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Multiplicity of solutions for resonant elliptic systems

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Abstract

We establish the existence and multiplicity of solutions for some resonant elliptic systems. The results are proved by applying minimax arguments and Morse theory.

Keywords: Resonant elliptic systems; Variational methods; Morse theory

1. Introduction

Let us consider the problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geqslant 3$ and $f \in C^1(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ is a subcritical nonlinearity, that is,

$$\left|f(x,s)\right| \leqslant c_1\left(1 + |s|^{p-1}\right)$$

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for all $(x,s) \in \Omega \times \mathbb{R}$ and for some $2 < p < 2^* = 2N/(N-2)$. We say that the problem (1.1) is asymptotically linear if there exists a function α such that

$$\lim_{|s| \to \infty} \frac{f(x,s)}{s} = \alpha(x).$$

It is well known (see [1,9,13,21]) that, in this case, the existence of solutions for (1.1) is related with the interaction between the limit function $\alpha(x)$ and the spectrum $\sigma(-\Delta, H_0^1(\Omega))$ of the linear problem

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

A special class of such problems is the resonant case where

$$\alpha(x) \equiv \lambda_k \tag{1.3}$$

for all $x \in \Omega$ and for some eigenvalue $\lambda_k \in \sigma(-\Delta, H_0^1(\Omega))$. This kind of problem (and its variants) is interesting and seems to be more difficult because the associated functional may not satisfy the classical Palais–Smale condition, which is important to prove the deformation theorems that we need for applying minimax arguments.

The aim of this paper is to study some classes of resonant elliptic systems. More specifically, we deal with the gradient system

$$(P) \qquad \begin{cases} -\Delta u = F_u(x,u,v) & \text{in } \Omega, \\ -\Delta v = F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $N \geqslant 3$. The function $F \in C^2(\Omega \times \mathbb{R}^2, \mathbb{R})$ satisfies the subcritical growth condition:

(F) there exist $c_1 > 0$ and $2 < p < 2^*$ such that

$$\left|\nabla F(x,z)\right| \leqslant c_1\left(1 + |z|^{p-1}\right), \quad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R}^2.$$

Under the above hypothesis, the weak solutions of the system (P) are precisely the critical points of the C^2-functional $I : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ given by

$$I(u,v) = \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) dx - \int_\Omega F(x,u,v)\, dx.$$

In order to obtain such critical points, we will impose some conditions in the behavior of the potential F at the infinity and at the origin. Before to presenting these assumptions, we need to introduce some notation. So, let us denote by $\mathcal{M}_2(\Omega)$ the set of all symmetric matrices of the form

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & c(x) \end{pmatrix},$$

where the functions $a, b, c \in C(\bar{\Omega}, \mathbb{R})$ satisfy

(M_1) A is cooperative, i.e., $b(x) \geqslant 0$ for all $x \in \bar{\Omega}$,
(M_2) $\max_{x\in\Omega} \max\{a(x), c(x)\} > 0$.

Given $A \in \mathcal{M}_2(\Omega)$, we can consider the weighted eigenvalue problem

$$(LP) \quad \begin{cases} -\Delta\binom{u}{v} = \lambda A(x)\binom{u}{v} & \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

In view of the conditions (M_1) and (M_2) above, we can use the spectral theory for compact operators (see [12]) and some results contained in [7] to obtain a sequence of eigenvalues

$$0 < \lambda_1(A) < \lambda_2(A) \leqslant \cdots \leqslant \lambda_k(A) \leqslant \cdots$$

such that $\lambda_k(A) \to \infty$ as $k \to \infty$ (see Section 2 for more details). The problem (*LP*) will substitute the eigenvalue problem (1.2) in the study of our system.

Since we need to control the behavior of F at infinity, we denote by $z = (u, v)$ an arbitrary vector of $\mathbb{R}^2$ and introduce the following condition:

(F_∞) there exist $A_\infty \in \mathcal{M}_2(\Omega)$ such that

$$\lim_{|z|\to\infty} \frac{2F(x,z) - \langle A_\infty(x)z, z\rangle}{|z|^2} = 0, \quad \text{uniformly for a.e. } x \in \Omega.$$

Note that the natural generalization of the resonant condition (1.3) is to suppose $A_\infty(x) \equiv \lambda_k I$ for some $\lambda_k \in \sigma(-\Delta, H_0^1(\Omega))$. Here we use a more general condition, which already appears for the scalar problem in [14], by assuming that $\lambda_k(A_\infty) = 1$, where $\lambda_k(A_\infty)$ is the kth positive eigenvalue of the weighted eigenvalue problem (*LP*).

In order to overcome the lack of compactness given by the resonant hypothesis, some extra conditions has been appeared in the literature (see [5,16]). Here we use the non-quadraticity condition introduced by Costa and Magalhães [10]. We then suppose that F satisfies

$$(NQ) \quad \lim_{|z|\to\infty} \big\{\nabla F(x,z)\cdot z - 2F(x,z)\big\} = \infty, \quad \text{uniformly for a.e. } x \in \Omega.$$

Under the above hypotheses we obtain the existence of a solution for the problem (P), as stated in the result below.

