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SOLUTIONS FOR A RESONANT ELLIPTIC SYSTEM WITH COUPLING IN $\mathbb{R}^N$

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ABSTRACT

Existence and multiplicity of solutions are established, via the Variational Method, for a class of resonant semilinear elliptic system in $\mathbb{R}^N$ under a local nonquadraticity condition at infinity. The main goal is to consider systems with coupling where one of the potentials does not satisfy any coercivity condition. The existence of solution is proved under a critical growth condition on the nonlinearity.

Key Words: Elliptic systems; Variational methods; Resonant problems

1. INTRODUCTION

This paper is concerned with the existence and multiplicity of solutions for the system

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$$\begin{cases} -\Delta u + a(x)u = F_u(x,u,v), & x \in \mathbb{R}^N, \\ -\Delta v + b(x)v = F_v(x,u,v), & x \in \mathbb{R}^N, \end{cases} \tag{P}$$

where $N \geq 3$ and the potentials a and b are positive continuous functions. In the scalar case Rabinowitz[23] showed the existence of a nontrivial solution by assuming that the nonlinearity is superlinear with subcritical growth and the potential is coercive. In Ref. [3] Bartsch & Wang consider the scalar case under assumptions similar to those in Ref. [23] and a condition weaker than coercivity for the potential. Proving that the associated functional satisfies the Palais-Smale condition they were able to establish existence and multiplicity of solutions. We should also mention the articles[27,28,4,8,16,19] where the scalar case is considered.

We observe that there exists an extensive bibliography in the study of elliptic systems on bounded domains (see Refs. [17,10,14,9] and references therein). In particular, we should mention the articles[7,25] where a condition on F similar to the one used here is assumed. For systems in $\mathbb{R}^N$ we refer the interested reader to the articles[11,2,12] where Hamiltonean elliptic systems are considered. In the case of gradient systems in $\mathbb{R}^N$, Costa[5] proves the existence of a nonzero solution for (P) under the coercivity of the potentials a and b, and a nonquadratic condition on F. One of the main goals of this article is to consider (P) in a class of resonant systems that allow us dealing with it without any coercivity condition on one of the potentials. More specifically, we suppose

(A_1) there are constants $a_0, b_0 > 0$ such that $a(x) \geq a_0, b(x) \geq b_0$ for all $x \in \mathbb{R}^N$,

(A_2) for every $M > 0$

$$\mu(\{x \in \mathbb{R}^N : b(x) \leq M\}) < \infty,$$

with μ denoting the Lebesgue measure in $\mathbb{R}^N$.

We observe that condition (A_2) was introduced by Bartsch & Wang[3] for the scalar case. To compensate the lack of coercivity on the potential a we suppose that the system is coupled in the following sense

$$(F_1) \quad \lim_{|z|\to\infty} \frac{F(x,z) - \lambda_k uv}{|z|^2} = 0, \quad \text{uniformly for a.e. } x \in \mathbb{R}^N,$$

where λ_k is a positive eigenvalue for the associated coupled linear problem

$$\begin{cases} -\Delta u + a(x)u = \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + b(x)v = \lambda u, & x \in \mathbb{R}^N. \end{cases} \tag{LP}$$

In our first result we establish the existence of a solution for the system (P) by verifying that the associated functional satisfies the hypothesis of a version of the Saddle Point Theorem[21] characterized by the fact that it requires a compactness condition with respect to the weak topology of the space. This fact allows us to deal with a nonlinearity satisfying the critical growth condition.

Considering $2^* = (2N)/(N-2)$ the critical Sobolev exponent and denoting by $\nabla F(x,z)$ the gradient of F with respect to the variable $z \in \mathbb{R}^2$, we assume

(F_2) $\quad F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$,

(F_3) there are constants $c_1, c_2 > 0$ and $2 \leq \sigma \leq 2^*$ such that

$$|\nabla F(x,z)| \leq c_1|z|^{\sigma-1} + c_2|z|, \quad \forall\,(x,z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

(F_4) there are constants $c_3, c_4 > 0$ and $\beta \in L^\infty(\mathbb{R}^N)$ such that

$$|F(x,z)| \leq c_3|u|\,|v| + c_4|v|^2 + \frac{\beta(x)}{2}|u|^2, \quad \forall\,(x,z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

where β satisfies

$$\limsup_{|x|\to\infty} \beta(x) = \beta_\infty < a_0.$$

In order to establish the existence of a solution for (P), based on a previous work,[13] we assume a local nonquadraticity condition on F: given $\gamma > 0$ we set

$$\Omega_\gamma = \left\{ x \in \mathbb{R}^N : b(x) < \frac{\gamma}{a_0}\lambda_k^2 \right\}$$

and suppose
(NQ) $\quad$ there exists $\gamma > (a_0/(a_0 - \beta_\infty))^2$ and $A \in L^1(\mathbb{R}^N)$ such that

$$\begin{cases} \lim\limits_{\substack{|u|\to\infty \\ |v|\to\infty}} \nabla F(x,z)\cdot z - 2F(x,z) = \infty, & \text{a.e. } x \in \Omega_\gamma, \\ \nabla F(x,z)\cdot z - 2F(x,z) \geq A(x), & \forall\,(x,z) \in \mathbb{R}^N \times \mathbb{R}^2, \end{cases}$$

where $a \cdot b$ denotes the usual inner product between $a,\ b \in \mathbb{R}^2$. The reader should be aware that Ω_γ, for this choice of γ, is a nonempty open subset of $\mathbb{R}^N$ (see Remark 3.3). Now we may state a result on the existence of solution.

Theorem 1.1. *Suppose* (A_1) *and* (A_2) *hold. If* F *satisfies* (F_1)–(F_4) *and* (NQ), *then problem* (P) *possesses a solution.*

In our next result we study the existence of a nontrivial solution for the system (P) when $F(x,0) \equiv \nabla F(x,0) \equiv 0$ and F satisfies

$(\widehat{F_2})$ $\quad F \in C^2(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$,

$(\widehat{F_3})$ there are constants $c_1, c_2 > 0$ and $2 \leq \sigma < 2^*$ such that

$$|D^2F(x,z)| \leq c_1|z|^{\sigma-2} + c_2, \quad \forall\,(x,z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

$(\widehat{F_4})$ there are constants $c_3, c_4, p, q > 0$ with $2p + 2^*q \geq 2(2^* - 1)$ and $p + q < 2^* - 1$ and $\beta \in L^\infty(\mathbb{R}^N)$ such that

$$|F_u(x,z)| \leq c_3|u|^p|v|^q + c_4|v| + \beta(x)|u|, \quad \forall\,(x,z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

where β satisfies

$$\limsup_{|x|\to\infty} \beta(x) = \beta_\infty < a_0,$$

(F_5) $D^2F(x,0) \equiv D^2F(0)$ and we have either

(i) $\quad F_{uu}(0),\ F_{vv}(0) \geq 0 \quad$ and $\quad \lambda_k < F_{uv}(0) + \sqrt{F_{uu}(0)F_{vv}(0)}$,

or

(ii) $\quad F_{uu}(0),\ F_{vv}(0) < 0, F_{uv}(0) > -\lambda_1$ and
$\lambda_{k-1} > F_{uv}(0) - \sqrt{F_{uu}(0)F_{vv}(0)}$.

