Multiple solutions for a semilinear problem with combined terms and nonlinear boundary condition

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Abstract

We consider the problem

$$-\Delta u + u = f(x,u) \text{ in } \Omega, \quad \frac{\partial u}{\partial \eta} = h(x)|u|^{q-2}u \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 3$, $1 \leq q < 2$ and h belongs to an appropriated Lebesgue space. In our main results we suppose that f is an asymptotically linear function and we obtain multiplicity of solutions when the norm of h is small. We also present a multiplicity result in the case that f is nonquadratic at infinity.

Key words: variational methods; elliptic problems; sublinear and asymptotically linear; resonant problems.

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1 Introduction

In this paper we consider the problem

$$(P)\qquad \begin{cases} -\Delta u + u = f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = h(x)|u|^{q-2}u & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 3$ and $\frac{\partial}{\partial \eta}$ is the outer normal derivative. The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with subcritical growth. More specifically, if we denote by σ' the Hölder conjugate of $\sigma > 1$, we assume that f satisfies the following condition

(f_0) there exist $2 < p < 2^*$, $a_1 > 0$ and $a \in L^{\sigma_p}(\Omega)$ such that

$$|f(x,s)| \leq a_1|s|^{p-1} + a(x), \quad \text{for a.e. } x \in \Omega,\ s \in \mathbb{R},$$

where $2^* := 2N/(N-2)$ and $\sigma_p := (2^*/p)'$.

Concerning the term on the boundary we assume that $1 \leq q < 2$ and

(h_0) $h \in L^{\sigma_q}(\partial\Omega)$, where $2_* := 2(N-1)/(N-2)$ and $\sigma_q := (2_*/q)'$.

We say that f is asymptotically linear if there exists a function k such that

$$\lim_{|s|\to\infty} \frac{f(x,s)}{s} = k(x).$$

In the Dirichlet case, it is well known (see [1–4]) that the existence of solution is related with the interaction between the limit function $k(x)$ and the spectrum of the operator $(-\Delta + \mathrm{Id})$ in $H_0^1(\Omega)$. In our case, we consider the asymptotic limit as a weight in the linear problem. So, we fix from now on $r > N/2$ and introduce the eigenvalue problem

$$(LP)\qquad \begin{cases} -\Delta u + u = \lambda k(x) u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases}$$

If the function k is positive on a set of positive measure, we can apply standard spectral theory to obtain a sequence of eigenvalues $\{\lambda_j(k)\}_{j\in\mathbb{N}}$ such that $\lambda_j(k) \to \infty$ as $j \to \infty$.

In our first result we also suppose that f verifies

(f_1) there exists $K_0 \in L^r(\Omega)$ such that

$$\lim_{s\to 0^+} \frac{2F(x,s)}{s^2} = K_0(x), \quad \text{uniformly for a.e. } x \in \Omega,$$

(f_2) there exists $k_\infty \in L^r(\Omega)$ such that

$$\lim_{s\to \infty} \frac{f(x,s)}{s} = k_\infty(x), \quad \text{uniformly for a.e. } x \in \Omega.$$

If we denote by $g^+(x) := \max\{g(x), 0\}$ the positive part of a given function g, we can state our first result as follows.

Theorem 1.1 *Suppose* (h_0), (f_0), (f_1) *and* (f_2) *with* $\lambda_1(k_\infty) < 1 < \lambda_1(K_0)$. *Suppose also that* $k_\infty(x) \geq 0$ *for a.e.* $x \in \Omega$ *and*

$(\widehat{f_0})$ *there exist* $b \in L^r(\Omega)$ *and* $c \in L^{(2^*)'}(\Omega)$ *such that*

$$|f(x,s)| \leq b(x)|s| + c(x), \quad \textit{for a.e. } x \in \Omega,\ s \geq 0.$$

Then there exists $m > 0$ *such that the problem* (P) *has two nonzero solutions whenever* $0 < \|h^+\|_{L^{\sigma_q}(\partial\Omega)} < m$. *Moreover, if* $1 < q < 2$, *the two solutions are positive on* Ω.

In our next result we allow the function f to be superlinear at infinity. So, we replace the condition (f_2) by the the so called nonquadraticity condition introduced by Costa and Magalhães in [5], namely

(f_3) there exists $a_2 > 1$ such that

$$\liminf_{s\to \infty} \frac{2F(x,s)}{s^2} \geq a_2 > 1, \quad \text{uniformly for a.e. } x \in \Omega;$$

(NQ) the following hold

(i) there exist $a_3 \geq 0$ and $\gamma > 0$ such that

$$\limsup_{s\to \infty} \frac{F(x,s)}{s^\gamma} \leq a_3, \quad \text{uniformly for a.e. } x \in \Omega.$$

(ii) there exist $a_4 > 0$ and $\mu > \max\{2_*, N(\gamma-2)/2\}$ such that

$$\liminf_{s\to \infty} \frac{f(x,s)s - 2F(x,s)}{s^\mu} \geq a_4, \quad \text{uniformly for a.e. } x \in \Omega;$$

We shall prove the following multiplicity result.

Theorem 1.2 *Suppose* (h_0), (f_0), (f_1) *with* $\lambda_1(K_0) > 1$, (f_3) *and* (NQ). *Then the same conclusions of Theorem 1.1 hold.*

We notice that $\lambda_1(1) = 1$, and therefore the condition (f_3) is related with the crossing of the first eigenvalue as $s \to \infty$. Moreover, as quoted in [5], the condition (NQ)(i) is clearly valid for $\gamma = p$ but it may be true for small values of γ.

We apply Critical Point Theory in the proof of our theorems. The main idea is firstly obtain a solution $u \in W^{1,2}(\Omega)$ with positive energy via the Mountain Pass Theorem. Later, we use a minimizing argument to get another solution with negative energy. The condition $\|h^+\|_{L^{\sigma_q}(\Omega)} > 0$ is used only to obtain the second solution $v = v_h$. So we can obtain some existence results even in the case $h \leq 0$ on $\partial\Omega$ (see Remark 2.1). As a byproduct of the minimizing argument, we also show that $v_h \to 0$ in $W^{1,2}(\Omega)$ as $\|h^+\|_{L^{\sigma_q}(\Omega)} \to 0$ (see Remark 2.2).