Theorem 1.1. *Suppose* (F), (F_∞) *and* (NQ) *holds. If* $\lambda_k(A_\infty) = 1$ *for some* $k \geqslant 2$, *then the problem* (P) *has at least one solution.*

Now we observe that, if $\nabla F(x, 0, 0) \equiv 0$, the problem (P) possesses the trivial solution $(u, v) = (0, 0)$. In this case, the key point is to assure the existence of nontrivial solutions for (P). With this aim, we need to introduce a condition that give us information about the behavior of F near the origin. More specifically, we suppose that

(F_0) there exist $A_0 \in \mathcal{M}_2(\Omega)$ such that

$$\lim_{|z|\to 0} \frac{2F(x,z) - \langle A_0(x)z, z\rangle}{|z|^2} = 0, \quad \text{uniformly for a.e. } x \in \Omega.$$

Our first result concerning the multiplicity of solutions for (P) can be stated as

Theorem 1.2. *Suppose $\nabla F(x,0,0) \equiv 0$ and (F), (F_0), (F_∞) and (NQ) holds. If $\lambda_k(A_\infty) = 1$ for some $k \geqslant 2$ and $\lambda_m(A_0) < 1 < \lambda_{m+1}(A_0)$ for some $m \neq k-1$, then the problem (P) has at least one nontrivial solution.*

In our next multiplicity result we suppose a nondegeneration condition and obtain the existence of two nontrivial solutions. In this case, we consider the complementary case $\lambda_1(A_0) > 1$ and prove

Theorem 1.3. *Suppose $\nabla F(x,0,0) \equiv 0$ and (F), (F_0), (F_∞) and (NQ) holds. If $\lambda_k(A_\infty) = 1$ for some $k \geqslant 3$ and $\lambda_1(A_0) > 1$, then the problem (P) has at least two nontrivial solutions provided $D^2F(x,z) \in \mathcal{M}_2(\Omega)$ for all $(x,z) \in \Omega \times \mathbb{R}^2$.*

In the above theorems we do not allow resonance at the first eigenvalue $\lambda_1(A_\infty)$. However, in this case we are able to prove that the functional is coercive and we obtain the existence of two nontrivial solutions for (P).

Theorem 1.4. *Suppose $\nabla F(x,0,0) \equiv 0$ and (F), (F_0) and (F_∞) holds. If $\lambda_m(A_0) < 1 < \lambda_{m+1}(A_0)$, then the problem (P) has at least two nontrivial solutions, provided either*

(i) $\lambda_1(A_\infty) > 1$, *or*
(ii) $\lambda_1(A_\infty) = 1$ *and* (NQ) *holds.*

For related results in the study of asymptotically linear elliptic systems we can cite [11,15]. Our results are not comparable and complement it, since we deal with the weighted linear problem (*LP*). It also complement the paper of Bartsch, Chang and Wang [4], where a Landesman–Lazer condition is assumed. Finally, we would like to cite [23], where a noncooperative system is studied with no compactness condition, and the recent paper of Li and Yang [17], where the case of asymptotically linear Hamiltonian systems was studied.

In the proof of our theorems we use some critical point theorems and Morse theory. As it is well known, this kind of theory is based on the existence of a linking structure and on deformation lemmas [2,3,9,20]. In general, to be able to derive such deformation results, it is supposed that the functional I satisfies a compactness condition. In this article, we use the (Ce) condition introduced by Cerami in [6]. We then recall that I satisfies the Cerami condition at level $c \in \mathbb{R}$ ($(\mathrm{Ce})_c$ for short), if any sequence $(z_n) \subset H_0^1(\Omega) \times H_0^1(\Omega)$ such that $I(z_n) \to c$ and $\|I'(z_n)\|(1 + \|z_n\|) \to 0$ possesses a convergent subsequence.

The paper is organized as follows: in Section 2, after presenting the abstract framework, we make a detailed discussion of the weighted eigenvalue problem (*LP*). In Section 3 we prove Theorem 1.1. Section 4 is devoted to the proof of Theorems 1.2 and 1.3. Theorem 1.4 is proved in Section 5.

2. Abstract framework

Hereafter we write $\int_\Omega u$ instead of $\int_\Omega u(x)\,dx$. Let H be the Hilbert space $H_0^1(\Omega) \times H_0^1(\Omega)$ equipped with the norm

$$\|z\|^2 = \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)$$

for all $z = (u, v) \in H$. By the Sobolev theorem we know that, for any $2 \leqslant \sigma \leqslant 2^*$ fixed, the embedding $H \hookrightarrow L^\sigma(\Omega) \times L^\sigma(\Omega)$ is continuous and therefore we can find a positive constant S_σ such that

$$\int_\Omega |z|^\sigma \leqslant S_\sigma \|z\|^\sigma. \tag{2.1}$$

Moreover, if $\sigma < 2^*$, the Rellich–Kondrachov theorem implies that the above embedding is also compact.