Under these conditions we are able to prove

Theorem 1.2. *Suppose* (A_1) *and* (A_2) *hold. If* F *satisfies* (F_1), $(\widehat{F_2})$–$(\widehat{F_4})$, (F_5), (NQ) *and* $F(x,0) \equiv \nabla F(x,0) \equiv 0$, *then problem* (P) *possesses a nonzero solution.*

Our final task is to verify the existence of multiple solutions for (P) under the assumption that the primitive is even with respect to the variable z. In order to obtain such result we apply a version of the Symmetric Mountain Pass Theorem of Ambrosetti–Rabinowitz.[1,22] Since we need a compactness condition with respect to the norm topology we assume (F_3) with $\sigma < 2^*$ and

$(\widehat{F_5})$ $\quad F(x,z) - \dfrac{1}{2}Az \cdot z = o(|z|^2), \quad$ as $|z| \to 0$, where $A = \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_2 & \mu_3 \end{bmatrix}$ is a symmetric matrix such that $\mu_1, \mu_3 < 0$, $\mu_2 > -\lambda_1$ and

$$\mu_2 - \sqrt{\mu_1 \mu_3} < \lambda_j < \lambda_k.$$

(F_6) $F(x, z)$ is even with respect to the variable $z \in \mathbb{R}^2$.
Now, we may state

Theorem 1.3. *Suppose* (A_1) *and* (A_2) *hold. If* F *satisfies* $F(x, 0) \equiv 0$, (F_1), (F_2), (F_3) *with* $\sigma < 2^*$, $(\widehat{F_4})$, $(\widehat{F_5})$, (F_6) *and* (NQ), *then problem* (P) *possesses* $k - j$ *pairs of nonzero solutions.*

As in Ref. [23], Theorems 1.1–1.3 will be proved by finding critical points for the associated functional defined on an appropriated Hilbert space. In Section 2, we state the abstract results that we need to prove our main theorems. There we also obtain the variational characterization of the eigenvalues of the coupled linear system (LP). In Section 3 we prove Theorem 1.1 and in Section 4 we present the proof of Theorem 1.2. Finally, in Section 5, we prove Theorem 1.3.

2. PRELIMINARIES

In this section we present some abstract results that will be used in the proofs of Theorems 1.1–1.3. We also study the linear problem associated to (P).

Let E be a real Hilbert space and $I : E \to \mathbb{R}$ a functional of class C^1. We recall that a sequence $(z_n) \subset E$ is said to be a Palais-Smale sequence if $I(z_n) \to c$ and $I'(z_n) \to 0$ as $n \to \infty$. As it is well known, minimax theorems are based on the existence of a linking structure and on deformation results. In general, in order to derive such deformation results, the functional must satisfy a compactness condition. In this article, we deal with a condition introduced by Silva in Ref. [26] and defined bellow.

Definition 2.1. The functional $I \in C^1(E, \mathbb{R})$ satisfies the strong Cerami condition [(SCe)] if any Palais-Smale sequence $(z_n) \subset E$ such that $\|z_n\| \|I'(z_n)\|$ is bounded possesses a convergent subsequence.

To establish the existence of a critical point for the functional we only need a version of the (SCe) condition for the weak topology.

Definition 2.2. The functional $I \in C^1(E, \mathbb{R})$ satisfies the strong Cerami condition for the weak topology [(SCe)'] if any Palais-Smale sequence $(z_n) \subset E$ such that $\|z_n\|\|I'(z_n)\|$ is bounded possesses a subsequence which converges weakly to a critical point of I.

Assuming the above condition, we have the following version of the abstract results in Refs. [24,26].

Theorem 2.3. *Let $E = V \oplus W$ be a real Hilbert space with V finite dimensional and $W = V^{\perp}$. Suppose $I \in C^1(E, \mathbb{R})$ satisfies (SCe)' and*

(I_1) *there exists $\beta \in \mathbb{R}$ such that $I(z) \leq \beta$, for all z* in *V,*

(I_2) *there exists $\gamma \in \mathbb{R}$ such that $I(z) \geq \gamma$, for all z in W.*

Then I possesses a critical point.

Proof. Arguing by contradiction, suppose that I does not have a critical point. Then, I satisfies (SCe). Indeed, in this case we do not have any Palais-Smale sequence $(z_n) \subset E$ such that $\|z_n\|\|I'(z_n)\|$ is bounded because, otherwise, I would have a critical point since it satisfies (SCe)'. Invoking Theorem 2.13 in Ref. [26], we obtain a critical point for I. This concludes the proof of the theorem. □

For the proof of Theorem 1.2 we apply the following version of the Lazer–Solimini's theorem[18] proved in Ref. [13] (see also Ref. [20] for a related result).

Theorem 2.4. *Let $E = V \oplus W$ be a real Hilbert space with V finite dimensional and $W = V^{\perp}$. Suppose $I \in C^1(E, \mathbb{R})$ satisfies (SCe), (I_1), (I_2) and*

(I_3) *the origin is a critical point of I, $D^2I(0)$ is a Fredholm operator and either* $\dim V < m(I, 0)$ *or* $\overline{m}(I, 0) < \dim V$.

Then I possesses a nonzero critical point.

Here, $m(I, z)$ [$\overline{m}(I, z)$] denotes the Morse index [augmented Morse index] of the functional I at the point z.

