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Multiplicity and concentration of solutions for elliptic systems with vanishing potentials ☆

Marcelo F. Furtado [a,*], Elves A.B. Silva [a], Magda S. Xavier [b]

[a] *Departamento de Matemática, Universidade de Brasília, 70910-900 Brasília – DF, Brazil*
[b] *Universidade Federal do Espírito Santo, Departamento de Matemática, 29075-910, Vitória – ES, Brazil*

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ABSTRACT

In this article we use variational methods to study a strongly coupled elliptic system depending on a positive parameter λ. We suppose that the potentials are nonnegative and the intersection of the sets where they vanish has positive measure. A technical condition, imposed on the product of the potentials, allows us to consider a setting where we do not assume any positive lower bound for the potentials. Considering the associated functional, defined on an appropriated subspace of $\mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$, we are able to establish results on the existence and multiplicity of solutions for the system when the parameter λ is sufficiently large. We also study the asymptotic behavior of these solutions when $\lambda \to \infty$.

1. Introduction

The goal of this paper is to study the existence, multiplicity and asymptotic behavior of solutions for the coupled elliptic system

$$\begin{cases} -\Delta u + \lambda a(x)u = \dfrac{p}{p+q}|u|^{p-2}u|v|^q, \\ -\Delta v + \lambda b(x)v = \dfrac{q}{p+q}|u|^p|v|^{q-2}v, \\ u, v \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases} \tag{S_λ}$$

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* Corresponding author.
E-mail addresses: mfurtado@unb.br (M.F. Furtado), elves@unb.br (E.A.B. Silva), magda@cce.ufes.br (M.S. Xavier).

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where $N \geqslant 3$, $\lambda > 0$ is a parameter, $p, q > 1$ and $p + q < 2^* := 2N/(N-2)$. Our hypotheses on the nonnegative potentials a and b are

(H_1) $a, b \in C(\mathbb{R}^N, [0,\infty))$, $\Omega_a := \operatorname{int} a^{-1}(0)$ and $\Omega_b := \operatorname{int} b^{-1}(0)$ have smooth boundaries, $\overline{\Omega}_a = a^{-1}(0)$, $\overline{\Omega}_b = b^{-1}(0)$, and $\Omega_a \cap \Omega_b$ is a nonempty set;
(H_2) there exists $M_0 > 0$ such that the set $F := \{x \in \mathbb{R}^N\colon a(x)b(x) \leqslant M_0\}$ has finite Lebesgue measure.

Note that we do not assume any positive lower bound for the potentials a and b. Hence we do not expect to find solutions for (S_λ) in the Sobolev space $H^1(\mathbb{R}^N)$. However, taking advantage of the strong coupling of the system and the hypothesis (H_2), we are able to use variational methods to study (S_λ) by considering the associated functional defined in a proper closed subspace of $\mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$. We also observe that the sets Ω_a and Ω_b may be unbounded and that the fact that $\Omega_a \cap \Omega_b$ is a nonempty set is essential for our results.

As in the scalar case [4], the main results in this article show that the semilinear elliptic system

$$\begin{cases} -\Delta u = \frac{p}{p+q}|u|^{p-2}u|v|^q & \text{in } \Omega_a, \\ -\Delta v = \frac{q}{p+q}|u|^p|v|^{q-2}v & \text{in } \Omega_b, \\ u \in H_0^1(\Omega_a), \quad v \in H_0^1(\Omega_b), \end{cases} \tag{L}$$

may be seen as a limit problem for (S_λ) when λ goes to infinity. It is worthwhile mentioning that, although Ω_a and Ω_b may be distinct open sets, the system (L) is variational. We also note that condition (H_2) implies that Ω_a and Ω_b have finite Lebesgue measure. So, we have the Sobolev compact imbedding $H^1(\Omega_a) \times H^1(\Omega_b) \hookrightarrow L^{r_1}(\Omega_a) \times L^{r_2}(\Omega_b)$, $1 \leqslant r_1, r_2 < 2^*$.

In order to state our results, we introduce the closed subspaces of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ associated with the potentials a and b:

$$X_a := \left\{u \in \mathcal{D}^{1,2}\big(\mathbb{R}^N\big)\colon \int_{\mathbb{R}^N} a(x)u^2\,dx < \infty\right\},$$

and

$$X_b := \left\{u \in \mathcal{D}^{1,2}\big(\mathbb{R}^N\big)\colon \int_{\mathbb{R}^N} b(x)u^2\,dx < \infty\right\}.$$

For any given $\lambda > 0$, we consider the Hilbert space $X := X_a \times X_b$ endowed with the norm

$$\big\|(u,v)\big\|_\lambda^2 := \int_{\mathbb{R}^N} \big(|\nabla u|^2 + |\nabla v|^2 + \lambda a(x)u^2 + \lambda b(x)v^2\big)\,dx.$$

Notice that $\|\cdot\|_0$ is the usual norm of the space $\mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$.

Associated with the problem (S_λ) we have the energy functional $I_\lambda : X \to \mathbb{R}$ defined by

$$I_\lambda(u,v) := \frac{1}{2}\big\|(u,v)\big\|_\lambda^2 - \frac{1}{p+q}\int_{\mathbb{R}^N} |u|^p|v|^q\,dx, \quad (u,v) \in X.$$

In view of the hypotheses (H_1) and (H_2), the functional I_λ is well defined and of class C^1. Furthermore, standard regularity theory implies that the critical points of I_λ are classical solutions of the problem (S_λ) (see Section 2).

