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SYSTEMS WITH COUPLING IN $\mathbb{R}^N$ FOR A CLASS OF NONCOERCIVE POTENTIALS

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Abstract. This paper deals with the existence and multiplicity of solutions to a class of resonant semilinear elliptic system in $\mathbb{R}^N$. The main goal is to consider systems with coupling where none of the potentials are coercive. The existence of solution is proved under a critical growth condition on the nonlinearity.

1. **Introduction.** In this article we study the existence and multiplicity of solutions for the problem

$$(P) \qquad \begin{cases} -\Delta u + a(x)u &= F_u(x,u,v), \quad x \in \mathbb{R}^N, \\ -\Delta v + b(x)v &= F_v(x,u,v), \quad x \in \mathbb{R}^N, \end{cases}$$

with $N \geq 3$ and the potentials a and b satisfy

(A_1) there are constants $a_0, b_0 > 0$ such that $a(x) \geq a_0, b(x) \geq b_0$ for all $x \in \mathbb{R}^N$,

(A_2) $\mu(\{x \in \mathbb{R}^N : a(x)b(x) < M\}) < \infty$, for every $M > 0$.

Here μ denotes the Lebesgue measure in $\mathbb{R}^N$. We also suppose that the system is coupled and resonant in the following sense

$$(F_1) \qquad \lim_{|z|\to\infty} \frac{F(x,z) - \lambda_k uv}{|z|^2} = 0, \text{ uniformly for a.e } x \in \mathbb{R}^N,$$

where $z = (u,v) \in \mathbb{R}^2$ and λ_k is a positive eigenvalue for the associated coupled linear problem

$$(LP) \qquad \begin{cases} -\Delta u + a(x)u &= \lambda v, \quad x \in \mathbb{R}^N, \\ -\Delta v + b(x)v &= \lambda u, \quad x \in \mathbb{R}^N. \end{cases}$$

Elliptic systems have been intensively studied in the literature (see [1, 9, 11] and references therein). In a recent paper [3], the authors studied the system (P), under the coupling condition (F_1), in which one of the potentials did not satisfy any coercivity condition. In this work we show that, actually, the existence of coupling allows us to consider a setting where none of the potentials are coercive.

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In our first result we prove the existence of one solution for problem (P). Denoting by $\nabla F(x,z)$ the gradient of F with respect to the variable $z \in \mathbb{R}^2$, we assume

(F_2) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$,

(F_3) there are constants $c_1, c_2 > 0$ and $2 \leq \sigma \leq 2^* = 2N/(N-2)$ such that

$$|\nabla F(x,z)| \leq c_1|z|^{\sigma-1} + c_2|z|, \ \forall\ (x,z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

(F_4) there are functions α and $\beta \in L^\infty(\mathbb{R}^N)$ and a constant $c_3 \geq 0$ such that

$$|F(x,z)| \leq c_3|u||v| + \frac{\alpha(x)}{2}|u|^2 + \frac{\beta(x)}{2}|v|^2, \ \forall\,(x,z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

where α and β satisfy

$$\limsup_{|x|\to\infty} \alpha(x) = \alpha_\infty < a_0 \quad \text{and} \quad \limsup_{|x|\to\infty} \beta(x) = \beta_\infty < b_0. \tag{1}$$

Given $\gamma > 0$, we set $\Omega_\gamma = \left\{x \in \mathbb{R}^N : a(x)b(x) < \gamma\lambda_k^2\right\}$, and we suppose a local nonquadraticity condition on F:

(NQ) there exists $\sqrt{\gamma} > \max\{a_0/(a_0-\alpha_\infty), b_0/(b_0-\beta_\infty)\}$ and $A \in L^1(\mathbb{R}^N)$ such that

$$\begin{cases} \lim\limits_{\substack{|u|\to\infty \\ |v|\to\infty}} \nabla F(x,z)\cdot z - 2F(x,z) = \infty, & \text{a.e. } x \in \Omega_\gamma, \\ \nabla F(x,z)\cdot z - 2F(x,z) \geq A(x), \ \forall\,(x,z) \in \mathbb{R}^N \times \mathbb{R}^2, \end{cases}$$

where $a \cdot b$ denotes the usual inner product between $a,\ b \in \mathbb{R}^2$. Now we may state our first result.