In Theorems 1.1 and 1.2 we look for non-negative solutions, in such way that the behavior of $f(x,s)$ for negative values of s is not important. In our next results we consider again the asymptotically linear case, but we are not worried with the sign of the solution. Hence we replace the condition (f_2) by the following one

$(\widehat{f_2})$ there exists $K_\infty \in L^r(\Omega)$ such that

$$\lim_{|s|\to\infty} \frac{2F(x,s)}{s^2} = K_\infty(x), \quad \text{uniformly for a.e. } x \in \Omega.$$

We are interested now in the resonant case, namely $\lambda_j(K_\infty) = 1$ for some $j \in \mathbb{N}$. It is well known that, in this case, the associated functional does not satisfy the usual compactness conditions. In order to overcome this difficult we use a version of the nonquadraticity condition [5] (see also [6]). We assume the following

$(\widehat{NQ})$ there exist $\Omega_0 \subset \Omega$ and $d \in L^1(\Omega)$ such that

(i) $\lim_{|s|\to\infty} [f(x,s)s - 2F(x,s)] = +\infty$ uniformly for a.e. $x \in \Omega_0$,

(ii) $[f(x,s)s - 2F(x,s)] \geq d(x)$, for a.e. $x \in \Omega,\ s \in \mathbb{R}$.

If we denote by $|A|$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}^N$, we can state our resonant result in the following way.

Theorem 1.3 *Suppose (h_0), $h \leq 0$ on $\partial\Omega$, (f_0) and $(\widehat{f_2})$ with $\lambda_j(K_\infty) = 1$ for some $j \in \mathbb{N}$. Then there exists $0 < \alpha < |\Omega|$ such that, if $(\widehat{NQ})$ holds with $|\Omega_0| > \alpha$, then the problem (P) possesses a solution. Moreover, if $j = 1$, the number α can be taken equals to zero.*

In the proof of the above theorem we apply the Saddle Point Theorem. The restriction on the sign of h is of technical nature. However, we emphasize that other results for problems with a negative parameter in the concave term can be found in some previous works (see [7,8] and references therein).

As a byproduct of the calculations performed in the proof of Theorem 1.3, we can also consider the complementary case $\lambda_1(K_\infty) > 1$. In this setting, we prove that the functional is coercive, and therefore we need no compactness assumptions neither restrictions on the sign of h, as can be viewed in the next result.

Theorem 1.4 *Suppose* (h_0), (f_0) *and* $(\widehat{f_2})$ *with* $\lambda_1(K_\infty) > 1$. *Then the problem* (P) *possesses a solution.*

We make now some comments on the motivation of our paper. Since we consider a concave term, the starting point of our study is the celebrated paper of Ambrosetti, Brezis and Cerami [9], where they considered the problem

$$-\Delta u = \lambda |u|^{q-2}u + |u|^{p-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

with $1 < q < 2$ and $2 < p < 2^*$. Among other results, they obtained the existence of two positive solutions provided $\lambda > 0$ is sufficiently small. After this work, many authors have considered the effect of concave-convex terms in Dirichlet problems. In 2004, Garcia-Azorero, Peral and Rossi [10] proved analogous results for the nonlinear problem

$$-\Delta u + u = |u|^{p-2}u \text{ in } \Omega, \quad \frac{\partial u}{\partial \eta} = \lambda |u|^{q-2}u \text{ on } \partial\Omega.$$

Another paper which is closely related with ours is that of Li, Wu and Zhou [11], where they studied the problem

$$-\Delta u = h(x)|u|^{q-2}u + f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

with $1 < q < 2$, $h \in L^\infty(\Omega)$ and f satisfying

$$\lim_{s\to\infty} \frac{f(x,s)}{s} = l > \mu_1, \tag{1.1}$$

where $\mu_1 > 0$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Under some other assumptions on the function f they proved the existence of two non-negative solutions for small values of $\|h\|_{L^\infty(\Omega)}$. These results are recently extended for the p-laplacian operator by de Paiva in [12].

In view of the connection of the papers [9,10], it is natural to ask if we can obtain similar results to that of [11] for the problem (P). The main results of our paper give a positive answer to this question. We emphasize that we allow that the asymptotic limit of the ratio $f(x,s)/s$ depends on x as well as the function $h(x)$ to be unbounded in Ω. Notice that (1.1) clearly implies our technical assumption $(\widehat{f_0})$ with b, $c \in L^\infty(\Omega)$ (a similar condition has already appeared in [12]). The ideas in dealing with asymptotical limits interacting with weighted linear problems has already been used in other papers (see [13,14,12] and references therein). Differently of the aforementioned works we do not suppose that h is bounded. In order to overcome this difficult we proceed as in [15] by making suitable applications of Hölder's inequality.

We point out that Theorem 1.1 is closely related to [11, Theorems 1.1. and 1.2]. Our second result Theorem 1.2 complements (and is not comparable with) the result of [11, Theorem 1.3]. It is worthwhile to mention that, although our problem is different from that considered in [11], the arguments developed here alow improvements in all the results of that paper. Moreover, differently of the aforementioned works, we also consider here the resonant and the coercive case. As a final comment, we notice that our approach enable us to obtain some partial results even in the linear case $q = 2$ (see Remarks 2.3 and 3.2).

In the next section we present the proof of our first two theorems. Section 3 is devoted to the proof of Theorems 1.3 and 1.4.

2 Two non-negative solutions

Throughout the paper we suppose that the functions f and h satisfy (f_0) and (h_0). For save notation, we write only $\int_\Omega g$ and $\int_{\partial\Omega} g$ instead of $\int_\Omega g(x)\mathrm{d}x$ and $\int_{\partial\Omega} g(x)\mathrm{d}\sigma$, respectively, where $\mathrm{d}\sigma$ is the measure on the boundary. For any $1 \le t \le \infty$, $\|g\|_t$ and $|g|_t$ denote the norms in $L^t(\Omega)$ and $L^t(\partial\Omega)$, respectively.

