x)\big(|y+tw_\varepsilon|^{2^*}-|y|^{2^*}-|tw_\varepsilon|^{2^*}\big)\,dx \\ &= 2^*\int\limits_{\mathbb{R}^N}\left(\int_0^1 K(x)\big(|tw_\varepsilon+sy|^{2^*-2}(tw_\varepsilon+sy)-|sy|^{2^*-2}sy\big)y\,ds\right)dx \\ &= 2^*\big(2^*-1\big)\int\limits_{\mathbb{R}^N}\left(\int_0^1 K(x)|sy+tw_\varepsilon\theta|^{2^*-2}tw_\varepsilon y\,ds\right)dx, \end{aligned}$$

where $0 \leqslant \theta(x) \leqslant 1$ is a measurable function. Recalling that the support of w_ε is contained in $B_2(0)$, we can use the above estimate, (4.2), (4.3) and the equivalence of norms in Y, to obtain

$$\begin{aligned}
\left|A(u)\right| &\leqslant c_1\left\{|y|_\infty^{2^*-1}|t||w_\varepsilon|_{L^1(\mathbb{R}^N)} + |y|_\infty|t|^{2^*-1}\int K(x)|w_\varepsilon|^{2^*-1}\right\} \\
&\leqslant \left\{|y|_{2^*,K}^{2^*-1}|t|O\left(\varepsilon^{\frac{N-2}{4}}\right) + |y|_{2^*,K}|t|^{2^*-1}O\left(\varepsilon^{\frac{N-2}{4}}\right)\right\}. \qquad (4.5)
\end{aligned}$$

Given $\delta > 0$, we can apply Young's inequality with exponent $s = 2^*/(2^*-1)$, $s' = 2^*$, to obtain $c_3 = c_3(\delta)$ such that

$$|y|_{2^*,K}^{2^*-1}|t|O\left(\varepsilon^{\frac{N-2}{4}}\right) \leqslant \delta|y|_{2^*,K}^{2^*} + c_3|t|^{2^*}O\left(\varepsilon^{\frac{N-2}{4}}\right)^{\frac{2N}{N-2}}.$$

Analogously, there exists $c_4 = c_4(\delta)$ satisfying

$$|y|_{2^*,K}|t|^{2^*-1}O\left(\varepsilon^{\frac{N-2}{4}}\right) \leqslant \delta|y|_{2^*,K}^{2^*} + c_4|t|^{2^*}O\left(\varepsilon^{\frac{N-2}{4}}\right)^{\frac{2N}{N+2}}.$$

By choosing $\delta > 0$ in such a way that $2\delta c_1 < 1/2$, we can replace the last two inequalities in (4.5) to get

$$\left|A(u)\right| \leqslant \frac{1}{2}|y|_{2^*,K}^{2^*} + |t|^{2^*}\left\{O\left(\varepsilon^{N/2}\right) + O\left(\varepsilon^{\frac{N-2}{N+2}\frac{N}{2}}\right)\right\}.$$

It follows from the definition of $A(u)$ and (3.9) that

$$\begin{aligned}
1 = |u|_{2^*,K}^{2^*} &\geqslant |tw_\varepsilon|_{2^*,K}^{2^*} + |t|^{2^*}O\left(\varepsilon^{\frac{N-2}{N+2}\cdot\frac{N}{2}}\right) + \frac{1}{2}|y|_{2^*,K}^{2^*} \\
&= |t|^{2^*}\left\{A_0 + O\left(\varepsilon^{\frac{N}{2}}\right) + O\left(\varepsilon^{\frac{N-2}{N+2}\cdot\frac{N}{2}}\right)\right\},
\end{aligned}$$

and therefore we cannot have $t \to \infty$ as $\varepsilon \to 0^+$. The lemma is proved. □

We are now able to prove that (4.1) holds in the first case.

Proof of Proposition 4.2 *(Case $\lambda_n < \lambda < \lambda_{n+1}$).* As quoted before, it suffices to verify (4.1). With this aim we take $u = y + tw_\varepsilon \in \Sigma_\varepsilon$ and use (4.3) to get

$$\begin{aligned}
\|u\|_K^2 &= \|y\|_K^2 + 2t(y, w_\varepsilon)_K + \|tw_\varepsilon\|_K^2 \\
&= \|y\|_K^2 + |y|_{2,K}O\left(\varepsilon^{\frac{N-2}{4}}\right) + \|tw_\varepsilon\|_K^2.
\end{aligned}$$