Theorem 1. *Suppose* (A_1) *and* (A_2) *hold. If* F *satisfies* $(F_1) - (F_4)$ *and* (NQ), *then problem* (P) *possesses a solution.*

When $F(x,0) \equiv \nabla F(x,0) \equiv 0$, we are able to establish the existence of a nontrivial solution by supposing that F satisfies

$(\widehat{F_2})$ $F \in C^2(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$,

$(\widehat{F_3})$ there are constants $c_1, c_2 > 0$ and $2 \leq \sigma < 2^*$ such that

$$|D^2F(x,z)| \leq c_1|z|^{\sigma-2} + c_2, \ \forall\ (x,z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

$(\widehat{F_4})$ there are constants $c_3, q_1, q_2 > 0$, $p_1, p_2 > 1$ and functions α and $\beta \in L^\infty(\mathbb{R}^N)$ such that

$$|F_u(x,z)| \leq c_3|u|^{p_1-1}|v|^{q_1} + \alpha(x)|u| + c_3|v|, \ \forall\,(x,z) \in \mathbb{R}^N \times \mathbb{R}^2,$$
$$|F_v(x,z)| \leq c_3|u|^{q_2}|v|^{p_2-1} + c_3|u| + \beta(x)|v|, \ \forall\,(x,z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

where $2 \leq p_i + q_i < 2^*$, $i = 1,2$ and α, β satisfy (1).

(F_5) $D^2F(x,0) \equiv D^2F(0)$ and we have either

(i) $F_{uu}(0) \geq 0,\ F_{vv}(0) \geq 0$ and $\lambda_k < F_{uv}(0) + \sqrt{F_{uu}(0)F_{vv}(0)}$,
or
(ii) $F_{uu}(0) < 0,\ F_{vv}(0) < 0$, $F_{uv}(0) > -\lambda_1$ and $\lambda_{k-1} > F_{uv}(0) - \sqrt{F_{uu}(0)F_{vv}(0)}$.

Under these conditions we are able to prove

Theorem 2. *Suppose (A_1) and (A_2) hold. If F satisfies (F_1), $(\widehat{F_2})-(\widehat{F_4})$, (F_5), (NQ) and $F(x,0)\equiv\nabla F(x,0)\equiv 0$, then problem (P) possesses a nonzero solution.*

In our final result we verify the existence of multiple solutions for (P) under the assumption that the primitive is even with respect to the variable z. Since we need a compactness condition with respect to the norm topology we assume (F_3) with $\sigma < 2^*$ and

$$(\widehat{F_5}) \qquad F(x,z)-\frac{1}{2}Az\cdot z = o(|z|^2), \quad \text{as} \quad |z|\to 0,$$

where $A=(a_{ij})$ is a symmetric 2×2 matrix such that $a_{11},a_{22}<0$, $a_{12}>-\lambda_1$ and $a_{12}-\sqrt{a_{11}a_{22}}<\lambda_j<\lambda_k$.

(F_6) $F(x,z)$ is even with respect to the variable $z\in\mathbb{R}^2$.

Now, we may state

Theorem 3. *Suppose (A_1) and (A_2) hold. If F satisfies $F(x,0)\equiv 0$, (F_1), (F_2), (F_3) with $\sigma<2^*$, $(\widehat{F_4})$, $(\widehat{F_5})$, (F_6) and (NQ), then problem (P) possesses $k-j$ pairs of nonzero solutions.*

The paper is organized as follows. In Section 2 we state the abstract results we need to prove our main theorems and study the coupled linear system (LP). In Section 3 we prove Theorem 1. Finally, in Section 4, we present the proofs of Theorems 2 and 3.

2. **Preliminaries and the Linear Problem.** Let E be a real Hilbert space and $I:E\to\mathbb{R}$ a functional of class C^1. In general, in order to apply minimax theorems, the functional must satisfy a compactness condition. In this article, we deal with a condition introduced by Silva in [10]. A sequence $(z_n)\subset E$ is said to be a strong Cerami sequence if $I(z_n)\to c$ and $I'(z_n)\to 0$ as $n\to\infty$, and $\|z_n\|\,\|I'(z_n)\|$ is bounded. We recall that $I\in C^1(E,\mathbb{R})$ satisfies the *strong Cerami condition* [(SCe)] if any strong Cerami sequence $(z_n)\subset E$ possesses a convergent subsequence.

To establish the existence of a critical point for the functional we only need a version of the (SCe) condition for the weak topology. We say that $I\in C^1(E,\mathbb{R})$ satisfies the *strong Cerami condition for the weak topology* [(SCe)'] if any strong Cerami sequence $(z_n)\subset E$ possesses a subsequence which converges weakly to a critical point of I.