We denote by H the Hilbert space $W^{1,2}(\Omega)$ endowed with the inner product

$$\langle u, v \rangle := \int_\Omega (\nabla u \cdot \nabla v + uv), \quad \text{for any } u, v \in H,$$

and by $\|\cdot\|$ its associated norm .

Since we are firstly interested in positive solutions we assume that $f(x, s) = 0$ for a.e. $x \in \Omega$, $s \le 0$. It follows from (f_0), (h_0) and standard arguments that the nonnegative weak solutions of (P) are precisely the critical points of the C^1-functional

$$I(u) := \frac{1}{2}\int_\Omega (|\nabla u|^2 + u^2) - \int_\Omega F(x, u^+) - \frac{1}{q}\int_{\partial\Omega} h(x)(u^+)^q, \quad \text{for any } u \in H.$$

We recall that I satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ ($(\mathrm{PS})_c$ for short) if any sequence $(u_n) \subset H$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_{H'} \to 0$

contains a convergent subsequence. Here, $\|I'(u)\|_{H'}$ denotes the norm of the Frechét derivative $I'(u)$ in the dual space H'.

Lemma 2.1 *Suppose f satisfies $(\widehat{f_0})$ and (f_2) with $\lambda_1(k_\infty) < 1$. Then the functional I satisfy the $(PS)_c$ condition for any $c \in \mathbb{R}$.*

Proof. Let $(u_n) \subset H$ be such that $I(u_n) \to c$ and $\|I'(u_n)\|_{H'} \to 0$. Since f has subcritical growth and $q < 2_*$, it suffices to show that (u_n) has a bounded subsequence. So we suppose, by contradiction, that $\|u_n\| \to \infty$ and set $v_n := \dfrac{u_n^+}{\|u_n\|}$.

Since $r > N/2$, we can choose $2 < t < 2^*$ such that

$$\frac{1}{r} + \frac{1}{t} + \frac{1}{2^*} = 1. \tag{2.1}$$

Up to a subsequence, we have that

$$\begin{cases} v_n \rightharpoonup v \text{ weakly in } H, \\ v_n \to v \text{ strongly in } L^t(\Omega), \\ v_n(x) \to v(x), \ |v_n(x)| \leq \psi_t(x) \text{ for a.e. } x \in \Omega, \end{cases} \tag{2.2}$$

for some non-negative function $v \in H$ and $\psi_t \in L^t(\Omega)$.

The boundedness of (v_n) and $I'(u_n) \to 0$ imply that

$$\begin{aligned} o_n(1) = \frac{I'(u_n)(v_n - v)}{\|u_n\|} &= \langle v_n, v_n - v\rangle - \frac{1}{\|u_n\|}\int_\Omega f(x, u_n^+)(v_n - v) \\ &\quad - \frac{1}{\|u_n\|}\int_{\partial\Omega} h(x)(u_n^+)^{q-1}(v_n - v), \end{aligned} \tag{2.3}$$

where $o_n(1)$ denotes a quantity approaching zero as $n \to \infty$. Hölder's inequality with exponents σ_q, $2_*/(q-1)$ and 2_*, provide

$$\begin{aligned} \frac{1}{\|u_n\|}\int_{\partial\Omega} |h(x)||u_n^+|^{q-1}|v_n - v| &= \frac{1}{\|u_n\|^{2-q}}\int_{\partial\Omega} |h(x)||v_n|^{q-1}|v_n - v| \\ &\leq \frac{1}{\|u_n\|^{2-q}} |h|_{\sigma_q} |v_n|_{2_*}^{q-1} |v_n - v|_{2_*} = o_n(1), \end{aligned}$$

and therefore, since $1 \leq q < 2$, we infer from (2.3) that

$$\langle v_n, v_n - v\rangle = \frac{1}{\|u_n\|}\int_\Omega f(x, u_n^+)(v_n - v) + o_n(1). \tag{2.4}$$

It follows from $(\widehat{f_0})$ and Hölder's inequality with exponents given in (2.1) that

$$\begin{aligned}\frac{1}{\|u_n\|}\int_\Omega f(x,u_n^+)(v_n-v) &\leq \int_\Omega b(x)|v_n||v_n-v| + \frac{1}{\|u_n\|}\int_\Omega c(x)|v_n-v| \\ &\leq \|b\|_r\|v_n\|_{2^*}\|v_n-v\|_t + \frac{1}{\|u_n\|}\|c\|_{(2^*)'}\|v_n-v\|_{2^*} \\ &= o_n(1),\end{aligned}$$

where we have used (2.2) and $\|u_n\| \to \infty$ in the last equality. This and (2.4) imply that $\langle v_n, v_n - v\rangle = o_n(1)$. Since $\|v_n\| = 1$, we obtain $v_n \to v \in H \setminus \{0\}$ strongly in H.

Let $\phi \in H$ be fixed. For n large we have that

$$\begin{aligned}\left|f(x,u_n^+)\frac{\phi}{\|u_n\|}\right| &\leq b(x)|v_n||\phi| + \frac{1}{\|u_n\|}c(x)|\phi| \\ &\leq b(x)\psi_t(x)|\phi| + c(x)|\phi|,\end{aligned} \tag{2.5}$$

with the right-hand side belonging to $L^1(\Omega)$. The first estimate in (2.5) implies that $f(x,u_n^+(x))\phi(x)/\|u_n\| \to 0$ almost everywhere in the set $\{x \in \Omega : v(x) = 0\}$. On the other hand, in the set $\{x \in \Omega : v(x) > 0\}$, we can use (f_2) and the definition of v_n to get

$$f(x,u_n^+(x))\frac{\phi(x)}{\|u_n\|} = \frac{f(x,u_n^+(x))}{u_n^+(x)}v_n(x)\phi(x) \to k_\infty(x)v(x)\phi(x), \quad \text{as } n \to \infty.$$

Thus, it follows from (2.5) and the Lebesgue Theorem that

$$\lim_{n\to\infty}\int_\Omega f(x,u_n^+)\frac{\phi}{\|u_n\|} = \int_\Omega k_\infty(x)v(x)\phi(x).$$

Recalling that $I'(u_n)\phi/\|u_n\| \to 0$ and arguing as in the first part of the proof, we conclude that the function v weakly solves the linear problem

$$-\Delta v + v = k_\infty(x)v \ \text{ in } \Omega.$$

Since $v \not\equiv 0$ and $k_\infty(x)v(x) \geq 0$ a.e. in Ω, it follows from the Maximum Principle (cf. [16, Theorem 8.19]) that $v > 0$ in Ω. But this implies that $\lambda_1(k_\infty) = 1$, which contradicts the hypothesis. This contradiction proves that (u_n) has a bounded subsequence and we have done. □

In the two results below we verify the geometric conditions for applying the Mountain Pass Theorem.