We proceed now with the study of the linear problem associated to (P). Let

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & c(x) \end{pmatrix} \in \mathcal{M}_2(\Omega)$$

and consider the eigenvalue problem with weight $A(x)$,

$$(LP) \quad \begin{cases} -\Delta u = \lambda(a(x)u + b(x)v) & \text{in } \Omega, \\ -\Delta v = \lambda(b(x)u + c(x)v) & \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

A simple calculation shows that λ is an eigenvalue of (LP) if and only if

$$T_A(u, v) = \frac{1}{\lambda}(u, v),$$

where $T_A : H \to H$ is the symmetric bounded linear operator defined by

$$\begin{aligned} \left\langle T_A(u, v), (\phi, \psi)\right\rangle &= \int_\Omega \left\langle A(x)\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \end{pmatrix}\right\rangle \\ &= \int_\Omega \left(a(x)u + b(x)v\right)\phi + \left(b(x)u + c(x)v\right)\psi. \end{aligned}$$

Since the coefficients of A are continuous functions and the embedding $H \hookrightarrow L^2(\Omega) \times L^2(\Omega)$ is compact, we can check that the operator T_A is also compact. Thus, we may invoke the spectral theory for compact operators to conclude that H possesses a Hilbertian basis formed by eigenfunctions of (LP).

Let us denote $z = (u, v)$ and

$$\frac{1}{\lambda_1(A)} = \mu_1(A) = \sup\left\{\langle T_A z, z\rangle \colon \|z\| = 1\right\}.$$

Recalling that A satisfies (M_1)–(M_2), we can use [7, Theorem 1.1] (see also [8, Theorem 1.1]) to conclude that the eigenvalue $\mu_1(A)$ is positive, simple and isolated in the spectrum of T_A. Moreover, if we denote by Φ_1^A the normalized eigenfunction associated to $\lambda_1(A)$, we can suppose that the two function coordinates of Φ_1^A are positive on Ω. By using induction, if we suppose that $\mu_1(A) > \mu_2(A) \geqslant \cdots \geqslant \mu_{k-1}(A)$ are the $k-1$ first eigenvalues of T_A and $\{\Phi_i^A\}_{i=1}^{k-1}$ are the associated normalized eigenfunctions, we can define

$$\frac{1}{\lambda_k(A)} = \mu_k(A) = \sup\bigl\{\langle T_A z, z\rangle\colon\ \|z\| = 1,\ z \in \bigl(\operatorname{span}\bigl\{\Phi_1^A, \ldots, \Phi_{k-1}^A\bigr\}\bigr)^{\perp}\bigr\}.$$

It is proved in [12, Proposition 1.3] that, if $\mu_k(A) > 0$, then it is an eigenvalue of T_A with associated normalized eigenfunction Φ_k^A. In view of the condition (M_2), we can argue as in the proof of [12, Proposition 1.11(c)] and conclude that $\mu_k(A) > 0$. Thus, we obtain a sequence of eigenvalues for (LP)

$$0 < \lambda_1(A) < \lambda_2(A) \leqslant \cdots \leqslant \lambda_k(A) \leqslant \cdots$$

such that $\lambda_k(A) \to \infty$ as $k \to \infty$. Moreover, if we set $V_k = \operatorname{span}\{\Phi_1^A, \ldots, \Phi_k^A\}$, we can decompose H as $H = V_k \oplus V_k^{\perp}$ and the following variational inequalities hold:

$$\|z\|^2 \leqslant \lambda_k(A) \int_\Omega \bigl\langle A(x)z, z\bigr\rangle, \quad \forall z \in V_k, \tag{2.2}$$

and

$$\|z\|^2 \geqslant \lambda_{k+1}(A) \int_\Omega \bigl\langle A(x)z, z\bigr\rangle, \quad \forall z \in V_k^{\perp}. \tag{2.3}$$

3. Proof of Theorem 1.1

In this section we present the proof of Theorem 1.1. As stated in the introduction, we will look for critical points of the C^2-functional $I : H \to \mathbb{R}$ given by

$$I(z) = \int_\Omega \bigl(|\nabla u|^2 + |\nabla v|^2\bigr) - \int_\Omega F(x, z).$$

The first step is to prove that I satisfies the Cerami condition at any level $c \in \mathbb{R}$. In order to verify this, we first note that, given $\varepsilon > 0$, we can use (F_∞) to obtain $R_\varepsilon > 0$ such that

$$\bigl|2F(x, z) - \bigl\langle A_\infty(x)z, z\bigr\rangle\bigr| \leqslant \varepsilon|z|^2, \quad \forall x \in \Omega,\ |z| \geqslant R_\varepsilon.$$

This and the continuity of F provide $M_\varepsilon > 0$ such that

$$F(x, z) \geqslant \frac{1}{2}\bigl\langle A_\infty(x)z, z\bigr\rangle - \frac{\varepsilon}{2}|z|^2 - M_\varepsilon \tag{3.1}$$

and

$$F(x, z) \leqslant \frac{1}{2}\bigl\langle A_\infty(x)z, z\bigr\rangle + \frac{\varepsilon}{2}|z|^2 + M_\varepsilon \tag{3.2}$$

for any $(x, z) \in \Omega \times \mathbb{R}^2$.

Lemma 3.1. *If* (F_∞) *and* (NQ) *hold, then* I *satisfies* $(\mathrm{Ce})_c$ *for any* $c \in \mathbb{R}$.