Next, we state the abstract results that will be used for the proof of our theorems (see also [8, 6, 5, 4, 7] for related results).

Theorem 4 ([3])**.** *Let $E=V\oplus W$ be a real Hilbert space with V finite dimensional and $W=V^\perp$. Suppose $I\in C^1(E,\mathbb{R})$ satisfies* (SCe)' *and*

(I_1) *there exists $\beta\in\mathbb{R}$ such that $I(z)\le\beta$, for all z in V,*

(I_2) *there exists $\gamma\in\mathbb{R}$ such that $I(z)\ge\gamma$, for all z in W.*

Then I possesses a critical point.

Theorem 5 ([2])**.** *Let $E=V\oplus W$ be a real Hilbert space with V finite dimensional and $W=V^\perp$. Suppose $I\in C^1(E,\mathbb{R})$ satisfies* (SCe), (I_1), (I_2) *and*

(I_3) *the origin is a critical point of I, $D^2I(0)$ is a Fredholm operator and either $\dim V<m(I,0)$ or $\overline{m}(I,0)<\dim V$.*

Then I possesses a nonzero critical point.

Theorem 6 ([8])**.** *Let $E = V \oplus W$ be a real Hilbert space with V finite dimensional and $W = V^{\perp}$. Suppose $I \in C^1(E, \mathbb{R})$ is even and satisfies $I(0) = 0$,* (SCe) *and*

(I_4) *there exists a finite dimensional closed subspace $\widehat{V}$ of E and $\beta \in \mathbb{R}$ such that $\widehat{V} \supset V$ and $I(z) \leq \beta$, for all z in $\widehat{V}$,*

(I_5) *there exists $\rho > 0$ such that $I(z) \geq 0$, for all z in $B_\rho(0) \cap W$.*

Then I possesses $\dim \widehat{V} - \dim V$ pairs of nontrivial critical points.

Remark 1. In Theorem 5, $m(I, z)$ [$\overline{m}(I, z)$] denotes the Morse index [augmented Morse index] of the functional I at the point z. Actually, in [8], Theorem 6 is stated for the Palais-Smale condition. The version for the (SCe) condition is based on a deformation lemma proved in [10].

For applying the abstract results we set $E = E_a \times E_b$ where

$$E_a = \left\{ u \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) : \int_{\mathbb{R}^N} (|\nabla u|^2 + a(x)u^2)\, dx < \infty \right\}$$

and

$$E_b = \left\{ v \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) : \int_{\mathbb{R}^N} (|\nabla v|^2 + b(x)v^2)\, dx < \infty \right\}$$

endowed with the inner product

$$\langle (u,v), (\phi, \psi) \rangle = \int_{\mathbb{R}^N} (\nabla u \nabla \phi + \nabla v \nabla \psi + a(x)u\phi + b(x)v\psi)\, dx,$$

and associated norm given by

$$\|z\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + a(x)u^2 + b(x)v^2)\, dx, \ \forall\ z = (u,v) \in E. \tag{2}$$

For $z \in E$ the functional

$$I(z) = \frac{1}{2}\|z\|^2 - \int_{\mathbb{R}^N} F(x,z)\, dx \tag{3}$$

is well defined and of class C^1 via (F_2) and (F_3). Moreover the critical points of I are precisely the weak solutions of the system (P).

The condition (A_1) and the Sobolev Theorem imply that the immersion $E \hookrightarrow L^s(\mathbb{R}^N, \mathbb{R}) \times L^s(\mathbb{R}^N, \mathbb{R})$ is continuous for $2 \leq s \leq 2^*$. However, since we are not supposing that a or b are coercive, this embedding may not be compact. This fact is compensated by the coupling of the system.

We state bellow a result proved in [3] that will be useful to verify the condition (SCe)'.

Lemma 1. *Suppose F satisfies $(F_2)-(F_3)$. If every strong Cerami sequence $(z_n) \subset E$ possesses a bounded subsequence, then I satisfies* (SCe)'.

Now, we proceed with the study of the associated coupled linear problem (LP). Standard calculations shows that λ is an eigenvalue of (LP) if, and only if,

$$T(u,v) = \frac{1}{\lambda}(u,v),$$

where $T : E \to E$ is the selfadjoint bounded linear operator defined by

$$\langle T(u,v), (\phi, \psi) \rangle = \int_{\mathbb{R}^N} (v\phi + u\psi)\, dx.$$

Moreover, the following result holds.