Theorem 1.3 will be proved by applying the following version of the symmetric Mountain Pass Theorem.[24]

Theorem 2.5. *Let $E = V \oplus W$ be a real Hilbert space with V finite dimensional and $W = V^{\perp}$. Suppose $I \in C^1(E, \mathbb{R})$ is even and satisfies $I(0) = 0$, (SCe) and*

(I_4) *there exists a finite dimensional closed subspace $\widehat{V}$ of E and $\beta \in \mathbb{R}$ such that $\widehat{V} \supset V$ and $I(z) \leq \beta$, for all z* in *$\widehat{V}$,*

(I_5) *there exists* $\rho > 0$ *such that* $I(z) \geq 0$, *for all* z *in* $B_\rho(0) \cap W$.

Then I *possesses* $\dim \widehat{V} - \dim V$ *pairs of nontrivial critical points.*

Actually, in Ref. [24], Theorem 2.5 is stated for the Palais-Smale condition. The version of the (SCe) condition is based on a deformation lemma proved in Ref. [26].

For applying the abstract results we set $E = E_a \times E_b$ where

$$E_a = \left\{ u \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) : \int_{\mathbb{R}^N} (|\nabla u|^2 + a(x)u^2)\, dx < \infty \right\}$$

and

$$E_b = \left\{ v \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) : \int_{\mathbb{R}^N} (|\nabla v|^2 + b(x)v^2)\, dx < \infty \right\}$$

endowed with the inner product

$$\langle (u,v), (\phi, \psi) \rangle = \int_{\mathbb{R}^N} (\nabla u \nabla \phi + \nabla v \nabla \psi + a(x)u\phi + b(x)v\psi)\, dx,$$

and associated norm given by

$$\|z\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + a(x)u^2 + b(x)v^2)\, dx, \quad \forall z = (u,v) \in E. \tag{2.1}$$

For $z \in E$ the functional

$$I(z) = \frac{1}{2}\|z\|^2 - \int_{\mathbb{R}^N} F(x,z)\, dx \tag{2.2}$$

is well defined and of class C^1 via (F_2) and (F_3). Moreover the critical points of I are precisely the weak solutions of the system (P).

The conditions (A_1), (A_2) and the Sobolev Theorem imply that the immersion $E \hookrightarrow L^s(\mathbb{R}^N, \mathbb{R}) \times L^s(\mathbb{R}^N, \mathbb{R})$ is continuous for $2 \leq s \leq 2^*$. In Ref. [3] it is proved that, in fact, the embedding $E_b \hookrightarrow L^s(\mathbb{R}^N, \mathbb{R})$ is compact for $2 \leq s < 2^*$. It is worthwhile mentioning that in our problem the embedding $E \hookrightarrow L^s(\mathbb{R}^N, \mathbb{R}) \times L^s(\mathbb{R}^N, \mathbb{R})$ may not be compact. This fact is compensated by the coupling of the system.

For the sake of completeness we prove a proposition that generalizes for the system (P) a well known fact for the scalar case.

Proposition 2.6. *Suppose F satisfies* (F_2)–(F_3). *Then every bounded sequence* $(z_n) \subset E$ *such that* $I'(z_n) \to 0$ *possesses a subsequence which converges weakly to a critical point of I.*

Proof. Let $(z_n) \subset E$ be a bounded sequence such that $I'(z_n) \to 0$. We may assume that

$$\begin{cases} z_n \rightharpoonup z & \text{in } E, \\ z_n(x) \to z(x), & \text{a.e. } x \in \mathbb{R}^N. \end{cases} \tag{2.3}$$

It suffices to show that, for every $w \in E$, we have

$$\int_{\mathbb{R}^N} \nabla F(x, z_n) \cdot w\, dx \to \int_{\mathbb{R}^N} \nabla F(x, z) \cdot w\, dx.$$

Given $\varepsilon > 0$ we set $\Omega^R = \mathbb{R}^N \backslash B_R(0)$, for $R > 0$, and use (F_3) and Holder's inequality to obtain

$$\begin{aligned} \int_{\Omega^R} |\nabla F(x, z_n) \cdot w|\, dx &\le c_1 \int_{\Omega^R} |z_n|^{2^*-1} |w|\, dx + c_2 \int_{\Omega^R} |z_n||w|\, dx \\ &\le c_1 \|z_n\|_{L^{2^*}}^{2^*-1} \|w\|_{L^{2^*}(\Omega^R)} + c_2 \|z_n\|_{L^2} \|w\|_{L^2(\Omega^R)}. \end{aligned}$$

Since (z_n) is bounded and $w \in L^2(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$, we obtain

$$\int_{\Omega^R} |\nabla F(x, z_n) \cdot w|\, dx < \frac{\varepsilon}{3}, \tag{2.4}$$

for R sufficiently large. Taking R larger if necessary, we may suppose that

$$\int_{\Omega^R} |\nabla F(x, z) \cdot w|\, dx < \frac{\varepsilon}{3}. \tag{2.5}$$

On other hand, by (2.3), (F_3) and the Lebesgue Convergence Theorem, there exists $n_0 \in N$ such that

$$\int_{B_R(0)} |\nabla F(x, z_n) \cdot w - \nabla F(x, z) \cdot w| < \frac{\varepsilon}{3}, \quad \forall\, n \ge n_0.$$

The above estimate, (2.4) and (2.5) show that

$$\int_{\mathbb{R}^N} |\nabla F(x, z_n) \cdot w - \nabla F(x, z) \cdot w|\, dx < \varepsilon, \quad \forall\, n \ge n_0.$$

This concludes the proof of the proposition. □

An immediate consequence of the above proposition is

Corollary 2.7. *Suppose F satisfies* $(F_2)-(F_3)$. *If every Palais-Smale sequence* $(z_n) \subset E$ *such that* $\|z_n\|\|I'(z_n)\|$ *is bounded possesses a bounded subsequence, then I satisfies* (SCe)'.

Now, we proceed with the study of the linear interchanged eigenvalue problem

$$\begin{cases} -\Delta u + a(x)u = \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + b(x)v = \lambda u, & x \in \mathbb{R}^N. \end{cases} \qquad (LP)$$

A simple calculation shows that λ is an eigenvalue of (LP) if, and only if,

$$T(u, v) = \frac{1}{\lambda}(u, v),$$

where $T : E \to E$ is a selfadjoint bounded linear operator defined by

$$\langle T(u, v), (\phi, \psi) \rangle = \int_{\mathbb{R}^N} (v\phi + u\psi)\, dx.$$

Moreover, the following result holds.