Lemma 2.2 *Suppose f satisfies (f_1) with $\lambda_1(K_0) > 1$. Then there exists $\rho > 0$ and $\alpha > 0$ such that $I(u) \geq \alpha > 0$, for any $u \in H$ with $\|u\| = \rho$, provided $|h^+|_{\sigma_q}$ is small enough.*

Proof. Given $\varepsilon > 0$, the hypothesis (f_1) provides $\delta > 0$ such that

$$F(x,s) \leq \frac{1}{2}(K_0(x) + \varepsilon)s^2, \quad \text{for a.e. } x \in \Omega,\ 0 \leq s \leq \delta.$$

By (f_0), there exist $c_1, c_2 > 0$ such that

$$F(x,s) \leq c_1|s|^p + c_2 a(x)|s|^p, \quad \text{for a.e. } x \in \Omega,\ s \geq \delta.$$

Hence, for some function $\widehat{a} \in L^{\sigma_p}(\Omega)$, we have that

$$F(x,s) \leq \frac{1}{2}(K_0(x) + \varepsilon)s^2 + \widehat{a}(x)|s|^p, \quad \text{for a.e. } x \in \Omega,\ s \geq 0. \tag{2.6}$$

This inequality, the variational characterization of $\lambda_1(K_0)$, the Sobolev embeddings and Hölder's inequality provide

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_\Omega K_0(x)u^2 - \frac{\varepsilon}{2}\int_\Omega u^2 - \int_\Omega \widehat{a}(x)|u|^p - \frac{1}{q}\int_{\partial\Omega} h(x)(u^+)^q \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\lambda_1(K_0)} - \varepsilon\right)\|u\|^2 - c_3\|\widehat{a}\|_{\sigma_p}\|u\|^p - c_4|h^+|_{\sigma_q}\|u\|^q. \end{aligned}$$

and therefore, for a small value of $\varepsilon > 0$, we get

$$I(u) \geq \frac{\nu}{2}\|u\|^2 - c_5\|u\|^p - c_4|h^+|_{\sigma_q}\|u\|^q, \tag{2.7}$$

where $c_4,\ c_5 > 0$ and $\nu := (1 - 1/\lambda_1(K_0) - \varepsilon) > 0$.

We now fix $0 < t < 1/(2-q)$ and notice that, if $\rho := |h^+|_{\sigma_q}^t$, the above inequality implies that, for any $u \in H$ such that $\|u\| = \rho$, there holds

$$I(u) \geq |h^+|_{\sigma_q}^{2t}\left(\frac{\nu}{2} - c_5|h^+|_{\sigma_q}^{t(p-2)} - c_4|h^+|_{\sigma_q}^{1+t(q-2)}\right).$$

Since $p > 2$ and $1 + t(q-2) > 0$ we see that the expression into the brackets above becomes positive whenever $|h^+|_{\sigma_q}$ is small enough. This concludes the proof. $\square$

Lemma 2.3 *Suppose that the conditions of Lemma 2.2 hold and let $\rho > 0$ be given by that lemma. If the function f also satisfies (f_2) with $\lambda_1(k_\infty) < 1$, then there exists $e \in H \setminus \overline{B}_\rho(0)$ such that $I(e) < 0$.*

Proof. For any given $\varepsilon > 0$ we can use (f_0) and (f_2) to obtain

$$F(x,s) \geq \frac{1}{2}(k_\infty(x) - \varepsilon)s^2 - \widehat{a}(x), \quad \text{for a.e. } x \in \Omega,\ s \geq 0,$$

for some function $\widehat{a} \in L^{\sigma_p}(\Omega)$.

Let $\varphi_1 := \varphi_1(k_\infty) > 0$ be the first eigenfunction associated to the linear problem (LP) with weight k_∞. The last inequality provides

$$I(t\varphi_1) \leq \frac{t^2}{2}\|\varphi_1\|^2 - \frac{t^2}{2}\int_\Omega (k_\infty(x) - \varepsilon)\varphi_1^2 + \|\widehat{a}\|_1 - \frac{t^q}{q}\int_{\partial\Omega} h(x)(\varphi_1)^q.$$

Since $\lambda_1(k_\infty)\int_\Omega k_\infty(x)\varphi_1 = \|\varphi_1\|^2$ and $1 \leq q < 2$, we obtain

$$\begin{aligned}\frac{I(t\varphi_1)}{t^2} &\leq \frac{1}{2}\|\varphi_1\|^2 - \frac{1}{2}\int_\Omega (k_\infty(x) - \varepsilon)\varphi_1^2 + \frac{1}{t^2}\|\widehat{a}\|_1 - \frac{t^{q-2}}{q}\int_{\partial\Omega} h(x)(\varphi_1)^q \\ &\leq \frac{1}{2}\left(1 - \frac{1}{\lambda_1(k_\infty)} + \varepsilon\right)\|\varphi_1\|^2 + o_t(1),\end{aligned}$$

as $t \to \infty$. Since $\lambda_1(k_\infty) < 1$, we can choose $\varepsilon > 0$ small to conclude that

$$\limsup_{t\to\infty} \frac{I(t\varphi_1)}{t^2} < 0.$$

It suffices now to take $e := t\varphi_1$ with $t > 0$ large enough. $\square$

We are ready to prove our first result.