Lemma 2. *Suppose* (A_1) *and* (A_2) *hold. Then* T *is a compact operator.*

Proof. Let $(z_n) = ((u_n, v_n)) \subset E$ be a sequence such that $z_n \rightharpoonup z = (u, v)$ in E (without loss of generality, we may suppose $z = (0, 0)$). Then there exists $M > 0$ such that, for all $n \in N$,

$$\|z_n\| \leq M, \ \|Tz_n\| \leq M. \tag{4}$$

By the condition (A_2), given $\varepsilon > 0$ we may choose $R > 0$ such that

$$\mu(C_\varepsilon) < \varepsilon, \tag{5}$$

where $C_\varepsilon = \{x \in \mathbb{R}^N \setminus B_R(0) : a(x)b(x) \leq \varepsilon^{-2}\}$. Now, writing $T = (T_1, T_2)$, by the definition of T, we have

$$\begin{aligned} 0 &\leq \|Tz_n\|^2 = \langle Tz_n, Tz_n\rangle \\ &= \int_{\mathbb{R}^N} v_n T_1 z_n \, dx + \int_{\mathbb{R}^N} u_n T_2 z_n \, dx. \end{aligned}$$

Since $v_n \to 0$ strongly in $L^2(B_R(0))$, for n sufficiently large, we obtain

$$\left| \int_{B_R(0)} v_n T_1 z_n \ dx \right| < \varepsilon. \tag{6}$$

Setting $D_\varepsilon = \mathbb{R}^N \setminus (B_R(0) \cup C_\varepsilon)$, we may use Holder's inequality and (4) to get

$$\left| \int_{D_\varepsilon} v_n T_1 z_n \ dx \right| \leq \varepsilon \int_{D_\varepsilon} |\sqrt{b(x)} v_n \sqrt{a(x)} T_1 z_n| \ dx \leq \varepsilon M^2. \tag{7}$$

On the other hand, invoking (4), (5) and using Holder's inequality one more time, we obtain $M_1 > 0$ such that

$$\begin{aligned} \left| \int_{C_\varepsilon} v_n T_1 z_n \ dx \right| &\leq \left[\int_{C_\varepsilon} |v_n|^2 \ dx \right]^{\frac{1}{2}} \|T_1 z_n\|_{L^2(\mathbb{R}^N)} \\ &\leq \mu(C_\varepsilon)^{\frac{1}{N}} \|v_n\|_{L^{2^*}(\mathbb{R}^N)} \|T_1 z_n\|_{L^2(\mathbb{R}^N)} \\ &\leq M_1 \varepsilon^{\frac{1}{N}}. \end{aligned} \tag{8}$$

In view of (6)-(8) and the fact that $\varepsilon > 0$ can be arbitrarily chosen, we have

$$\int_{\mathbb{R}^N} v_n T_1 z_n \ dx \to 0, \quad \text{as} \quad n \to \infty.$$

Analogous argument shows that

$$\int_{\mathbb{R}^N} u_n T_2 z_n \ dx \to 0, \quad \text{as} \quad n \to \infty.$$

Consequently, $Tz_n \to 0$, as $n \to \infty$. The lemma is proved. □

Observing that $(u, -v)$ is an eigenfunction associated with the eigenvalue $-\lambda$ whenever (u, v) is an eigenfunction associated to λ, we invoke Lemma 2 and the spectral theory for compact operators to conclude that (LP) possesses a sequence $\{\lambda_m\}_{m \in \mathbb{Z}^*}$ of eigenvalues

$$\cdots \leq \lambda_{-m} \leq \cdots \leq \lambda_{-2} \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots,$$

such that $\lambda_{\pm m} \to \pm\infty$ as $m \to \infty$.

3. **Proof of Theorem 1.** In this section we prove Theorem 1 by verifying that the functional I defined in (3) satisfies the hypotheses of Theorem 4.

Considering k given by (F_1) we set $V = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_{k-1}\}$ and $W = V^{\perp}$ (without loss of generality $\lambda_{k-1} < \lambda_k$ and $V = \emptyset$ if $k = 1$). The following lemma is a variant of Lemma 3.1 in [3].