Lemma 2.8. *Suppose* (A_1) *and* (A_2) *hold. Then T is a compact operator.*

Proof. Let $(z_n) = (u_n, v_n) \subset E$ be a sequence such that $z_n \rightharpoonup z = (u, v)$ in E. Writing $T = (T_1, T_2)$ and using the compact embedding of E_b in $L^2(\mathbb{R}^N)$ we have that $v_n \to v$ in $L^2(\mathbb{R}^N)$ and $T_2(z_n - z) \to 0$ in $L^2(\mathbb{R}^N)$. Now, by the definition of T,

$$\begin{aligned} 0 &\le \big\langle T(z_n - z), T(z_n - z) \big\rangle \\ &= \int_{\mathbb{R}^N} (v_n - v) T_1(z_n - z)\, dx + \int_{\mathbb{R}^N} (u_n - u) T_2(z_n - z)\, dx \\ &\le \|v_n - v\|_{L^2} \|T_1(z_n - z)\|_{L^2} + \|u_n - u\|_{L^2} \|T_2(z_n - z)\|_{L^2}. \end{aligned}$$

Since T and $\|u_n - u\|_{L^2}$ are bounded we conclude that $T(z_n) \to T(z)$. The lemma is proved. □

Observing that $(u, -v)$ is an eigenfunction associated with the eigenvalue $-\lambda$ whenever (u, v) is an eigenfunction associated to λ, we invoke Lemma 2.8 and the spectral theory for compact operators to conclude that (LP) possesses a sequence $\{\lambda_m\}_{m \in Z^*}$ of eigenvalues

$$\cdots \le \lambda_{-m} \le \cdots \le \lambda_{-2} \le \lambda_{-1} < 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m \le \cdots,$$

such that $\lambda_{\pm m} \to \pm\infty$ as $m \to \infty$. Furthermore, denoting by $\{\varphi_m\}_{m \in Z^*}$ the orthonormal basis of eigenfunctions associated to the sequence $\{\lambda_m\}_{m \in Z^*}$, the variational characterization of the eigenvalues provides the inequalities

$$\frac{1}{2}\|z\|^2 \leq \lambda_m \int_{\mathbb{R}^N} uv\,dx, \quad \forall\, z = (u, v) \in \operatorname{span}\{\varphi_1, \varphi_2, \ldots, \varphi_m\}, \tag{2.6}$$

and

$$\frac{1}{2}\|z\|^2 \geq \lambda_{m+1} \int_{\mathbb{R}^N} uv\,dx, \quad \forall\, z = (u, v) \in (\operatorname{span}\{\varphi_1, \varphi_2, \ldots, \varphi_m\})^{\perp}, \tag{2.7}$$

where λ_m is a positive eigenvalue. For negative eigenvalues we have analogous inequalities. In particular,

$$\frac{1}{2}\|z\|^2 \geq -\lambda_1 \int_{\mathbb{R}^N} uv\,dx, \quad \forall\, z = (u, v) \in \overline{\operatorname{span}\{\varphi_{-1}, \varphi_{-2}, \ldots, \varphi_{-m}, \ldots\}}. \tag{2.8}$$

Finally we observe that, by the orthogonality of the eigenfunctions and the definition of T we have that, if $\varphi_l = (u_l, v_l)$ and $\varphi_m = (u_m, v_m)$, with $l \neq m$, then

$$\int_{\mathbb{R}^N} (u_l v_m + u_m v_l)\,dx = 0. \tag{2.9}$$

3. PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1 by verifying that the functional I defined in (2.2) satisfies the hypotheses of Theorem 2.3.

Considering k given by (F_1) we set $V = \operatorname{span}\{\varphi_1, \varphi_2, \ldots, \varphi_{k-1}\}$ and $W = V^{\perp}$ (without loss of generality $\lambda_{k-1} < \lambda_k$ and $V = \phi$ if $k = 1$). In order to show that I satisfies (SCe)', we use the following technical result.

Lemma 3.1. *Suppose F satisfies (F_2) and (F_4). Then, given $R > 0$ and $\varepsilon > 0$, there exists $M = M(R) > 0$ such that*

$$\int_{\{|z| \leq R\}} F(x, z)\,dx \leq M + \left(\varepsilon + \frac{\beta_\infty}{2a_0}\right)\|z\|^2, \quad \forall\, z \in E. \tag{3.1}$$

Proof. Given $\widehat{\varepsilon} > 0$, the compact embedding of E_b in $L^2(\mathbb{R}^N)$ and the Monotone Convergence Theorem imply that there exists $R_1 > 0$ such that

$$\int_{\mathbb{R}^N \setminus B_{R_1}(0)} |v|^2 \, dx \leq \widehat{\varepsilon}, \quad \forall \, \|z\| = 1. \tag{3.2}$$

Taking R_1 larger if necessary, using the above estimate and Holder's inequality we get

$$\int_{\mathbb{R}^N \setminus B_{R_1}(0)} |u||v| \leq \widehat{\varepsilon}, \quad \forall \, \|z\| = 1. \tag{3.3}$$

Invoking (F_4), we also may suppose that

$$\beta(x) \leq \beta_\infty + \widehat{\varepsilon}, \quad \forall \, |x| > R_1.$$

Defining the sets

$$\Omega^1 = \{|z| \leq R\} \cap B_{R_1}(0) \quad \text{and} \quad \Omega^2 = \{|z| \leq R\} \cap (\mathbb{R}^N \setminus B_{R_1}(0)),$$

we can use (3.2), (3.3), the above inequality, (F_2) and (F_4) to obtain

$$\begin{aligned} \int_{\{|z| \leq R\}} F(x,z) \, dx &\leq \int_{\Omega^1} F(x,z) \, dx + \int_{\Omega^2} \left(c_3 |u||v| + c_4 |v|^2 + \frac{\beta(x)}{2} |u|^2 \right) dx \\ &\leq M + \left(c_3 \widehat{\varepsilon} + c_4 \widehat{\varepsilon} + \frac{\widehat{\varepsilon}}{2a_0} + \frac{\beta_\infty}{2a_0} \right) \|z\|^2. \end{aligned}$$

Taking $\widehat{\varepsilon} > 0$ sufficiently small we conclude the proof of the lemma. □

The following result provides the compactness for the functional I. Hereafter we use the notation

$$\|z\|_\Omega^2 = \int_\Omega \left(|\nabla u|^2 + a(x)u^2 + |\nabla v|^2 + b(x)v^2 \right) dx$$

for a measurable set $\Omega \subset \mathbb{R}^N$ and $z = (u, v) \in E$. We also denote by S a positive constant such that $\|z\|_{L^2}^2 \leq S\|z\|^2$, for all $z \in E$.