Since an analogous estimate holds for $|u|_{2,K}^2$ and $\|y\|_K^2 \leqslant \lambda_n|y|_{2,K}^2$, we obtain

$$\|u\|_K^2 - \lambda|u|_{2,K}^2 \leqslant (\lambda_n - \lambda)|y|_{2,K}^2 + |y|_{2,K}O\left(\varepsilon^{\frac{N-2}{4}}\right) + \|tw_\varepsilon\|_K^2 - \lambda|tw_\varepsilon|_{2,K}^2. \qquad (4.6)$$

We now recall that $-as^2 + bs \leqslant \frac{-b^2}{4a}$ whenever $a > 0$ and $s \in \mathbb{R}$. Thus, the above expression implies that

$$\|u\|_K^2 - \lambda|u|_{2,K}^2 \leqslant \frac{1}{4(\lambda - \lambda_n)}O\left(\varepsilon^{\frac{N-2}{2}}\right) + Q_\lambda(tw_\varepsilon)|tw_\varepsilon|_{2^*,K}^{2^*}, \qquad (4.7)$$

where

$$Q_\lambda(v) := \frac{\|v\|_K^2 - \lambda |v|_{2,K}^2}{|v|_{2^*,K}^{2^*}},$$

for any $v \in H^1(\alpha) \setminus \{0\}$.

Since $Q_\lambda(w_\varepsilon) = Q_\lambda(u_\varepsilon)$ and $\lambda > \lambda_1/2$, we can use the calculations performed in [4, pp. 1166–1167] to get

$$Q_\lambda(w_\varepsilon) = \begin{cases} S + \varepsilon^{\alpha/2} d + O(\varepsilon^\alpha), & \text{if } N > 2\alpha + 2, \\ S + \varepsilon^{\alpha/2} d + O(\varepsilon^\alpha |\log \varepsilon|) + O(\varepsilon^{\frac{N-2}{2}}), & \text{if } N = 2\alpha + 2, \\ S + \varepsilon^{\alpha/2} d + O(\varepsilon^{\frac{N-2}{2}}), & \text{if } \alpha + 2 < N < 2\alpha + 2, \\ S + \varepsilon^{\alpha/2} |\log \varepsilon| d + O(\varepsilon^{\frac{N-2}{2}}), & \text{if } N = \alpha + 2, \end{cases}$$

with $d < 0$ being a negative number. Hence, for $\mu > 0$ small, $N \geqslant \alpha + 2$ and γ given by

$$\gamma := \min\left\{\alpha - \mu, \frac{N-2}{2}\right\} > 0, \tag{4.8}$$

we have that

$$Q_\lambda(t w_\varepsilon) = Q_\lambda(w_\varepsilon) \leqslant S + d\varepsilon^{\alpha/2 - \mu} + O(\varepsilon^\gamma). \tag{4.9}$$

On the other hand, by using the Mean Value Theorem we obtain $\theta = \theta(x) \in (0, 1)$ such that

$$1 = \int K(x) |t w_\varepsilon + y|^{2^*} = \int K(x) \left\{ |t w_\varepsilon|^{2^*} + 2^* |t w_\varepsilon + \theta y|^{2^*-2} (t w_\varepsilon + \theta y) y \right\}.$$

Thus, we can use the equivalence of norms in Y, Lemma 4.4 and (4.2), to get

$$1 \geqslant |t w_\varepsilon|_{2^*,K}^{2^*} + 2^* \int K(x) |t w_\varepsilon|^{2^*-1} |y| \geqslant |t w_\varepsilon|_{2^*,K}^{2^*} - |y|_{2,K} O\left(\varepsilon^{\frac{N-2}{4}}\right),$$

from which it follows that

$$|t w_\varepsilon|_{2^*,K}^{2^*} \leqslant 1 + |y|_{2,K} O\left(\varepsilon^{\frac{N-2}{4}}\right).$$

By replacing the above estimate and (4.9) in (4.7), we conclude that

$$\begin{aligned} \|u\|_K^2 - \lambda |u|_{2,K}^2 &\leqslant \frac{1}{4(\lambda - \lambda_n)} O\left(\varepsilon^{\frac{N-2}{2}}\right) + S + d\varepsilon^{\alpha/2-\mu} + O\left(\varepsilon^\gamma\right) \\ &= S + \varepsilon^{\alpha/2-\mu}\left(d + O\left(\varepsilon^{\gamma - \frac{\alpha}{2} + \mu}\right) + O\left(\varepsilon^{\frac{N-2-\alpha}{2} + \mu}\right)\right). \end{aligned}$$

Recalling that $N - \alpha - 2 \geqslant 0$ and using the definition of γ, we conclude that $\min\{\gamma - \alpha/2 + \mu, (N - 2 - \alpha)/2 + \mu\} > 0$. Since $d < 0$, it follows from the above expression that, for any $\varepsilon > 0$ small, there hods

$$\|u\|_K^2 - \lambda |u|_{2,K}^2 < S, \quad \forall u \in \Sigma_\varepsilon.$$

This concludes the proof in this first case. □

We now consider the case where $\lambda = \lambda_n$ is an eigenvalue. Notice that, in this setting, estimation (4.6) does not hold and therefore we need slightly change the argument. So, for any $\varepsilon > 0$ we define

$$\widetilde{w}_\varepsilon := w_\varepsilon - (w_\varepsilon, \varphi_n)_K \varphi_n.$$

The next lemma shows that this new function has the same properties of w_ε.