Proof of Theorem 1.1. In view of Lemma 2.1, there exits $m > 0$ such that the conclusion of that lemma holds whenever $|h^+|_{\sigma_q} < m$. It follows from Lemma 2.2, Lemma 2.3 and the Mountain Pass Theorem [17] that I possesses a critical point u such that $I(u) \geq \alpha > 0$.

In order to obtain the second solution we use a minimization argument. Let $\rho > 0$ be given by Lemma 2.1 and consider $(v_n) \subset \overline{B_\rho(0)}$ such that

$$I(v_n) \to d := \inf_{\overline{B_\rho(0)}} I < \infty. \tag{2.8}$$

We have that $v_n \rightharpoonup v \in \overline{B_\rho(0)}$ weakly in H. Moreover, since $p < 2^*$ and $q < 2_*$, we can easily conclude that $I(v) \leq \liminf_{n\to\infty} I(v_n) \to d$ and therefore $I(v) = d$.

We claim that $d < 0$. If this is true we infer from Lemma 2.2 that $v \in B_\rho(0)$ and therefore $I'(v) = 0$ and $I(v) < 0$.

In order to prove the claim we choose $\phi \in H$ such that $\displaystyle\int_{\partial\Omega} h(x)(\phi^+)^q > 0$. Arguing as in (2.6) we obtain

$$F(x,s) \geq \frac{1}{2}(K_0(x) - \varepsilon)s^2 - \widehat{a}(x)|s|^p, \quad \text{for a.e. } x \in \Omega,\ s \geq 0,$$

with $\widehat{a} \in L^{\sigma_p}(\Omega)$. Thus, for any $t > 0$, we have that

$$I(t\phi) \leq \frac{t^2}{2}\left(\|\phi\|^2 - \int_\Omega (K_0(x) - \varepsilon)(\phi^+)^2\right) + t^p \int_\Omega \widehat{a}(x)(\phi^+)^p - \frac{t^q}{q}\int_{\partial\Omega} h(x)(\phi^+)^q.$$

Since $q < 2 < p$, we conclude that

$$\limsup_{t \to 0^+} \frac{I(t\phi)}{t^q} \leq -\int_{\partial\Omega} h(x)(\phi^+)^q < 0,$$

and therefore $I(t\phi) < 0$ for $t > 0$ sufficiently small. This implies that $d < 0$, as claimed.

We shall verify that the solutions obtained are positive if $1 < q < 2$. Indeed, for the first solution u we have that

$$\int_\Omega (\nabla u \nabla \varphi + u\varphi) = \int_\Omega f(x, u^+)\varphi + \int_{\partial\Omega} h(x)(u^+)^{q-1}\varphi, \quad \text{for any } \varphi \in H.$$

Thus, we can take $\varphi = u^- := \max\{-u, 0\}$ and recall that $f(x, s) = 0$ for $s \leq 0$ to conclude that $\|u^-\| = 0$. Hence, $u \geq 0$ in Ω and it follows from the Maximum Principle that $u > 0$ in Ω. The argument for the second solution v is analogous. $\square$

We now consider the nonquadratic case given by condition (NQ). In this new setting we shall use a compactness condition weaker than Palais-Smale. So, we recall that I satisfies the Cerami condition at level $c \in \mathbb{R}$, ($(\mathrm{Ce})_c$ for short) if any sequence $(u_n) \subset H$ such that $I(u_n) \to c$ and $(1 + \|u_n\|)\|I'(u_n)\|_{H'} \to 0$ contains a convergent subsequence.

Lemma 2.4 *If (NQ) holds then I satisfies the $(Ce)_c$ condition at any level $c \in \mathbb{R}$.*

Proof. The proof follows the same lines of [5, Lemma 1]. We present it here for the sake of completeness.

Let $(u_n) \subset H$ be such that $I(u_n) \to c \in \mathbb{R}$ and $(1 + \|u_n\|)\|I^{'}(u_n)\|_{H'} \to 0$. It follows from (f_0) and (NQ)(ii) that

$$c_1|s|^\mu - \widehat{a}(x) \leq f(x, s)s - 2F(x, s), \quad \text{for a.e. } x \in \Omega,\ s \geq 0,$$

for some $c_1 > 0$ and $\widehat{a} \in L^{\sigma_p}(\Omega)$. Since we may suppose that $2_* \leq \mu < 2^*$, Hölder's inequality implies that

$$\int_{\partial\Omega} h(x)(u_n^+)^q \leq |h|_{\sigma_q}\|u_n\|_{2_*}^q \leq c_2\|u_n\|_\mu^q,$$

with $c_2 > 0$. The two above inequalities and $1 \le q < 2$ imply that

$$
\begin{aligned}
2I(u_n) - I'(u_n)u_n &= \int_\Omega \left(f(x,u_n^+)u_n^+ - 2F(x,u_n^+)\right) + \left(\frac{q-2}{q}\right) \int_{\partial\Omega} h(x)(u_n^+)^q \\
&\ge c_1 \|u_n\|_\mu^\mu - \|\widehat{a}\|_1 - c_3 \|u_n\|_\mu^q,
\end{aligned}
$$

with $c_3 > 0$. Recalling that $\mu \ge 2_* > 2 > q$, we conclude that (u_n) is bounded in $L^\mu(\Omega)$.