Proof. Let $(z_n) \subset H$ be such that $I(z_n) \to c$ and $\|I'(z_n)\|_{H^*}(1 + \|z_n\|) \to 0$. Since the nonlinearity F has subcritical growth, standard arguments [20] show that the lemma is proved if we can show that (z_n) has a bounded subsequence. Suppose, by contradiction, that $\|z_n\| \to \infty$ as $n \to \infty$. Then, there exists $M > 0$ such that

$$\liminf_{n\to\infty} \int_\Omega H(x, z_n) = \liminf_{n\to\infty} \int_\Omega \big(2I(z_n) - \langle I'(z_n), z_n\rangle\big) \leqslant M, \tag{3.3}$$

where $H(x, z_n) = \nabla F(x, z_n) \cdot z_n - 2F(x, z_n)$. We obtain a contradiction by the following claim: there exists $\hat{\Omega} \subset \Omega$ with positive measure, such that up to a subsequence, $|u_n(x)| \to +\infty$ or $|v_n(x)| \to +\infty$ as $n \to +\infty$, for a.e. $x \in \hat{\Omega}$.

Assuming the claim, by Fatou's lemma and (NQ), we have

$$\liminf \int_\Omega H(x, z_n) \geqslant \int_\Omega \liminf H(x, z_n) = \infty,$$

which contradicts (3.3).

Now we proceed with the proof of the claim. Given $\varepsilon > 0$, by (3.2) and $I(z_n) \to c$ we have, for n sufficiently large,

$$\begin{aligned}\frac{1}{2}\|z_n\|^2 &\leqslant (c+1) + \frac{1}{2}\int_\Omega \big\langle A_\infty(x)z_n, z_n\big\rangle + \frac{\varepsilon}{2}\int_\Omega |z_n|^2 + M_\varepsilon|\Omega| \\ &\leqslant M + \frac{1}{2}\int_\Omega \big\langle A_\infty(x)z_n, z_n\big\rangle + \frac{\varepsilon S_2}{2}\|z_n\|^2.\end{aligned} \tag{3.4}$$

Defining $\hat{z}_n = (\hat{u}_n, \hat{v}_n) = \frac{1}{\|z_n\|}(|u_n|, |v_n|)$, we may assume that $\hat{u}_n \to \hat{u}$, $\hat{v}_n \to \hat{v}$ strongly in $L^2(\Omega)$ and $\hat{u}_n(x) \to \hat{u}(x)$, $\hat{v}_n(x) \to \hat{v}(x)$ for a.e. $x \in \Omega$. Thus, dividing (3.4) by $\|z_n\|^2$, taking $n \to \infty$, $\varepsilon \to 0$, we conclude that

$$1 \leqslant \int_\Omega \big\langle A_\infty(x)\hat{z}, \hat{z}\big\rangle, \tag{3.5}$$

where $\hat{z} = (\hat{u}, \hat{v})$. By denoting

$$A_\infty(x) = \begin{pmatrix} a_\infty(x) & b_\infty(x) \\ b_\infty(x) & c_\infty(x) \end{pmatrix}$$

and recalling that $b_\infty(x) \geqslant 0$, we can use (3.5) and Young's inequality to obtain

$$\begin{aligned}1 &\leqslant \int_\Omega \big(a_\infty(x)\hat{u}^2 + 2b_\infty(x)\hat{u}\hat{v} + c_\infty(x)\hat{v}^2\big) \\ &\leqslant \int_\Omega \big(a_\infty(x) + b_\infty(x)\big)\hat{u}^2 + \int_\Omega \big(b_\infty(x) + c_\infty(x)\big)\hat{v}^2,\end{aligned}$$

and therefore there exists $\hat{\Omega} \subset \Omega$ such that $\hat{u}(x) \neq 0$ or $\hat{v}(x) \neq 0$, a.e. $x \in \hat{\Omega}$. The claim is now proved by observing that we are assuming that $\|z_n\| \to +\infty$ as $n \to +\infty$. $\square$

Lemma 3.2. *If* (F_∞) *and* (NQ) *hold, then there exists* $M_\infty > 0$ *such that*

$$F(x,z) - \frac{1}{2}\langle A_\infty(x)z, z\rangle \leqslant M_\infty, \quad \forall (x,z) \in \Omega \times \mathbb{R}^2.$$

Proof. Defining $G(x,z) = F(x,z) - \frac{1}{2}\langle A_\infty(x)z, z\rangle$, we have

$$\nabla G(x,z)\cdot z - 2G(x,z) = \nabla F(x,z)\cdot z - 2F(x,z).$$

For any fixed $\bar{z} \in \mathbb{R}^2$ with $|\bar{z}| = 1$, it follows from (NQ) that, for every $M > 0$, there is $R_M > 0$ such that

$$\nabla G(x, s\bar{z})\cdot (s\bar{z}) - 2G(x, s\bar{z}) \geqslant M, \quad \forall |s| \geqslant R_M.$$