Lemma 3. *Suppose (A_1) and (A_2) hold. If F satisfies (F_2) and (F_4), then, given $R > 0$ and $\varepsilon > 0$, there exists $M = M(R) > 0$ such that*

$$\int_{\{|z|\le R\}} F(x,z)\,dx \le M + \left(\varepsilon + \frac{\alpha_\infty}{2a_0}\right)\|u\|_{E_a}^2 + \left(\varepsilon + \frac{\beta_\infty}{2b_0}\right)\|v\|_{E_b}^2, \tag{9}$$

for all $z = (u, v) \in E$.

Proof of Theorem 1. Conditions (F_2) and (F_3) imply that $I \in C^1(E, \mathbb{R})$ and the critical points of I are the weak solutions of (P). The geometrical conditions (I_1) and (I_2) are proved by arguments similar to those used in the proof of Theorem 1.1 in [3]. Thus, we need only to show that I satisfies (SCe)' condition.

Let $(z_n) \subset E$ be such that $I(z_n) \to c$, $I'(z_n) \to 0$ and $\|z_n\|\,\|I'(z_n)\|$ is bounded. In view of Lemma 1 it suffices to verify that (z_n) possesses a bounded subsequence. Arguing by contradiction, we suppose that $\|z_n\| \to \infty$. Since $I(z_n) \to c$ and $\|z_n\|\,\|I'(z_n)\|$ is bounded there exists $M > 0$ such that

$$\liminf \int_{\mathbb{R}^N} H(x, z_n)\,dx = \liminf\,[2I(z_n) - I'(z_n)z_n] \le M, \tag{10}$$

where $H(x, z_n) = \nabla F(x, z_n) \cdot z_n - 2F(x, z_n)$. We obtain a contradiction by the following claim: there exists $\hat{\Omega} \subset \Omega_\gamma$ with $\mu(\hat{\Omega}) > 0$, such that up to subsequences, $|u_n(x)| \to +\infty$ and $|v_n(x)| \to +\infty$ as $n \to +\infty$, for almost every $x \in \hat{\Omega}$.

Assuming the claim, by Fatou's lemma and (NQ), we have

$$\liminf \int_{\mathbb{R}^N} H(x, z_n)\,dx \ge \int_{\mathbb{R}^N} \liminf H(x, z_n)\,dx = \infty,$$

which contradicts (10).

Now we proceed with the proof of the claim. Given $\varepsilon > 0$, by (F_1), there exists $R > 0$ such that

$$F(x,z) \le \lambda_k uv + \varepsilon|z|^2,\ \forall\, x \in \mathbb{R}^N,\ |z| > R.$$

Thus, for n sufficiently large, we have

$$\frac{1}{2}\|z_n\|^2 \le (c+1) + \int_{\{|z|>R\}} \left(\lambda_k|u_n||v_n| + \varepsilon|z_n|^2\right)\,dx + \int_{\{|z|\le R\}} F(x, z_n)\,dx,$$

and therefore

$$\frac{1}{2}(1 - 2S\varepsilon)\|z_n\|^2 \le M_1 + \lambda_k \int_{\mathbb{R}^N} |u_n||v_n|\,dx + \int_{\{|z|\le R\}} F(x, z_n)\,dx,$$

where S is a positive constant such that $\|z\|_{L^2}^2 \le S\|z\|^2$, for all $z \in E$. Recalling that $\|z_n\|^2 = \|u_n\|_{E_a}^2 + \|v_n\|_{E_b}^2$, we can use Lemma 3 to obtain

$$\frac{\nu_1}{2}\|u_n\|_{E_a}^2 + \frac{\nu_2}{2}\|v_n\|_{E_b}^2 \le M_2 + \lambda_k \int_{\mathbb{R}^N} |u_n||v_n|\,dx, \tag{11}$$

where $\nu_1 = (1 - 2\varepsilon(S+1) - \alpha_\infty/a_0)$ and $\nu_2 = (1 - 2\varepsilon(S+1) - \beta_\infty/b_0)$. Furthermore, taking $\varepsilon > 0$ sufficiently small, we may suppose that $\nu_1, \nu_2 > 1/\sqrt{\gamma}$.