Proposition 3.2. *Suppose F satisfies (F_1)–(F_4) and (NQ). Then I satisfies (SCe)'.*

Proof. Let $(z_n) \subset E$ be such that $I(z_n) \to c$, $I'(z_n) \to 0$ and $\|z_n\|\|I'(z_n)\|$ is bounded. In view of Corollary 2.7, we need only to verify that (z_n) possesses a bounded subsequence. Arguing by contradiction, we suppose that $\|z_n\| \to \infty$. Since $I(z_n) \to c$ and $\|z_n\|\|I'(z_n)\|$ is bounded there exists $M > 0$ such that

$$\liminf \int_{\mathbb{R}^N} H(x, z_n)\, dx = \liminf \big[2I(z_n) - I'(z_n)z_n\big] \le M, \tag{3.4}$$

where $H(x, z_n) = \nabla F(x, z_n) \cdot z_n - 2F(x, z_n)$. On the other hand, for n sufficiently large, we have

$$\frac{1}{2}\|z_n\|^2 \le (c+1) + \int_{\mathbb{R}^N} F(x, z_n)\, dx. \tag{3.5}$$

Given $\varepsilon > 0$, the hypothesis (F_1) implies that there exists $R > 0$ such that

$$F(x, z) \le \lambda_k uv + \varepsilon|z|^2, \quad \forall\, x \in \mathbb{R}^N,\ |z| > R.$$

Using this inequality and (3.5), we get

$$\frac{1}{2}\|z_n\|^2 \le M_1 + \int_{\{|z|>R\}} \big(\lambda_k|u_n||v_n| + \varepsilon|z_n|^2\big)\, dx + \int_{\{|z|\le R\}} F(x, z_n)\, dx,$$

and therefore

$$\frac{1}{2}(1 - 2S\varepsilon)\|z_n\|^2 \le M_1 + \lambda_k \int_{\mathbb{R}^N} |u_n||v_n|\, dx + \int_{\{|z|\le R\}} F(x, z_n)\, dx.$$

Now we use Lemma 3.1 to obtain

$$\frac{\nu}{2}\|z_n\|^2 \le M_2 + \lambda_k \int_{\mathbb{R}^N} |u_n||v_n|\, dx, \tag{3.6}$$

where $\nu = 1 - 2\varepsilon(S+1) - (\beta_\infty/a_0)$.

Let γ be given by (NQ). Since $\gamma > (a_0/(a_0 - \beta_\infty))^2$ we choose $\varepsilon > 0$ sufficiently small such that $\nu\gamma > 1$ and $\nu\gamma > \nu^{-1}$. Taking $\delta > 0$ such that $\nu\gamma > 1 + \delta > \nu^{-1}$, we can use (3.6) and Young's inequality to get

$$\frac{\nu}{2}\left(\|z_n\|^2_{\Omega_\gamma} + \int_{\mathbb{R}^N\setminus\Omega_\gamma} |\nabla z_n|^2\, dx\right) + G(z_n) \le M_2 + \lambda_k \int_{\Omega_\gamma} |u_n||v_n|\, dx, \tag{3.7}$$

where $G(z_n) = \int_{\mathbb{R}^N\setminus\Omega_\gamma} (H_1(x)u_n^2 + H_2(x)v_n^2)\, dx$, with

$$H_1(x) = \frac{1}{2}\left(\nu a(x) - \frac{a_0}{1+\delta}\right) \quad \text{and} \quad H_2(x) = \frac{1}{2}\left(\nu b(x) - \frac{\lambda_k^2(1+\delta)}{a_0}\right). \tag{3.8}$$

Hypothesis (A_1), the definition of Ω_γ and the choice of δ provide $\nu_1, \nu_2 > 0$ such that

$$H_1(x) \geq \frac{\nu_1}{2} a(x), \qquad H_2(x) \geq \frac{\nu_2}{2} b(x), \qquad \forall\, x \in \mathbb{R}^N \backslash \Omega_\gamma.$$

Setting $\nu_0 = (1/2)\min\{\nu,\ \nu_1,\ \nu_2\}$, the above expression and (3.7) imply

$$\nu_0 \|z_n\|^2 \leq M_2 + \lambda_k \int_{\Omega_\gamma} |u_n||v_n|\, dx. \tag{3.9}$$

Defining $\widehat{z}_n = (\widehat{u}_n, \widehat{v}_n) = 1/(\|z_n\|)(|u_n|, |v_n|)$, we may assume that

$$\begin{cases} \widehat{u}_n \rightharpoonup \widehat{u} & \text{in } L^2(\Omega_\gamma), \\ \widehat{v}_n \to \widehat{v} & \text{in } L^2(\Omega_\gamma)\,. \end{cases}$$

Hence, by (3.9), we get

$$\nu_0 \leq \lambda_k \int_{\Omega_\gamma} \widehat{u}\, \widehat{v}.$$

Since we are supposing that $\|z_n\|$ goes to infinity, the above expression implies that $\lim_{n\to\infty} |u_n(x)| = \infty$ and $\lim_{n\to\infty} |v_n(x)| = \infty$ on a subset of Ω_γ with positive measure. Finally, using the Fatou's lemma and (NQ), we have

$$\liminf \int_{\mathbb{R}^N} H(x, z_n)\, dx \geq \int_{\mathbb{R}^N} \liminf H(x, z_n)\, dx = \infty.$$

This contradicts (3.4) and concludes the proof of the proposition. □

Remark 3.3. By taking $F(x,z) = \lambda_k uv$ and $z_n = n\varphi_k$, the proof of Proposition 3.2 up to (3.9) implies that $\Omega_\gamma \neq \emptyset$ for every $\gamma > 1$.

The next result is a version of Lemma 3.1 in Ref. [6] (see also Ref. [13]) and will be used to check condition (I_2) for the functional I.