On the other hand, using (NQ)(i) we obtain $c_4 > 0$ and $\widetilde{a} \in L^{\sigma_p}(\Omega)$ such that

$$
F(x,s) \le c_4 s^\gamma + \widetilde{a}(x), \quad \text{for a.e. } x \in \Omega,\ s \ge 0.
$$

This, $I(u_n) \to c$, interpolation and Hölder's inequality, the Sobolev embedding and the boundedness of (u_n) in $L^\mu(\Omega)$ provide

$$
\begin{aligned}
\frac{1}{2}\|u_n\|^2 &\le c + o_n(1) + c_4 \|u_n\|_\gamma^\gamma + \|\widetilde{a}\|_1 + \frac{1}{q}\int_{\partial\Omega} h(x)(u_n^+)^q \\
&\le c_5 + c_4 \|u_n\|_\mu^{\gamma(1-t)} \|u_n\|_{2^*}^{\gamma t} + c_6 |h|_{\sigma_q} \|u_n\|^q \\
&\le c_5 + c_7 \|u_n\|^{\gamma t} + c_6 |h|_{\sigma_q} \|u_n\|^q,
\end{aligned}
\tag{2.9}
$$

with c_5, c_6, $c_7 > 0$ and $t \in [0,1]$ satisfying

$$
\frac{1}{\gamma} = \frac{1-t}{\mu} + \frac{t}{2^*}.
$$

Since $\mu > N(\gamma-2)/2$, a straightforward calculation shows that $\gamma t < 2$. Hence (u_n) is bounded in H and the result follows. $\square$

We are now ready to prove our second theorem.

Proof of Theorem 1.2. Arguing as in the proof of Theorem 1.1 we obtain $\rho > 0$ and $\alpha > 0$ such that $I(u) \ge \alpha > 0$, for any $u \in H$ with $\|u\| = \rho$, provided $|h^+|_{\sigma_q}$ is small enough.

For any $\varepsilon > 0$ we can use (f_3) and (f_0) to obtain

$$
F(x,s) \ge \frac{1}{2}(a_2 - \varepsilon)s^2 - \widehat{a}(x), \quad \text{for a.e. } x \in \Omega,\ s \ge 0,
$$

with $a_2 > 1$ and $\widehat{a} \in L^{\sigma_p}(\Omega)$. Thus, for any $t > 0$, we have that

$$
I(t) \le \frac{t^2}{2}|\Omega|(1 - a_2 + \varepsilon) + \|\widehat{a}\|_1 - \frac{t^q}{q}\int_{\partial\Omega} h(x).
$$

Choosing $\varepsilon > 0$ small and using $1 \leq q < 2$, $a_2 > 1$, we obtain

$$\limsup_{t\to\infty} \frac{I(t)}{t^2} < 0.$$

Hence, if we denote by e the constant function $e(x) = t$ with $t > 0$ large enough, we have that $I(e) < 0$ and $\|e\| > \rho$.

It follows from Lemma 2.4, the above considerations and the Mountain Pass Theorem that I possesses a nonzero critical point $u \in H$ such that $I(u) \geq \alpha > 0$. The second solution can be obtained by minimization as done in the proof of Theorem 1.1. If $1 < q < 2$, we can argue as before to conclude that the solutions are positive in Ω. The theorem is proved. $\square$

Remark 2.1 *In the proof of Theorems 1.1 and 1.2, the condition* $|h^+|_{\sigma_q} > 0$ *was used only to show that the infimum given in (2.8) is negative. However, if* $h \leq 0$ *on* $\partial\Omega$*, we can proceed as above and obtain one nontrivial solution for the problem. Indeed, it suffices to notice that, in this case, we can argue as in the first part of the proof of Lemma 2.2 to obtain*

$$I(u) \geq \frac{1}{2}\left(1 - \frac{1}{\lambda_1(K_0)} - \varepsilon\right)\|u\|^2 - c_3\|\widehat{a}\|_{\sigma_p}\|u\|^p,$$

where we have used that $h(x)(u^+)^q \leq 0$ *on* $\partial\Omega$*. Since* $p > 2$*, the above equation implies that the origin is a local minimum for* I*. Thus, we can proceed as before to obtain a solution* $u \in H$ *such that* $I(u) > 0$*.*

Remark 2.2 *As in [15], we can study the asymptotic behavior of the second solution* $v = v_h$ *obtained in Theorems 1.1 and 1.2. Indeed, notice that it satisfies* $v_h \in B_\rho(0)$*, with* $\rho = |h^+|_{\sigma_q}^t$ *given by Lemma 2.2. Thus, we have that* $v_h \to 0$ *as* $|h^+|_{\sigma_q} \to 0$*.*

Remark 2.3 *Finally, we should point out some further results on the case* $q = 2$*. So, we suppose that is the case and assume that the conditions of Theorem 1.1 hold. The proof of Lemma 2.1 can be done in the same way. In Lemma 2.2, the expression (2.7) becomes*

$$I(u) \leq \left(\frac{\nu}{2} - c_4|h^+|_{\sigma_2}\right)\|u\|^2 - c_5\|u\|^p,$$

and therefore the lemma holds for small values of $|h^+|_{\sigma_2}$*. Also, Lemma 2.3 is true if we additionally suppose that* $h \geq 0$ *on* $\partial\Omega$*, since in this case we have that*

$$\frac{I(t\varphi_1)}{t^2} \leq \frac{1}{2}\|\varphi_1\|^2 - \frac{1}{2}\int_\Omega (k_\infty(x) - \varepsilon)\varphi_1^2 + \frac{1}{t^2}\|\widehat{a}\|_1.$$

Thus, we can use the Mountain Pass Theorem to obtain a positive solution with positive energy.

Concerning Theorem 1.2 in the case $q = 2$, it suffices to notice that expression (2.9) becomes

$$\left(\frac{1}{2} - c_6 |h|_{\sigma_2}\right) \|u_n\|^2 \leq c_5 + c_7 \|u_n\|^{\gamma t}.$$

Thus, we can proceed as above to obtain a positive solution if we additionally suppose that $|h|_{\sigma_2}$ is small.

3 The resonant and coercive cases

In this section we prove Theorems 1.3 and 1.4. Since we are not looking for signed solutions, we consider from now on the functional I defined as

$$I(u) := \frac{1}{2}\int_\Omega (|\nabla u|^2 + u^2) - \int_\Omega F(x,u) - \frac{1}{q}\int_{\partial\Omega} h(x)|u|^q.$$

We start by showing that the local nonquadraticity condition $(\widehat{NQ})$ is suffices to get compactness, provided the function h is non-positive.