Let γ be given by (NQ), since $\sqrt{a(x)b(x)} \geq \lambda_k\sqrt{\gamma}$ in $\mathbb{R}^N \setminus \Omega_\gamma$, we may use Young's inequality to obtain

$$\lambda_k \int_{\mathbb{R}^N\setminus\Omega_\gamma} |u_n||v_n|\ dx \leq \frac{1}{2\sqrt{\gamma}} \int_{\mathbb{R}^N\setminus\Omega_\gamma} \left(a(x)|u_n|^2 + b(x)|v_n|^2\right)\ dx.$$

Splitting the integrals in (11) over the sets Ω_γ and $\mathbb{R}^N \setminus \Omega_\gamma = \Gamma$ and using the above estimate, we get

$$\begin{aligned} \frac{\nu_1}{2}\left(\|u_n\|^2_{E_a(\Omega_\gamma)} + \int_\Gamma |\nabla u_n|^2\ dx\right) \quad &+ \quad \frac{\nu_2}{2}\left(\|v_n\|^2_{E_b(\Omega_\gamma)} + \int_\Gamma |\nabla v_n|^2\ dx\right) \\ &+ \quad \frac{1}{2}(\nu_1 - \gamma^{-1/2}) \int_\Gamma a(x)|u_n|^2\ dx \\ &+ \quad \frac{1}{2}(\nu_2 - \gamma^{-1/2}) \int_\Gamma b(x)|v_n|^2\ dx \\ &\leq \quad M_2 + \lambda_k \int_{\Omega_\gamma} |u_n||v_n|\ dx. \end{aligned}$$

Therefore, setting $\nu_0 = \frac{1}{2}\min\{\nu_1 - \gamma^{-1/2}, \nu_2 - \gamma^{-1/2}\} > 0$, we have

$$\nu_0 \|z_n\|^2 \leq M_2 + \lambda_k \int_{\Omega_\gamma} |u_n||v_n|\ dx. \tag{12}$$

Now we set $C = \{x \in \mathbb{R}^N \setminus B_{R_1}(0) : \sqrt{a(x)b(x)} \leq \lambda_k\nu_0^{-1}\}$. By (A_2) and the argument used in Lemma 2, we find $R_1 > 0$ such that

$$\lambda_k \int_C |u_n||v_n|\ dx \leq \frac{\nu_0}{4}\|z_n\|^2. \tag{13}$$

Moreover, by Young's inequality, we have

$$\lambda_k \int_{\mathbb{R}^N\setminus(B_{R_1}(0)\cup C)} |u_n||v_n|\ dx \leq \frac{\nu_0}{2}\|z_n\|^2. \tag{14}$$

¿From (12)-(14), we get

$$\frac{\nu_0}{4}\|z_n\|^2 \leq M_2 + \lambda_k \int_{\Omega_\gamma\cap B_{R_1}(0)} |u_n||v_n|\ dx. \tag{15}$$

Defining $\widehat{z}_n = (\widehat{u}_n, \widehat{v}_n) = \dfrac{1}{\|z_n\|}(|u_n|, |v_n|)$, we may assume that

$$\begin{cases} \widehat{u}_n \to \widehat{u} \text{ in } L^2(\Omega_\gamma \cap B_{R_1}(0)), \\ \widehat{v}_n \to \widehat{v} \text{ in } L^2(\Omega_\gamma \cap B_{R_1}(0)). \end{cases}$$

Hence, by (15), we get

$$\frac{\nu_0}{4} \leq \lambda_k \int_{\Omega_\gamma\cap B_{R_1}(0)} \widehat{u}\widehat{v}\ dx,$$

and therefore there exists $\hat{\Omega} \subset \Omega_\gamma$, with positive measure, such that $\widehat{u}(x) \neq 0$ and $\widehat{v}(x) \neq 0$, a.e. $x \in \hat{\Omega}$. The claim is now proved by observing that we are assuming that $\|z_n\| \to +\infty$ as $n \to +\infty$. $\square$

4. **Proofs of Theorems 2 and 3.** We begin by observing that we may suppose without loss of generality that 0 is an isolated critical point of I. Conditions (F_1), $(\widehat{F_3})$, $(\widehat{F_4})$, $F(x,0) \equiv 0$ and $\nabla F(x,0) \equiv 0$ imply that (F_4) holds. Thus, by Proposition 3.5 in [3], the geometrical conditions (I_1) and (I_2) are satisfied. In order to verify that $D^2I(0)$ is a Fredholm operator we first note that

$$D^2I(0)(z,z) = \|z\|^2 - F_{uu}(0)\int_{\mathbb{R}^N} u^2\,dx - F_{vv}(0)\int_{\mathbb{R}^N} v^2\,dx - 2F_{uv}(0)\int_{\mathbb{R}^N} uv\,dx.$$

Since, by $(\widehat{F_4})$, $F_{uu}(0) \le \alpha_\infty < a_0$ and $F_{vv}(0) \le \beta_\infty < b_0$, the above expression implies that $D^2I(0)$ is of the type $L-K$, where L is an isomorphism and K is compact.