Lemma 4.5. *The following estimates hold as $\varepsilon \to 0^+$*

$$\int K(x)|\widetilde{w}_\varepsilon|^{2^*-1} = O\big(\varepsilon^{\frac{N-2}{4}}\big), \quad |\widetilde{w}_\varepsilon|_{L^1(\mathbb{R}^N)} = O\big(\varepsilon^{\frac{N-2}{4}}\big),$$

$$\max\big\{(y, \widetilde{w}_\varepsilon)_K, (y, \widetilde{w}_\varepsilon)_{2,K}\big\} = |y|_{2,K}\, O\big(\varepsilon^{\frac{N-2}{4}}\big),$$

and

$$Q_\lambda(tw_\varepsilon) = Q_\lambda(w_\varepsilon) \leqslant S + d\varepsilon^{\alpha/2-\mu} + O\big(\varepsilon^\gamma\big), \tag{4.10}$$

where $d < 0$, $\mu > 0$ is small and γ is given by (4.8)*.*

Proof. We have that

$$\int K(x)|\widetilde{w}_\varepsilon|^{2^*-1} \leqslant c_1 \int K(x)|w_\varepsilon|^{2^*-1} + c_1\big|(w_\varepsilon, \varphi_n)_K\big|^{2^*-1} \int K(x)|\varphi_n|^{2^*-1}.$$

The exponential decay of φ_n implies that the last integral is finite. Since $2^* - 1 > 1$, it follows from the above inequality, (4.2) and (4.3) that

$$\int K(x)|\widetilde{w}_\varepsilon|^{2^*-1} = O\big(\varepsilon^{\frac{N-2}{4}}\big),$$

and therefore the first statement holds. By using the exponential decay of φ_n and Lemma 4.3 we can prove the second and the third statement in a similar way. We omit the details.

It remains to check the last assertion. Notice that, in view of the definition of $\widetilde{w}_\varepsilon$ and Lemma 4.3, we have that

$$\begin{aligned}\|\widetilde{w}_\varepsilon\|_K^2 &= \|w_\varepsilon\|_K^2 + (w_\varepsilon, \varphi_n)_K^2 \|\varphi_n\|_K^2 - 2(w_\varepsilon, \varphi_n)_K^2 \\ &\leqslant \|w_\varepsilon\|_K^2 + O\big(\varepsilon^{\frac{N-2}{2}}\big)\end{aligned}$$

and

$$|\widetilde{w}_\varepsilon|_{2,K}^2 = |w_\varepsilon|_{2,K}^2 + O\big(\varepsilon^{\frac{N-2}{2}}\big).$$

In order to estimate $|\widetilde{w}_\varepsilon|_{2^*,K}^2$ in terms of $|w_\varepsilon|_{2^*,K}^2$, we write

$$
\begin{aligned}
|\widetilde{w}_\varepsilon|_{2^*,K}^{2^*} - |w_\varepsilon|_{2^*,K}^{2^*} &= \int_{\mathbb{R}^N} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} \left| w_\varepsilon - s(w_\varepsilon, \varphi_n)_K \varphi_n \right|^{2^*} \mathrm{d}s\,\mathrm{d}x \\
&\leqslant c_2 \left| (w_\varepsilon, \varphi_n)_K \right| \int_{\mathbb{R}^N} K(x) |w_\varepsilon|^{2^*-1}\,\mathrm{d}x \\
&\quad + c_3 \left| (w_\varepsilon, \varphi_n)_K \right|^{2^*} \int_{\mathbb{R}^N} K(x) |\varphi_n|^{2^*}\,\mathrm{d}x \\
&= O\big(\varepsilon^{\frac{N-2}{4}}\big) O\big(\varepsilon^{\frac{N-2}{4}}\big) + O\big(\varepsilon^{\frac{N-2}{4}}\big)^{2^*} = O\big(\varepsilon^{\frac{N-2}{2}}\big),
\end{aligned}
$$

where we have used the exponential decay of φ_n and Lemma 4.3. It follows from the above expression that, for some $\theta \in (0,1)$, there holds