Thus,

$$\frac{d}{ds}\left[\frac{G(x,s\bar{z})}{s^2}\right] = \frac{\nabla G(x,s\bar{z})\cdot(s\bar{z}) - 2G(x,s\bar{z})}{s^3} \geqslant \frac{M}{s^3}.$$

Integrating the above expression over the interval $[t,T] \subset [R_M, +\infty)$, we obtain

$$\frac{G(x,t\bar{z})}{t^2} \leqslant \frac{G(x,T\bar{z})}{T^2} + \frac{M}{2}\left[\frac{1}{T^2} - \frac{1}{t^2}\right].$$

Letting $T \to +\infty$, we conclude that $G(x,t\bar{z}) \leqslant -M/2$, for $t \geqslant R_M$, for a.e. $x \in \Omega$. In a similar way, we have $G(x,t\bar{z}) \leqslant -M/2$, for $t \leqslant -R_M$ and a.e. $x \in \Omega$. Hence

$$\lim_{|z|\to\infty} G(x,z) = -\infty, \quad \text{uniformly for a.e. } x \in \Omega.$$

The lemma follows from the above equality and the continuity of G. □

We are now ready to prove our first theorem.

Proof of Theorem 1.1. Let $k \geqslant 2$ be given by the condition (F_∞). For any $1 \leqslant j \leqslant k-1$, let $\Phi_j^{A_\infty}$ be the normalized eigenfunction associated to the jth positive eigenvalue $\lambda_j(A_\infty)$, as explained in Section 2. If we define

$$V = \operatorname{span}\{\Phi_1^{A_\infty}, \ldots, \Phi_{k-1}^{A_\infty}\} \quad \text{and} \quad W = V^\perp,$$

we have that $H = V \oplus W$ and $\dim V = k-1 < \infty$. We claim that the functional I satisfies the geometry of the Saddle Point Theorem, that is,

Claim 1. $I(z) \to -\infty$ as $\|z\| \to \infty$, $z \in V$.

Claim 2. There exists $\gamma \in \mathbb{R}$ such that $I(z) \geqslant \gamma$, for all z in W.

Assuming that the above claims are true, we can use Lemma 3.1 and the Saddle Point Theorem [20, Theorem 4.6] (see also [22, Theorem 2.13]) to get a critical point for I. As explained before, this critical point is a solution of (P) and we have done.

It remains to prove the claims. Let us first consider the first one. Since $k \geqslant 2$ and $\lambda_1(A_\infty) < \lambda_2(A_\infty)$, we may suppose, without loss of generality, that $\lambda_{k-1}(A_\infty) < \lambda_k(A_\infty) = 1$. Thus, the inequality (2.2) implies that, for any $z \in V \setminus \{0\}$,

$$\|z\|^2 \leqslant \lambda_{k-1}(A_\infty) \int_\Omega \langle A_\infty(x)z, z\rangle < \int_\Omega \langle A_\infty(x)z, z\rangle.$$

Recalling that V is finite dimensional, we obtain $\delta > 0$ such that

$$\|z\|^2 - \int_\Omega \langle A_\infty(x)z, z\rangle \leqslant -\delta\|z\|^2, \quad \forall z \in V.$$

Thus, we can use (3.1) and (2.1) to get

$$\begin{aligned} I(z) &\leqslant \frac{1}{2}\|z\|^2 - \frac{1}{2}\int_\Omega \langle A_\infty(x)z, z\rangle + \frac{\varepsilon}{2}\int_\Omega |z|^2 + M_\varepsilon|\Omega| \\ &\leqslant \frac{1}{2}(-\delta + \varepsilon S_2)\|z\|^2 + M_\varepsilon|\Omega|. \end{aligned}$$

By taking $\varepsilon = \delta/(2S_2)$, we conclude that $I(z) \to -\infty$ as $\|z\| \to \infty$, $z \in V$.

In order the verify the second claim, we take $z \in W$ and use $\lambda_k(A_\infty) = 1$, (2.3) and Lemma 3.2, to get

$$\begin{aligned} I(z) &= \frac{1}{2}\|z\|^2 - \frac{\lambda_k(A_\infty)}{2}\int_\Omega \langle A_\infty(x)z, z\rangle - \int_\Omega \left(F(x,z) - \frac{1}{2}\langle A_\infty(x)z, z\rangle\right) \\ &\geqslant -M_\infty|\Omega|. \end{aligned}$$

Hence Claim 2 is true and the theorem is proved. □

4. Proofs of Theorems 1.2 and 1.3

In this section we prove our multiplicity results concerning the resonance at higher eigenvalues. Since we will apply Morse theory, it is useful to recall some concepts. Let $z_0 \in H$ be an isolated critical point of I, $c = I(z_0)$ and j be a nonnegative integer. We define the jth critical group of I at z_0 as being

$$C_j(I, z_0) = H_j\big(I^c, I^c \setminus \{z_0\}\big),$$

where $I^c = \{u \in H\colon\ I(u) \leqslant c\}$ and $H_j(I^c, I^c \setminus \{z_0\})$ denotes the jth relative singular homology group with coefficients in $(\mathbb{Z}, +)$ (see [9] for more details). The critical groups will enable us to distinguish the critical points obtained by different kinds of links.