It is proved in [3] that the hypothesis (F_5) implies the Morse index estimates stated in condition (I_3). Thus, in view of Theorem 5, we need only to verify that the functional I satisfies (SCe).

Let $(z_n) \subset E$ be a strong Cerami sequence. In view of the proof of Theorem 1, we may suppose that (z_n) is bounded and $z_n \rightharpoonup z$, with z a critical point of I. Furthermore, up to a subsequence, we have

$$\begin{cases} u_n \rightharpoonup u \text{ in } E_a,\ v_n \rightharpoonup v \text{ in } E_b, \\ u_n \to u,\ v_n \to v \text{ in } L^s_{loc}(\mathbb{R}^N) \text{ for } 2 \le s < 2^*. \end{cases} \tag{16}$$

We will show that the first inequality in the condition $(\widehat{F_4})$ implies $u_n \to u$ in E_a. Since $I'(z) = 0$, $I'(z_n) \to 0$ and (u_n) is bounded, we have

$$\begin{aligned} \|u_n - u\|^2_{E_a} &= I'(z_n)(u_n-u,0) + \int_{\mathbb{R}^N}(u_n-u)(F_u(x,z_n)-F_u(x,z))\,dx \\ &\le o(1) + \int_{\mathbb{R}^N}\left(F_u(x,z_n)(u_n-u) + F_u(x,z)(u-u_n)\right)\,dx, \end{aligned} \tag{17}$$

as $n\to\infty$. Choosing $0<\delta<a_0-\alpha_\infty$, we claim that

$$\int_{\mathbb{R}^N} F_u(x,z_n)(u_n-u)\,dx \le o(1) + \left(\frac{\alpha_\infty+\delta}{a_0}\right)\|u_n-u\|^2_{E_a}, \quad \text{as} \quad n\to\infty. \tag{18}$$

Indeed, defining $G(z_n) = \int_{\mathbb{R}^N} F_u(x,z_n)(u_n-u)\,dx$, we may use $(\widehat{F_4})$ to obtain $R>0$ such that

$$\begin{aligned} G(z_n) \le\ & c_3\int_{\mathbb{R}^N}|u_n|^{p_1-1}|v_n|^{q_1}|u_n-u|\,dx + \int_{B_R(0)}\alpha(x)|u_n||u_n-u|\ dx \\ & + (\alpha_\infty+\delta)\int_{\mathbb{R}^N\setminus B_R(0)}\left(|u||u_n-u| + |u_n-u|^2\right)\ dx \\ & + c_3\int_{\mathbb{R}^N}|v_n||u_n-u|\,dx. \end{aligned} \tag{19}$$

First note that, by the local convergence in (16),

$$\int_{B_R(0)}\alpha(x)|u_n||u_n-u|\ dx \to 0, \quad \text{as} \quad n\to\infty. \tag{20}$$

In order to estimate the first integral in (19), given $\varepsilon>0$, we choose $R_1>0$ such that $\mu(C_\varepsilon)<\varepsilon$ where $C_\varepsilon = \{x \in \mathbb{R}^N \setminus B_{R_1}(0) : a(x)b(x) \le \varepsilon^{-2}\}$. For this value of R_1, taking $r_1 = (p_1+q_1)/(p_1-1)$, $r_2 = (p_1+q_1)/q_1$ and $r_3 = p_1+q_1$, where

$1/r_1+1/r_2+1/r_3=1$, we may invoke the local convergence as above and Holder's inequality to get

$$\int_{B_{R_1}(0)} |u_n|^{p_1-1}|v_n|^{q_1}|u_n-u|\ dx<\varepsilon, \tag{21}$$

for n sufficiently large. Setting $D_\varepsilon=\mathbb{R}^N\setminus(B_{R_1}(0)\cup C_\varepsilon)$ and using Holder's inequality and (16), we find $M_1>0$ such that

$$\int_{D_\varepsilon} |u_n|^{p_1-1}|v_n|^{q_1}|u_n-u|\ dx\le M_1\left[\int_{D_\varepsilon}\left(|u_n|^{p_1-1}|v_n|^{q_1}\right)^{\frac{p_1+q_1}{p_1+q_1-1}}\ dx\right]^{\frac{p_1+q_1-1}{p_1+q_1}}.$$