Lemma 3.4. *Suppose F satisfies* (F_1) *and* (NQ). *Then*

$$F(x,z) - \lambda_k uv \leq -\frac{A(x)}{2}, \quad \forall\, z \in \mathbb{R}^2, \quad \text{a.e. } x \in \mathbb{R}^N.$$

Proof. Defining $G(x,z) = F(x,z) - \lambda_k uv$, we have

$$\nabla G(x,z) \cdot z - 2G(x,z) = \nabla F(x,z) \cdot z - 2F(x,z).$$

Thus, for any $s > 0$ and $\overline{z} \in \mathbb{R}^2$ such that $|\overline{z}| = 1$, by (NQ), we have

$$\frac{d}{ds}\left[\frac{G(x, s\overline{z})}{s^2}\right] = \frac{\nabla G(x, s\overline{z}) \cdot (s\overline{z}) - 2G(x, s\overline{z})}{s^3} \geq \frac{A(x)}{s^3}.$$

Integrating over $[s, t] \subset (0, \infty)$, we get

$$\frac{G(x, s\bar{z})}{s^2} \leq \frac{G(x, t\bar{z})}{t^2} - \frac{A(x)}{2}\left[\frac{1}{s^2} - \frac{1}{t^2}\right].$$

Taking the limit as t goes to infinity on the above expression and using (F_1), we conclude that

$$G(x, s\bar{z}) \leq -\frac{A(x)}{2}, \quad \forall\, s > 0,\ \bar{z} \in \mathbb{R}^2 \text{ s.t. } |\bar{z}| = 1, \quad \text{a.e. } x \in \mathbb{R}^N.$$

The argument for $s < 0$ is similar, hence the lemma is proved. □

The next proposition establishes the geometric conditions for the associated functional.

Proposition 3.5. *Suppose F satisfies (F_1), (F_2), (F_4) and (NQ). Then the functional I satisfies (I_1) and (I_2).*

Proof. For any $z \in W$ we have, by (2.7) and Lemma 3.4,

$$I(z) = \frac{1}{2}\|z\|^2 - \lambda_k \int_{\mathbb{R}^N} uv\,dx - \int_{\mathbb{R}^N} (F(x, z) - \lambda_k uv)\,dx \geq -\frac{\|A\|_{L^1}}{2}.$$

Consequently, I satisfies (I_2).

In order to verify (I_1) we first observe that (2.6) and the definition of V provide $\delta > 0$ such that

$$\frac{1}{2}\|z\|^2 - \lambda_k \int_{\mathbb{R}^N} uv\,dx \leq -\delta\|z\|^2, \quad \forall\, z \in V.$$

Hence,

$$I(z) \leq -\delta\|z\|^2 + \int_{\mathbb{R}^N} (\lambda_k uv - F(x, z))\,dx, \quad \forall\, z \in V.$$

Given $\varepsilon > 0$, we may use (F_1) to obtain $R > 0$ such that

$$|F(x, z) - \lambda_k uv| \leq \varepsilon |z|^2, \quad \forall\, x \in \mathbb{R}^N,\ |z| > R.$$

and therefore

$$I(z) \leq -\delta\|z\|^2 + \varepsilon \int_{\mathbb{R}^N} |z|^2\,dx + \int_{\{|z| \leq R\}} (\lambda_k |u||v| - F(x, z))\,dx, \quad \forall\, z \in V.$$

Since $\dim V < \infty$, the above inequality, (F_2), (F_4) and a similar argument to the one employed in the proof of Lemma 3.1 imply that there exist $M_1 = M_1(R) > 0$ such that

$$I(z) \leq M_1 + (-\delta + \varepsilon(S + 1))\|z\|^2.$$

Taking $\varepsilon > 0$ sufficiently small we have that $I(z) \to -\infty$ as $\|z\| \to \infty$, $z \in V$. This concludes the proof of the proposition. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Conditions (F_2) and (F_3) imply that $I \in C^1(E, \mathbb{R})$ and the critical points of I are the weak solutions of (P). In view of Proposition 3.2 the functional I satisfies (SCe)'. Furthermore, by Proposition 3.5, I satisfies the geometric conditions (I_1) and (I_2). The proof of Theorem 1.1 is concluded by invoking Theorem 2.3. □

4. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2. We begin by showing that I satisfies (SCe).

Proposition 4.1. *Suppose F satisfies $F(x,0) \equiv 0$, (F_1), $(\widehat{F_2}) - (\widehat{F_4})$ and (NQ). Then the functional I satisfies (SCe).*

Proof. Let $(z_n) \subset E$ be a Palais-Smale sequence such that $\|z_n\| \times \|I'(z_n)\| \leq M < \infty$. Since $F(x,0) \equiv 0$ we can use (F_1), $(\widehat{F_3})$ and $(\widehat{F_4})$ to show that condition (F_4) holds. Hence, in view of Proposition 3.2, we may suppose that (z_n) is bounded and $z_n \rightharpoonup z$, with z a critical point of I. Furthermore, up to a subsequence, we have

$$\begin{cases} u_n \rightharpoonup u & \text{in } E_a, \\ u_n \to u & \text{in } L^s_{\text{loc}}(\mathbb{R}^N),\ 2 \leq s < 2^*, \\ u_n(x) \to u(x), & \text{a.e. } x \in \mathbb{R}^N, \\ v_n \to v & \text{in } E_b, \\ v_n \to v & \text{in } L^s(\mathbb{R}^N),\ 2 \leq s < 2^*. \end{cases} \tag{4.1}$$

Our objective is to verify that $u_n \to u$ in E_a. Recalling that $I'(z_n) \to 0$ and $I'(z) = 0$, we get

$$\|u_n - u\|^2_{E_a} = \|u_n\|^2_{E_a} - 2\langle u_n, u\rangle + \|u\|^2_{E_a}$$
$$\leq o(1) + \int_{\mathbb{R}^N} F_u(x, z_n)(u_n - u)\, dx + \int_{\mathbb{R}^N} F_u(x, z)(u - u_n)\, dx, \tag{4.2}$$

as n goes to infinity. Given $0 < \varepsilon < \beta_\infty - a_0$, we claim that

$$\int_{\mathbb{R}^N} F_u(x, z_n)(u_n - u)\, dx \leq o(1) + \left(\frac{\beta_\infty + \varepsilon}{a_0}\right)\|u_n - u\|^2_{E_a}, \qquad \text{as } n \to \infty. \tag{4.3}$$

Indeed, defining $G(z_n) = \int_{\mathbb{R}^N} F_u(x, z_n)(u_n - u)\, dx$, we may use $(\widehat{F_4})$ to obtain $R > 0$ such that

$$\begin{aligned} G(z_n) \leq{}& c_3 \int_{\mathbb{R}^N} |u_n|^p |v_n|^q |u_n - u|\, dx + c_4 \int_{\mathbb{R}^N} |v_n||u_n - u|\, dx \\ &+ \int_{B_R(0)} \beta(x)|u_n||u_n - u|\, dx \\ &+ (\beta_\infty + \varepsilon) \int_{\mathbb{R}^N \setminus B_R(0)} \left(|u||u_n - u| + |u_n - u|^2\right) dx. \end{aligned} \tag{4.4}$$