Proof of Theorem 1.2. Let w be the solution given by Theorem 1.1. It is sufficient to show that $w \neq 0$. With this aim we first note that, since w was obtained by the Saddle Point Theorem with the finite dimensional subspace having dimension $k-1$, we know by [9, Theorem 1.5 of Chapter 2] that

$$C_{k-1}(I, w) \neq 0. \tag{4.1}$$

Recalling that $F \in C^2(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R})$ and using some calculations, we can see that the matrix $A_0(x)$ given by the condition (F_0) is precisely the second derivative $D^2F(x,0)$. Thus, the condition $\lambda_m(A_0) < 1 < \lambda_{m+1}(A_0)$ implies that 0 is a nondegenerated critical point of I and the Morse index of I at 0, which we denote by $m(I,0)$, is equal to m. Thus, by applying [9, Theorem 4.1 of Chapter 1], we get

$$C_j(I,0) = \begin{cases} \mathbb{Z}, & \text{if } j = m, \\ 0, & \text{otherwise.} \end{cases}$$

Since we are assuming that $m \neq k-1$, the above equation and (4.1) show that $w \neq 0$ and the theorem is proved. □

Before to presenting the proof of Theorem 1.3, we need the following result.

Lemma 4.1. *Suppose $\nabla F(x,0,0) \equiv 0$ and (F), (F_0) and (F_∞) holds. If $\lambda_1(A_0) > 1$ and $\lambda_k(A_\infty) = 1$ for some $k \geqslant 3$, then the functional I has the mountain pass geometry, that is, $I(0) = 0$ and*

(i) *there exist $\alpha, \rho > 0$ such that $I(z) \geqslant \alpha$, for all $z \in H$ such that $\|z\| = \rho$,*
(ii) *there exists $e \in H$ such that $\|e\| > \rho$ and $I(e) \leqslant 0$.*

Proof. Given $\varepsilon > 0$, we can use (F_0) to obtain $\delta_\varepsilon > 0$ such that

$$\left|2F(x,z) - \langle A_0(x)z, z\rangle\right| \leqslant \varepsilon |z|^2, \quad \forall x \in \Omega,\ |z| < \delta_\varepsilon.$$

Moreover, condition (F) provides $A_\varepsilon > 0$ such that

$$\left|F(x,z)\right| \leqslant A_\varepsilon |z|^p, \quad \forall x \in \Omega,\ |z| \geqslant \delta_\varepsilon.$$

It follows from the two above inequalities that

$$F(x,z) \leqslant \frac{1}{2}\langle A_0(x)z, z\rangle + \frac{\varepsilon}{2}|z|^2 + A_\varepsilon |z|^p \tag{4.2}$$

for any $(x,z) \in \Omega \times \mathbb{R}^2$.

Thus, for any $z \in H$, we can use the above inequality, (2.3) and (2.1) to get

$$\begin{aligned} I(z) &\geqslant \frac{1}{2}\|z\|^2 - \frac{1}{2}\int_\Omega \langle A_0(x)z, z\rangle - \frac{\varepsilon}{2}\int_\Omega |z|^2 - A_\varepsilon \int_\Omega |z|^p \\ &\geqslant \frac{1}{2}\left(1 - \frac{1}{\lambda_1(A_0)} - \varepsilon S_2\right)\|z\|^2 - A_\varepsilon S_p \|z\|^p. \end{aligned}$$

Recalling that $\lambda_1(A_0) > 1$, we can take $\varepsilon > 0$ small in such way that $(1 - 1/\lambda_1(A_0) - \varepsilon S_2) = \nu > 0$, and conclude that

$$I(z) \geqslant \frac{\nu}{2}\|z\|^2 - A_\varepsilon S_p \|z\|^p = \left(\frac{\nu}{2} - A_\varepsilon S_p \|z\|^{p-2}\right)\|z\|^2.$$

Thus, we can check that the item (i) holds for $\rho = (\nu/(4A_\varepsilon S_p))^{1/(p-2)}$ and $\alpha = \rho^2\nu/4$.

Let $\Phi_1^{A_\infty}$ be the normalized eigenfunction associated to $\lambda_1(A_\infty)$. In view of (3.1) and (2.3), we have

$$\begin{aligned}
I\big(t\Phi_1^{A_\infty}\big) &= \frac{1}{2}\big\|t\Phi_1^{A_\infty}\big\|^2 - \int_\Omega F\big(x, t\Phi_1^{A_\infty}\big) \\
&\leqslant \frac{t^2}{2} - \frac{t^2}{2}\int_\Omega \big\langle A_\infty(x)\Phi_1^{A_\infty}, \Phi_1^{A_\infty}\big\rangle + \frac{t^2\varepsilon}{2}\int_\Omega \big|\Phi_1^{A_\infty}\big|^2 + M_\varepsilon|\Omega| \\
&= \frac{t^2}{2}\left(1 - \frac{1}{\lambda_1(A_\infty)} + \varepsilon\int_\Omega \big|\Phi_1^{A_\infty}\big|^2\right) + M_\varepsilon|\Omega|.
\end{aligned}$$