$$
\begin{aligned}
|\widetilde{w}_\varepsilon|_{2^*,K}^{2} &= \big(|\widetilde{w}_\varepsilon|_{2^*,K}^{2^*} \big)^{2/2^*} = \big(|w_\varepsilon|_{2^*,K}^{2^*} + O\big(\varepsilon^{\frac{N-2}{2}}\big) \big)^{2/2^*} \\
&= \big(|w_\varepsilon|_{2^*,K}^{2^*} \big)^{2/2^*} + \frac{2}{2^*} \big(|w_\varepsilon|_{2^*,K}^{2^*} + \theta O\big(\varepsilon^{\frac{N-2}{2}}\big) \big)^{\frac{2}{2^*}-1} O\big(\varepsilon^{\frac{N-2}{2}}\big).
\end{aligned}
$$

If follows from (3.9) that $0 < \lim_{\varepsilon \to 0^+} |w_\varepsilon|_{2^*,K}^{2^*} < \infty$. Thus, we conclude that

$$
|\widetilde{w}_\varepsilon|_{2^*,K}^{2} = |w_\varepsilon|_{2^*,K}^{2} + O\big(\varepsilon^{\frac{N-2}{2}}\big).
$$

All together, the above estimates provide

$$
Q_\lambda(\widetilde{w}_\varepsilon) \leqslant \frac{\|w_\varepsilon\|_K^2 - \lambda |w_\varepsilon|_{2,K}^2 + O(\varepsilon^{\frac{N-2}{2}})}{|w_\varepsilon|_{2^*,K}^2 + O(\varepsilon^{\frac{N-2}{2}})} = Q_\lambda(w_\varepsilon) + \frac{O(\varepsilon^{\frac{N-2}{2}})}{|w_\varepsilon|_{2^*,K}^2 + O(\varepsilon^{\frac{N-2}{2}})}.
$$

The statement (4.10) is now a consequence of the above inequality and (4.9). This finishes the proof. □

Let us now prove Proposition 4.2 in the resonant case $\lambda = \lambda_n$.

Proof of Proposition 4.2 *(Case $\lambda = \lambda_n$).* As in the first case, it suffices to show that

$$
\widetilde{m}_\varepsilon := \max_{u \in \widetilde{\Sigma}_\varepsilon} \big(\|u\|_K^2 - \lambda_n |u|_{2,K}^2 \big) < S,
$$

where

$$
\widetilde{\Sigma}_\varepsilon := \big\{ u = y + t\widetilde{w}_\varepsilon \colon\ y \in Y,\ t \in \mathbb{R},\ |u|_{2^*,K} = 1 \big\}.
$$

Let $u = y + t\widetilde{w}_\varepsilon \in \widetilde{\Sigma}_\varepsilon$ and notice that the function $y \in Y$ can be written as

$$
y = \widetilde{y} + (y, \varphi_n)_K \varphi_n.
$$

Since $(\varphi_n, \widetilde{w}_\varepsilon)_K = (\varphi_n, \widetilde{w}_\varepsilon)_{2,K} = 0$ and $\|\varphi_n\|_K^2 = \lambda_n |\varphi_n|_{2,K}^2$, we have that

$$\|u\|_K^2 - \lambda_n |u|_{2,K}^2 = \|\widetilde{y}\|_K^2 - \lambda_n \|\widetilde{y}_n\|_{2,K}^2 + 2(\widetilde{y}, t\widetilde{w}_\varepsilon)_K - 2\lambda_n(\widetilde{y}, t\widetilde{w}_\varepsilon)_{2,K} \\ + Q_\lambda(t\widetilde{w}_\varepsilon)|t\widetilde{w}_\varepsilon|_{2^*,K}^2.$$

Lemma 4.5 and the same argument employed in the proof of Lemma 4.4 show that $t = O(1)$ as $\varepsilon \to 0^+$. Thus, we can use the above inequality, $\widetilde{y} \in \operatorname{span}\{\varphi_1, \ldots, \varphi_{n-1}\}$ and Lemma 4.4 to get

$$\|u\|_K^2 - \lambda_n |u|_{2,K}^2 \leqslant \frac{1}{4(\lambda_{n-1} - \lambda_n)} O\big(\varepsilon^{\frac{N-2}{2}}\big) + Q_\lambda(t\widetilde{w}_\varepsilon)|t\widetilde{w}_\varepsilon|_{2^*,K}^2.$$

It follows from the boundedness of t, (4.10) and the same argument used in the first case that, for $\varepsilon > 0$ small enough, there holds

$$\|u\|_K^2 - \lambda |u|_{2,K}^2 < S, \quad \forall u \in \widetilde{\Sigma}_\varepsilon.$$

The proposition is proved. $\square$

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