Supposing without loss of generality that $q_1>p_1-1$, we apply Holder's inequality and (16) one more time to find $M_2>0$ such that

$$\int_{D_\varepsilon} |u_n|^{p_1-1}|v_n|^{q_1}|u_n-u|\ dx\le M_2\left[\int_{D_\varepsilon}(|u_nv_n|)^{\frac{p_1+q_1}{2}}\ dx\right]^{\frac{2(p_1-1)}{p_1+q_1}}. \tag{22}$$

Now we take $t\in(0,1]$ such that $p_1+q_1=2t+(1-t)2^*$, obtaining by interpolation and the definition of D_ε

$$\begin{aligned}\int_{D_\varepsilon}|u_n||v_n|^{\frac{p_1+q_1}{2}}\ dx &\le \left(\int_{D_\varepsilon}|u_nv_n|\ dx\right)^t\left(\int_{\mathbb{R}^N}|u_nv_n|^{2^*/2}\,dx\right)^{1-t}\\ &\le \varepsilon^t\left(\|u_n\|^2_{E_a}+\|v_n\|^2_{E_b}\right)^t\left(\|u_n\|^{2^*}_{L^{2^*}}+\|v_n\|^{2^*}_{L^{2^*}}\right)^{1-t}.\end{aligned}$$

By the above expression, (22) and the bound of (z_n), we find $r>0$ and $M_3>0$ such that, for every n,

$$\int_{D_\varepsilon} |u_n|^{p_1-1}|v_n|^{q_1}|u_n-u|\ dx\le M_3\varepsilon^r. \tag{23}$$

On the other hand, by Holder's inequality and (16), for every n,

$$\begin{aligned}&\int_{C_\varepsilon}|u_n|^{p_1-1}|v_n|^{q_1}|u_n-u|\ dx\le\\ &\left(\int_{\mathbb{R}^N}|u_n|^{p_1+q_1}\,dx\right)^{\frac{p_1-1}{p_1+q_1}}\left(\int_{\mathbb{R}^N}|v_n|^{p_1+q_1}\,dx\right)^{\frac{q_1}{p_1+q_1}}\left(\int_{\mathbb{R}^N}|u_n-u|^{2^*}\,dx\right)^{\frac{1}{2^*}}\mu(C_\varepsilon)^s\\ &<M_4\ \varepsilon^s,\end{aligned} \tag{24}$$

where $s=\frac{2^*-(p_1+q_1)}{2^*(p_1+q_1)}>0$. Hence it follows from (21), (23) and (24)

$$\int_{\mathbb{R}^N}|u_n|^{p_1-1}|v_n|^{q_1}|u_n-u|\,dx\to 0,\quad\text{as } n\ \to\infty. \tag{25}$$

Analogously

$$\int_{\mathbb{R}^N}|v_n||u_n-u|\,dx\to 0\ \text{ and }\ \int_{\mathbb{R}^N\setminus B_R(0)}|u||u_n-u|\ dx\to 0, \tag{26}$$

as n goes to infinity. Furthermore

$$(\alpha_\infty + \delta) \int_{\mathbb{R}^N \setminus B_R(0)} |u_n - u|^2 \, dx \le \left(\frac{\alpha_\infty + \delta}{a_0} \right) \|u_n - u\|^2_{E_a} .$$

This and equations (19),(20), (25) and (26) prove the claim. In a similar way

$$\int_{\mathbb{R}^N} F_u(x, z)(u - u_n) \, dx \to 0, \quad \text{as} \quad n \to \infty.$$

The above inequality, (17) and (18) shows that

$$\left(1 - \frac{\alpha_\infty + \delta}{a_0} \right) \|u_n - u\|^2_{E_a} \le o(1), \quad \text{as} \quad n \to \infty.$$

By our choice of δ we conclude that $u_n \to u$ in E_a.

Proceeding in a similar way we can use the second inequality in $(\widehat{F_4})$ to show that $v_n \to v$ in E_b. Thus, we conclude that the functional I satisfies (SCe). □

For the proof of Theorem 3 we just refer the reader to the article [3] where the conditions (I_4) and (I_5) have been established. In our setting the condition (SCe) is settled by the argument above.

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