Lemma 3.1 *Suppose f satisfies $(\widehat{f_2})$ and $h \leq 0$ on $\partial\Omega$. Then there exists $0 < \alpha < |\Omega|$ such that, if $(\widehat{NQ})$ holds with $|\Omega_0| > \alpha$, then I satisfies $(Ce)_c$ for any $c \in \mathbb{R}$.*

Proof. Let $(u_n) \subset H$ be such that $I(u_n) \to c \in \mathbb{R}$ and $(1 + \|u_n\|)\|I'(u_n)\|_{H'} \to 0$. As in Lemma 2.1, it suffices to verify that (u_n) has a bounded subsequence.

If we set $G(x,s) := f(x,s)s - 2F(x,s)$ we can use $(\widehat{NQ})$ and $h \leq 0$ to obtain

$$\begin{aligned}\liminf_{n\to\infty} \int_\Omega G(x,u_n) &\leq \liminf_{n\to\infty} \int_\Omega \left(G(x,u_n) + \frac{(q-2)}{q} h(x)|u_n|^q\right) \\ &= \liminf_{n\to\infty} \left(2I(u_n) - I'(u_n)u_n\right) = 2c.\end{aligned} \tag{3.1}$$

On the other hand, given $\varepsilon > 0$, it follows from $(\widehat{f_2})$ and (f_0) that

$$\frac{1}{2}(K_\infty(x) - \varepsilon)s^2 - \widehat{a}(x) \leq F(x,s) \leq \frac{1}{2}(K_\infty(x) + \varepsilon)s^2 + \widehat{a}(x), \tag{3.2}$$

for a.e. $x \in \Omega$, $s \in \mathbb{R}$ and some $\widehat{a} \in L^{\sigma_p}(\Omega)$. This and $2I(u_n) \to 2c$ imply that

$$\|u_n\|^2 \leq \int_\Omega K_\infty(x) u_n^2 + \varepsilon \int_\Omega u_n^2 + \frac{2}{q}\int_{\partial\Omega} |h(x)||u_n|^q + c_1 + o_n(1), \tag{3.3}$$

with $c_1 = 2c + 2\|\widehat{a}\|_1$. If we set $v_n := \dfrac{u_n}{\|u_n\|}$ we may suppose that, up to a subsequence,

$$\begin{cases} v_n \rightharpoonup v \ \ \text{weakly in } H, \\ v_n \to v \ \ \text{strongly in } L^2(\Omega) \text{ and } L^t(\Omega), \\ v_n(x) \to v(x), \ |v_n(x)| \le \psi_t(x) \text{ for a.e. } x \in \Omega, \end{cases} \tag{3.4}$$

for some $v \in H$ and $\psi_t \in L^t(\Omega)$, where $t > 1$ satisfies (2.1).

If we divide (3.3) by $\|u_n\|^2$ and use Hölder's inequality and the trace Sobolev embedding we get

$$1 \le \int_\Omega K_\infty(x) v_n^2 + \varepsilon \int_\Omega v_n^2 + c_2 |h|_{\sigma_q} \|u_n\|^{q-2} + o_n(1). \tag{3.5}$$

Notice that

$$\left| \int_\Omega K_\infty(x)(v_n^2 - v^2) \right| \le \|K_\infty\|_r \|v_n - v\|_t \|v_n - v\|_{2^*},$$

and therefore we infer from (3.4) that $\int_\Omega K_\infty(x) v_n^2 \to \int_\Omega K_\infty(x) v^2$ as $n \to \infty$. Hence, we can use $1 \le q < 2$ and take the limit as $n \to \infty$, $\varepsilon \to 0$ in (3.5) to obtain

$$1 \le \int_\Omega K_\infty(x) v^2. \tag{3.6}$$

At this point we claim that, if $|\Omega \setminus \Omega_0|$ is small, there exists $\widetilde{\Omega} \subset \Omega_0$ with positive measure such that $v(x) \neq 0$ for a.e. $x \in \widetilde{\Omega}$. Hence $|u_n(x)| \to \infty$ for a.e. $x \in \widetilde{\Omega}$ and we can use $h \le 0$, $(\widehat{NQ})$ and Fatou's lemma to obtain

$$2c \ge \liminf \int_\Omega G(x, u_n) \ge \int_\Omega \liminf G(x, u_n) = \infty,$$

which contradicts (3.1). Thus, the sequence (u_n) is bounded and the lemma is proved.

In order to prove the claim we denote $S := \inf\{\|u\|^2 : u \in H, \|u\|_{2^*} = 1\}$, fix $t_0 > 1$ such that

$$\frac{1}{r} + \frac{1}{2^*/2} + \frac{1}{t_0} = 1$$

and set

$$\alpha := |\Omega| - \left(\frac{S}{\|K_\infty\|_r} \right)^{t_0} > 0.$$

Arguing by contradiction we suppose that $v(x) = 0$ for a.e. $x \in \Omega_0$. The expression (3.6), Hölder's inequality and the definition of S provide

$$1 \le \int_{\Omega\setminus\Omega_0} K_\infty(x)v^2 \le \|K_\infty\|_{L^r(\Omega\setminus\Omega_0)}\|v\|_{2^*}^2|\Omega\setminus\Omega_0|^{\frac{1}{t_0}} \le \frac{1}{S}\|K_\infty\|_r|\Omega\setminus\Omega_0|^{\frac{1}{t_0}} < 1,$$

whenever $|\Omega_0| > \alpha$. This contradiction concludes the proof. $\square$

Remark 3.1 *If $\lambda_1(K_\infty) = 1$ the above result holds with $\alpha = 0$, that is, the condition $(\widehat{NQ})$ with no restriction on the (positive) measure of Ω_0 is suffices to get compactness. Indeed, in this case it follows from (3.6) that*

$$1 \le \int_\Omega K_\infty(x)v^2 = \lambda_1(K_\infty)\int_\Omega K_\infty(x)v^2 \le \|v\|^2 = 1,$$

and therefore v is an eigenfunction associated to the the first eigenvalue. Hence, v has constant sign in Ω and we can take $\widetilde{\Omega} = \Omega_0$ to obtain the desired contradiction.

We are ready to prove our main results on the resonant case.

Proof of Theorem 1.3. We first consider the case of resonance at higher eigenvalues, namely $j = m+1$, for some $m \in \mathbb{N}$. Without loss of generality we may suppose that $\lambda_m(K_\infty) < 1$.