First note that, by the local convergence in (4.1),

$$\int_{B_R(0)} \beta(x)|u_n||u_n - u|\, dx \to 0, \quad \text{as } n \to \infty. \tag{4.5}$$

Since $u_n(x) \to u(x)$ for almost everywhere $x \in \mathbb{R}^N$ and $|u_n|^p|u_n - u|$ is bounded in $L^{2^*/(p+1)}(\mathbb{R}^N)$, we have that $|u_n|^p|u_n - u| \rightharpoonup 0$ in $L^{2^*/(p+1)}(\mathbb{R}^N)$ (see Lemma 4.8 in Ref. [15]). Hence,

$$\int_{\mathbb{R}^N} |u_n|^p |v_n|^q |u_n - u|\, dx \to 0, \quad \text{as } n \to \infty, \tag{4.6}$$

because $|v_n|^q \to |v|^q$ in $L^{(2^*/(p+1))'}(\mathbb{R}^N)$ from $2 \leq (2^*q)/(2^* - (p+1)) < 2^*$ and (4.1). Analogously

$$\int_{\mathbb{R}^N} |v_n||u_n - u|\, dx \to 0, \qquad \int_{\mathbb{R}^N \setminus B_R(0)} |u||u_n - u|\, dx \to 0, \tag{4.7}$$

as n goes to infinity. For the second term of the last integral in the right hand side of (4.4) we have

$$(\beta_\infty + \varepsilon) \int_{\mathbb{R}^N \setminus B_R(0)} |u_n - u|^2\, dx \leq \left(\frac{\beta_\infty + \varepsilon}{a_0}\right)\|u_n - u\|^2_{E_a}.$$

This and equations (4.4)–(4.7) prove the claim. In a similar way

$$\int_{\mathbb{R}^N} F_u(x, z)(u - u_n)\, dx \to 0, \quad \text{as } n \to \infty.$$

The above inequality, (4.2) and (4.3) shows that

$$\left(1-\frac{\beta_\infty+\varepsilon}{a_0}\right)\|u_n-u\|_{E_a}^2 \leq o(1), \quad \text{as } n\to\infty.$$

By our choice of ε we conclude that $u_n \to u$ in E_a and therefore the proposition is proved. □

Proof of Theorem 1.2. Since $\nabla F(x,0)\equiv 0$, we may suppose, without loss of generality, that 0 is an isolated critical point of I. By Proposition 4.1, I satisfies (SCe). As mentioned before, conditions (F_1), $(\widehat{F_3})$, $(\widehat{F_4})$ and $F(x,0)\equiv 0$ imply that (F_4) holds. Thus, by Proposition 3.5, the geometrical conditions (I_1) and (I_2) are satisfied. In order to verify that $D^2I(0)$ is a Fredholm operator we first note that

$$D^2I(0)(z,z) = \|z\|^2 - F_{uu}(0)\int_{\mathbb{R}^N} u^2\,dx - \int_{\mathbb{R}^N}\left(2F_{uv}(0)uv + F_{vv}(0)v^2\right)dx.$$

Since, by $(\widehat{F_4})$, $F_{uu}(0)\leq \beta_\infty < a_0$, the above expression implies that $D^2I(0)$ is of the type $L-K$, where L is an isomorphism and K is compact.

In view of Theorem 2.4, we need only to verify that $m(I,0) > \dim V$ or $\overline{m}(I,0) < \dim V$. Suppose first that (F_5) (ii) holds and

$$F_{uv}(0) - \sqrt{F_{uu}(0)F_{vv}(0)} < \lambda_{k-1} < F_{uv}(0) + \sqrt{F_{uu}(0)F_{vv}(0)}. \tag{4.8}$$

For $z=(u,v)\in(\mathrm{span}\{\varphi_1,\varphi_2,\ldots,\varphi_{k-2}\})^\perp = W_{k-2}$ we have, by (2.7),

$$D^2I(0)(z,z) \geq \lambda_{k-1}\int_{\mathbb{R}^N} uv\,dx - \int_{\mathbb{R}^N} D^2F(0)(z,z)\,dx = \int_{\mathbb{R}^N} Q(z)\,dx,$$

where $Q(z)$ is a quadratic form represented by the positive definite symmetric matrix

$$Q = \begin{bmatrix} -F_{uu}(0) & \lambda_{k-1}-F_{uv}(0) \\ \lambda_{k-1}-F_{uv}(0) & -F_{vv}(0)\end{bmatrix}.$$

Thus, $\overline{m}(I,0) < \dim V$.

In the case that

$$\lambda_{k-1} \geq F_{uv}(0) + \sqrt{F_{uu}(0)F_{vv}(0)}, \tag{4.9}$$

given $z=(u,v)\in W_{k-2}$, we write $z=z^-+z^+$ with

$$z^- = (u^-,v^-)\in \overline{\mathrm{span}\{\varphi_{-1},\varphi_{-2},\ldots,\varphi_{-j},\ldots\}}$$

and

$$z^+ = (u^+, v^+) \in \overline{\text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_j, \ldots\}} \cap W_{k-2}.$$

By (2.7)–(2.9), we get

$$\begin{aligned} D^2 I(0)(z,z) \geq & -2(\lambda_1 + F_{uv}(0)) \int_{\mathbb{R}^N} u^- v^- \, dx \\ & + 2(\lambda_{k-1} - F_{uv}(0)) \int_{\mathbb{R}^N} u^+ v^+ \, dx \\ & - \int_{\mathbb{R}^N} \left(F_{uu}(0)u^2 + F_{vv}(0)v^2\right) dx. \end{aligned}$$

This and the hypothesis (F_5) (ii) implies that $D^2 I(0)(z,z) > 0$ for all $z \in W_{k-2}$ with $z \neq 0$. Hence $\overline{m}(I,0) < \dim V$ when (F_5) (ii) holds.