Since $\lambda_1(A_\infty) < \lambda_k(A_\infty) = 1$, we can choose $\varepsilon > 0$ small in such way that $(1 - 1/\lambda_1(A_\infty) + \varepsilon\int_\Omega |\Phi_1^{A_\infty}|^2) = \kappa < 0$, and therefore

$$I\big(t\Phi_1^{A_\infty}\big) \leqslant \frac{t^2}{2}\kappa + M_\varepsilon|\Omega| \to -\infty,$$

as $t \to \infty$. Thus, for sufficiently large $t_0 > \rho$ we have $I(t_0\Phi_1^{A_\infty}) \leqslant 0$. This proves item (ii) and concludes the proof of the lemma. $\square$

Proof of Theorem 1.3. Let w be the nontrivial solution given by Theorem 1.2. In view of Lemmas 4.1 and 3.1, we can apply the Mountain Pass Theorem [20, Theorem 2.2] to obtain a nontrivial solution $\bar{w}$ of the problem (P). In order to prove the theorem, it suffices to show that $\bar{w} \neq w$.

Firstly, we note that

$$C_1(I, \bar{w}) \neq 0.$$

Now, by the Shifting Theorem [9, Corollary 5.1 of Chapter 1] we have that $m(I, \bar{w}) \leqslant 1$. If $m(I, \bar{w}) = 1$ then, by the Shifting theorem, we have

$$C_j(I, \bar{w}) = \begin{cases} \mathbb{Z}, & \text{if } j = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

If $m(I, \bar{w}) = 0$ then $\bar{w}$ is a degenerated critical point and the assumption that $D^2F(x, \bar{w}) \in \mathcal{M}_2(\Omega)$ implies that $\dim \operatorname{Ker} I''(\bar{w}) = 1$ (see [7]). Again, we can use [9, Theorem 1.6 of Chapter 2] to conclude that (4.3) also holds in this case.

Recalling that $k > 2$, we conclude that $C_{k-1}(I, \bar{w}) = 0$ and it follows from (4.1) that $\bar{w} \neq w$. The theorem is proved. $\square$

5. Proof of Theorem 1.4

In this final section we consider the resonance at the first eigenvalue $\lambda_1(A_\infty)$. We start by proving that, in this case, the functional I is coercive on H.

Lemma 5.1. *If (F_∞) holds and we have either*

(i) $\lambda_1(A_\infty) > 1$, *or*
(ii) $\lambda_1(A_\infty) = 1$ *and* (NQ) *holds,*

then the functional I is coercive on H, i.e., $I(z) \to \infty$ whenever $\|z\| \to \infty$.

Proof. We first present the proof of (i). Given $\varepsilon > 0$ small we can use (3.2), (2.3) and $\lambda_1(A_\infty) > 1$ to obtain

$$\begin{aligned} I(z) &\geqslant \frac{1}{2}\|z\|^2 - \frac{1}{2}\int_\Omega \langle A_\infty(x)z, z\rangle - \varepsilon \int_\Omega |z|^2 - M_\varepsilon|\Omega| \\ &\geqslant \frac{1}{2}\left(1 - \frac{1}{\lambda_1(A_\infty)} - \varepsilon S_2\right)\|z\|^2 - M_\varepsilon|\Omega| \\ &\geqslant \frac{\nu}{2}\|z\|^2 - M_\varepsilon|\Omega| \end{aligned} \tag{5.1}$$

for some $\nu > 0$. Thus, $I(z) \to +\infty$ as $\|z\| \to +\infty$ and the lemma is proved in this first case.

For the second case (ii) we note that, if

$$G(x, z) = F(x, z) - \frac{1}{2}\langle A_\infty(x)z, z\rangle,$$

then we can argue as in the proof of Lemma 3.2 and conclude that

$$\lim_{|z|\to\infty} G(x, z) = -\infty, \quad \text{uniformly for a.e. } x \in \Omega. \tag{5.2}$$

Now we suppose, by contradiction, that the lemma is false. Then there exists $(z_n) \subset H$ such that $\|z_n\| \to \infty$ as $n \to \infty$ and $I(z_n) \leqslant C$, for some $C \in \mathbb{R}$. By writing $z_n = (u_n, v_n)$, we define the new sequence $\hat{z}_n = \frac{z_n}{\|z_n\|} = (\hat{u}_n, \hat{v}_n)$. Passing to a subsequence if necessary, we may suppose that

$$\begin{aligned} &\hat{z}_n \rightharpoonup \hat{z} = (\hat{u}, \hat{v}) \quad \text{weakly in } H, \\ &\hat{z}_n \to \hat{z} \quad \text{in } L^2(\Omega) \times L^2(\Omega), \\ &\hat{z}_n(x) \to \hat{z}(x) \quad \text{for a.e. } x \in \Omega. \end{aligned} \tag{5.3}$$

In view of Lemma 3.2, we have that

$$\begin{aligned} C \geqslant I(z_n) &= \frac{1}{2}\|z_n\|^2 - \frac{1}{2}\int_\Omega \langle A_\infty(x)z_n, z_n\rangle - \int_\Omega G(x, z_n) \\ &\geqslant \frac{1}{2}\|z_n\|^2 - \frac{1}{2}\int_\Omega \langle A_\infty(x)z_n, z_n\rangle - M_\infty|\Omega|. \end{aligned}$$