Let $0 < \alpha < |\Omega|$ be given by the previous lemma and suppose that $|\Omega_0| > \alpha$. Considering $\varphi_i := \varphi_i(K_\infty)$ the i-th eigenfunction of the linear problem (LP) with weight K_∞, we set

$$V := \operatorname{span}\{\varphi_1, \dots, \varphi_m\}, \quad W := V^\perp.$$

With this definition we have that $H = V \oplus W$ and we claim that the functional I satisfies the following

(i) $I(u) \to -\infty$ as $\|u\| \to \infty$, $u \in V$;
(ii) there exists $\beta \in \mathbb{R}$ such that $I(u) \ge \beta$ for all $u \in W$.

Assuming the above statements and recalling that I satisfies the $(\mathrm{Ce})_c$ condition at any level $c \in \mathbb{R}$, we may invoke the Saddle Point Theorem [17] (see also [18,19]) to obtain a critical point of I.

It remains to prove (i) and (ii). We first notice that the variational characterization of $\lambda_m(K_\infty)$ provides

$$\|u\|^2 \le \lambda_m(K_\infty)\int_\Omega K_\infty(x)u^2 < \int_\Omega K_\infty(x)u^2, \quad \text{for any } u \in V \setminus \{0\}.$$

Since V is finite dimensional, we obtain $\delta > 0$ such that

$$\|u\|^2 - \int_\Omega K_\infty(x)u^2 \le -\delta\,\|u\|^2, \quad \text{for any } u \in V.$$

Thus, given $\varepsilon > 0$, we can use the first inequality in (3.2), Hölder's inequality and the Sobolev embedding to obtain

$$\begin{aligned} I(u) &\le \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_\Omega K_\infty(x)u^2 + \frac{\varepsilon}{2}\int_\Omega |u|^2 + \|\widehat{a}\|_1 - \frac{1}{q}\int_{\partial\Omega} h(x)|u|^q \\ &\le \frac{1}{2}(-\delta+\varepsilon)\|u\|^2 + \|\widehat{a}\|_1 + c_1 |h|_{\sigma_q}\|u\|^q, \end{aligned} \tag{3.7}$$

with $c_1 > 0$. Since $1 \le q < 2$, we can choose $\varepsilon = \delta/2$ to conclude that statement (i) holds.

We shall verify (ii). By the variational characterization of $\lambda_{m+1}(K_\infty) = 1$, we get

$$\|u\|^2 \ge \lambda_{m+1}(K_\infty)\int_\Omega K_\infty(x)u^2 = \int_\Omega K_\infty(x)u^2, \quad \text{for any } u \in W.$$

Recalling that $h \le 0$, we obtain

$$\begin{aligned} I(u) &= \frac{1}{2}\left(\|u\|^2 - \int_\Omega K_\infty(x)u^2\right) - \int_\Omega \left(F(x,u) - \frac{1}{2}K_\infty(x)u^2\right) - \frac{1}{q}\int_{\partial\Omega} h(x)|u|^q \\ &\ge -\int_\Omega \left(F(x,u) - \frac{1}{2}K_\infty(x)u^2\right). \end{aligned}$$

Arguing as in [6, Lemma 3.5] (see also [5]) we can use $(\widehat{NQ})$ to prove that

$$F(x,s) - \frac{1}{2}K_\infty(x)s^2 \le -\frac{d(x)}{2},$$

for a.e. $x \in \Omega$, $s \in \mathbb{R}$. Hence,

$$I(u) \ge -\frac{\|d\|_1}{2} = \beta, \quad \text{for any } u \in W,$$

and the theorem is proved in the first case.

We now suppose that $\lambda_1(K_\infty) = 1$. It follows from $(\widehat{NQ})$ and Remark 3.1 that I satisfies the Cerami condition at any level. Moreover, the same argument employed in the first part of the proof shows that I is bounded from below. Thus, standard arguments imply that the infimum of I is attained at a critical point $u \in H$. This concludes the proof of the theorem. □

Proof of Theorem 1.4. Given $\varepsilon > 0$ small we can use the second inequality in (3.2), the variational characterization of $\lambda_1(K_\infty)$ and Hölder's inequality to obtain

$$\begin{aligned} I(u) &\ge \frac{1}{2}\left(1 - \frac{1}{\lambda_1(K_\infty)} - \varepsilon\right)\|u\|^2 - \|\widehat{a}\|_1 - c_1 |h|_{\sigma_q}\|u\|^q \\ &= \frac{\nu}{2}\|u\|^2 - \|\widehat{a}\|_1 - c_1 |h|_{\sigma_q}\|u\|^q, \end{aligned} \tag{3.8}$$

with $\nu = (1 - 1/\lambda_1(K_\infty) - \varepsilon) > 0$. Since $1 \leq q < 2$ we conclude that $I(u) \to +\infty$ as $\|u\| \to +\infty$, that is, I is coercive on H. Arguing as in the second part of the proof of Theorem 1.3 we obtain a critical point of I. $\square$

Remark 3.2 *As in the previous section, we can obtain some results in the case $q = 2$. We first show that Theorem 1.3 holds if $q = 2$ and $|h|_{\sigma_2}$ is small. Indeed, first notice that the proof of Lemma 3.1 holds in this new setting. In the case of resonance at higher eigenvalues expression (3.7) becomes*

$$I(u) \leq \frac{1}{2}(-\delta + \varepsilon + c_1 |h|_{\sigma_2})\|u\|^2 + \|\widehat{a}\|_1,$$

and therefore the statement (i) of the proof of Theorem 1.3 is true if $|h|_{\sigma_2}$ is small enough. The rest of the proof follows as before.

Concerning Theorem 1.4, the expression (3.8) becomes

$$I(u) \leq \left(\frac{\nu}{2} - c_1 |h|_{\sigma_2}\right) \|u\|^2 - \|\widehat{a}\|_1,$$

and therefore the theorem also holds if $q = 2$ and $|h|_{\sigma_2}$ is small.

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