We claim that $m(I,0) > \dim V$ when the condition (F_5) (i) is satisfied. Indeed, if $\lambda_k < F_{uv}(0)$ we have, by (2.6),

$$D^2 I(0)(z,z) \leq 2(\lambda_k - F_{uv}(0)) \int_{\mathbb{R}^N} uv \, dx - \int_{\mathbb{R}^N} \left(F_{uu}(0)u^2 + F_{vv}(0)v^2\right) dx < 0$$

for all $z = (u,v) \in \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_k\} \backslash \{0\}$, since $\int_{\mathbb{R}^N} uv \, dx > 0$. Thus, $m(I,0) > \dim V$. In the case that

$$F_{uv}(0) \leq \lambda_k < F_{uv}(0) + \sqrt{F_{uu}(0) F_{vv}(0)}$$

we may have $F_{uu}(0) > 0$ and $F_{vv}(0) > 0$. By analyzing a quadratic form as before we also have $m(I,0) > \dim V$. The proof is complete. □

Remark 4.2. Condition (F_5) in the previous result can be replaced by related conditions. For example, if $F_{uu}(x,0)$ and $F_{vv}(x,0)$ are non-negative functions on $L^{N/2}(\mathbb{R}^N)$ and $\lambda_k < F_{uv}(x,0)$, then the same argument employed before shows that $m(I,0) > \dim V$. If we suppose that $F_{uu}(x,0), F_{uv}(x,0)$ and $F_{vv}(x,0)$ are in $L^{N/2}(\mathbb{R}^N)$ and satisfy $F_{uu}(x,0) > 0$, $F_{vv}(x,0) > 0$ and

$$F_{uv}(x,0) - \sqrt{F_{uu}(x,0) F_{vv}(x,0)} < \lambda_k < F_{uv}(x,0) + \sqrt{F_{uu}(x,0) F_{vv}(x,0)},$$

for all $x \in \mathbb{R}^N$, then we also have $m(I,0) > \dim V$.

5. PROOF OF THEOREM 1.3

In this section we will prove Theorem 1.3. In view of (F_6) and $F(x,0) \equiv 0$ we have that I is even and satisfies $I(0) = 0$. Moreover, by Proposition 4.1, the functional I satisfies (SCe).

Considering k, j given by (F_1) and $(\widehat{F}_5)$ we set $V = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_{j-1}\}$, $W = V^{\perp}$ and $\widehat{V} = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_{k-1}\}$. By the definition of $\widehat{V}$ and Proposition 3.5, condition (I_4) is satisfied. Thus, in view of Theorem 2.5, we need only to verify that (I_5) holds. With this purpose, for the matrix A given by $(\widehat{F}_5)$ and $z \in E$, we define

$$\psi(z) = \|z\|^2 - \int_{\mathbb{R}^N} Az \cdot z \, dx.$$

With this setting, we can use $(\widehat{F}_5)$ and the subcritical growth (F_3) to obtain

$$I(z) = \frac{1}{2}\psi(z) - \int_{\mathbb{R}^N} \left(F(x,z) - \frac{1}{2} Az \cdot z \right) dx = \frac{1}{2}\psi(z) + o(\|z\|^2), \tag{5.1}$$

as $\|z\| \to 0$. Now we claim that there exists $\alpha > 0$ such that

$$\psi(z) \geq \alpha \|z\|^2, \quad \forall\, z \in W. \tag{5.2}$$

This fact and (5.1) imply that condition (I_5) holds and therefore we may invoke Theorem 2.5 to obtain $\dim \widehat{V} - \dim V = k - j$ pairs of distinct non-zero critical points for I.

It remains to prove the claim. Consider first the case that

$$\mu_2 - \sqrt{\mu_1 \mu_3} < \lambda_j < \mu_2 + \sqrt{\mu_1 \mu_3}. \tag{5.3}$$

This assumption and the hypothesis $(\widehat{F}_5)$ imply that the matrix

$$Q = \begin{bmatrix} -\mu_1 & \lambda_j - \mu_2 \\ \lambda_j - \mu_2 & -\mu_3 \end{bmatrix}$$

is positive definite, and therefore there exists $\alpha_1 > 0$ such that $Qz \cdot z \geq \alpha_1 |z|^2$, for all $z \in \mathbb{R}^2$. Now, we define

$$W^1 = \text{span}\{\varphi_j, \varphi_{j+1}, \ldots, \varphi_{j+l}\},$$

where $\lambda_j = \lambda_{j+1} = \cdots = \lambda_{j+l} < \lambda_{j+l+1}$, and $W^2 = W \cap (W^1)^{\perp}$. By this setting we have the orthogonal decomposition $W = W^1 \oplus W^2$. Hence, for any $z = z^1 + z^2 \in W^1 \oplus W^2$, we get

$$\begin{aligned} \psi(z) &= \|z^1\|^2 + \|z^2\|^2 - 2\lambda_j \int_{\mathbb{R}^N} \left(u^1 v^1 + u^2 v^2\right) dx + \int_{\mathbb{R}^N} Qz \cdot z \, dx \\ &\geq \left(1 - \frac{\lambda_j}{\lambda_{j+l+1}}\right) \|z^2\|^2 + \alpha_1 \int_{\mathbb{R}^N} \left((u^1 + u^2)^2 + (v^1 + v^2)^2\right) dx, \end{aligned}$$

where we are writing $z^i = (u^i, v^i) \in W^i$, for $i = 1, 2$. Defining $\alpha_2 = (1 - (\lambda_j/\lambda_{j+l+1})) > 0$, we can use the above expression to obtain

$$\psi(z) \geq \alpha_2 \|z^2\|^2 + \alpha_1 \|z\|_{L^2}^2, \quad \forall z = z^1 + z^2 \in W. \tag{5.4}$$

Now, arguing by contradiction, we suppose that (5.2) is false for every $\alpha > 0$. Then there exists a sequence $(z_n) \subset W$ such that $\|z_n\| = 1$ and $\psi(z_n) \to 0$ as $n \to \infty$. In view of (5.4) we have that $\|z_n^2\| \to 0$ and $\|z_n\|_{L^2} \to 0$. Since W^1 is finite dimensional, we may suppose that

$$\begin{cases} z_n^1 \to z^1 & \text{in } E, \\ z_n^1 \to z^1 & \text{in } L^2(\mathbb{R}^N), \end{cases}$$

with $\|z^1\| = 1$. Recalling that in W^1 the norms $\|\cdot\|$ and $\|\cdot\|_{L^2}$ are equivalent we conclude that $\|z^1\|_{L^2} > 0$ and therefore

$$0 = \lim_{n\to\infty} \|z_n\|_{L^2} \geq \lim_{n\to\infty} \|z_n^1\|_{L^2} = \|z^1\|_{L^2} > 0.$$

This contradiction concludes the proof in the case that (5.3) holds.

Arguing as in the proof of Theorem 1.2 and as above we can prove that the claim is also true in the complementary case $\lambda_j \geq \mu_2 + \sqrt{\mu_1 \mu_3}$. Theorem 1.3 is now proved. □

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