Dividing this expression by $\|z_n\|^2$, letting $n \to \infty$ and using (5.3), we obtain

$$1 \leqslant \int_\Omega \langle A_\infty(x)\hat{z}, \hat{z}\rangle.$$

On the other hand, recalling the weak convergence in (5.3) and that $\lambda_1(A_\infty) = 1$, we can use (2.3) to get

$$1 \leqslant \int_\Omega \langle A_\infty(x)\hat{z}, \hat{z}\rangle \leqslant \|\hat{z}\|^2 \leqslant \liminf_{n\to\infty} \|\hat{z}_n\|^2 = 1,$$

from which follows that $\|\hat{z}\| = 1$. This and the above expression show that $\hat{z} = \pm\Phi_1^{A_\infty}$, the first eigenfunction of (*LP*). Since in both cases the components of $\hat{z}$ have the same constant sign, we conclude that $|u_n(x)| \to \infty$ and $|v_n(x)| \to \infty$ as $n \to \infty$. This fact, $\lambda_1(A_\infty) = 1$, (2.3) and (5.2) imply that

$$C \geqslant I(z_n) = \frac{1}{2}\|z_n\|^2 - \frac{1}{2}\int_\Omega \langle A_\infty(x)z_n, z_n\rangle - \int_\Omega G(x, z_n)$$
$$\geqslant -\int_\Omega G(x, z_n) \to \infty,$$

as $n \to \infty$. This is a contradiction and the lemma is proved. □

In order to verify that I has a local linking at the origin, we take A_0 be the function given by hypothesis (F_0). Recalling the spectral theory of the operator T_{A_0} discussed in Section 2, we set

$$V = \operatorname{span}\{\Phi_1^{A_0}, \Phi_2^{A_0}, \ldots, \Phi_m^{A_0}\} \quad \text{and} \quad W = V^\perp.$$

Thus, we have the direct decomposition $H = V \oplus W$. Moreover, the following holds.

Lemma 5.2. *If (F_0) holds and $\lambda_m(A_0) < 1 < \lambda_{m+1}(A_0)$, then the functional I has a local link at the origin, i.e.,*

(i) *there exists $\rho_1 > 0$ such that $I(z) \leqslant 0$, for all $z \in V \cap B_{\rho_1}(0)$,*
(ii) *there exists $\rho_2 > 0$ such that $I(z) > 0$, for all nonzero $z \in W \cap B_{\rho_2}(0)$.*

Proof. Given $\varepsilon > 0$, we can use (F_0) and (F) as in the proof of Theorem 1.3, to obtain $\delta_\varepsilon, A_\varepsilon > 0$ such that

$$F(x, z) \geqslant \frac{1}{2}\langle A_0(x)z, z\rangle - \frac{\varepsilon}{2}|z|^2 - A_\varepsilon|z|^p \tag{5.4}$$

for any $(x, z) \in \Omega \times \mathbb{R}^2$.

By taking $\varepsilon > 0$ sufficiently small, we can use (5.4), (2.2), (2.1) and $\lambda_k(A_0) < 1$ to obtain

$$I(z) \leqslant \frac{1}{2}\|z\|^2 - \frac{1}{2}\int_\Omega \langle A_0(x)z, z\rangle + \frac{\varepsilon}{2}\int_\Omega |z|^2 + A_\varepsilon \int_\Omega |z|^p$$
$$\leqslant \frac{1}{2}\left(1 - \frac{1}{\lambda_k(A_0)} + \varepsilon S_2\right)\|z\|^2 + A_\varepsilon S_p\|z\|^p$$
$$\leqslant \left(\frac{\kappa}{2} + A_\varepsilon S_p\|z\|^{p-2}\right)\|z\|^2$$

for some $\kappa < 0$ and for all $z \in V$. Hence, we have that the condition (i) holds for $\rho_1 = (-\kappa/2A_\varepsilon S_p)^{1/(p-2)} > 0$.

In order to verify (ii), we choose $\varepsilon > 0$ small and use (4.2), (2.3), (2.1) and $\lambda_{k+1}(A_0) > 1$, to get

$$\begin{aligned} I(z) &\geqslant \frac{1}{2}\left(1 - \frac{1}{\lambda_{k+1}(A_0)} - \varepsilon S_2\right)\|z\|^2 - A_\varepsilon S_p \|z\|^p \\ &\geqslant \left(\frac{\mu}{2} + A_\varepsilon S_p \|z\|^{p-2}\right)\|z\|^2 \end{aligned}$$

for some $\mu > 0$ and for all $z \in W$. As before, we can check that (ii) holds for $\rho_2 = (\mu/2A_\varepsilon S_p)^{1/(p-2)} > 0$. The lemma is proved. □

Proof of Theorem 1.4. In view of Lemmas 5.1 and 5.2, we can invoke the Three Critical Point Theorem [18] (see also [19, Theorem 2.1]) to obtain two nontrivial critical points for I. As before, these critical points are nontrivial solutions of (P) and the theorem is proved. □

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