In our first result we consider the existence and behavior of least energy solutions of (S_λ). We recall that a least energy solution of our problem is a critical point of I_λ associated with the lowest positive critical level of this functional. In our setting, once proved the existence of a least energy solution, we may always find a positive least energy solution. Here we observe that we call $z = (u, v)$ a positive function if the functions u and v are positive almost everywhere in $\mathbb{R}^N$.

Theorem 1.1. *Suppose (H_1) and (H_2) hold. Then there is $\Lambda > 0$ such that, for all $\lambda \geqslant \Lambda$, the system (S_λ) possesses a positive least energy solution z_λ. Furthermore, if $(\lambda_n) \subset \mathbb{R}$ is such that $\lambda_n \to \infty$ and (z_{λ_n}) is a sequence of positive least energy solution of (S_{λ_n}), then (z_{λ_n}) converges in $\mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ along a subsequence to a positive least energy solution of (L).*

In our next result we use the symmetry of our problem to establish multiplicity of solutions for large values of λ. More specifically, we shall prove

Theorem 1.2. *Suppose (H_1) and (H_2) hold. Then , for any given $k \in \mathbb{N}$, there exists $\Lambda_k > 0$ such that, for each $\lambda \geqslant \Lambda_k$, the system (S_λ) possesses at least k pairs of nonzero solutions.*

As in the case of the least energy solutions found in Theorem 1.1, the solutions derived from Theorem 1.2 have uniformly-bounded energy with respect to λ. This allows us to show that these solutions converge in $\mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ toward solutions of (L) as $\lambda \to \infty$. More generally, we have the following concentration result.

Theorem 1.3. *Let $(\lambda_n) \subset \mathbb{R}$ be such that $\lambda_n \to \infty$ and (z_{λ_n}) be a sequence of solutions of (S_{λ_n}) such that $\liminf_{n\to\infty} I_{\lambda_n}(z_{\lambda_n}) < \infty$. Then (z_{λ_n}) converges in $\mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ along a subsequence to a solution of (L).*

The results presented in this article are motivated by that obtained in [4,5] (see also [3]) for the scalar case, where it is considered the potential $c_\lambda(x) = \lambda c(x) + 1$ with c being such that the set $\{x \in \mathbb{R}^N\colon\ c(x) \leqslant M_0\}$ has finite Lebesgue measure, for some $M_0 > 0$. Concerning our multiplicity result we follow a different approach from [5]. Instead of considering the Ljusternik–Schnirelmann category of some set related with the limit problem, here we use the symmetry of the nonlinearity to derive the existence of multiple solutions.

We observe that there exists an extensive bibliography in the study of elliptic systems on bounded domains (see [15,16,19,9,17,8,11] and references therein). In the case of gradient systems in the whole $\mathbb{R}^N$, in [7] the author proves the existence of a nonzero solution for (P) under the coercivity of the potentials a and b, and a nonquadratic condition on the nonlinearity. A related result for noncoercive potentials is proved in [12] (see also [14] for the superlinear case). We should also mention the recent papers [13,1] where some existence results of positive solutions for weakly coupled system are established. We would like to emphasize that, instead of the aforementioned works, the coupling in our system (S_λ) allows us to consider potentials which are not bounded from below by positive constants. We may have one of the potentials going to zero as $|x| \to \infty$ provided the other one goes to infinity at an appropriated rate.

The paper is organized in the following way. In Section 2 we present technical results which will be used throughout the work. We also investigate the behavior of the Palais–Smale sequences when λ goes to infinity. We prove Theorem 1.1 in Section 3. The final Section 4 is devoted to the proof of Theorems 1.2 and 1.3.

2. Preliminaries

In this section we present some preliminaries for the proof of Theorem 1.1. In this paper, we denote by B_R the open ball in $\mathbb{R}^N$ of radius $R > 0$ and center at the origin. For any given set $K \subset \mathbb{R}^N$, we set $K^c := \mathbb{R}^N \setminus K$ and we write $\mathcal{L}(K)$ for the Lebesgue measure of K whenever this set is measurable. $C_0^\infty(K)$ denotes the set of all functions $u : K \to \mathbb{R}$ of class C^∞ with compact support contained in the open set $K \subset \mathbb{R}^N$. If $u \in L^s(K)$, $s \geqslant 1$, we set $u_+ := \max\{u, 0\}$, $u_- := \max\{-u, 0\}$ and write $\|u\|_{L^s(K)}$ for the L^s-norm of u. In order to simplify the notation, we write $\int_K u$ instead of $\int_K u(x)\,dx$. We also omit the set K whenever $K = \mathbb{R}^N$. Finally, we use the symbols c_i, $i \in \mathbb{N}$, to represent positive constants.

We start with two technical results.

Lemma 2.1. *For any given measurable set $K \subset \mathbb{R}^N$ there exists a constant $c > 0$ such that*

$$\int_K |u|^p|v|^q \leqslant c\big\|(u,v)\big\|_0^{p+q-2+2^*t/r}\left(\int_K |uv|\right)^{\beta}, \quad \text{for all } (u,v) \in X,$$

where $r := 2^*/(2^* - p - q + 2) > 1$, and $t \in (0,1)$ satisfies $r = 2^*t/2 + (1-t)$ and $\beta := (1-t)/r$.

Proof. From the definition of $r > 1$ we have that

$$\frac{p-1}{2^*} + \frac{q-1}{2^*} + \frac{1}{r} = 1. \tag{2.1}$$

Hölder's inequality and the imbedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ imply that

$$\begin{aligned}
\int_K |u|^p|v|^q &\leqslant \int_K |u|^{p-1}|v|^{q-1}|uv| \\
&\leqslant \left(\int_K |u|^{2^*}\right)^{(p-1)/2^*}\left(\int_K |v|^{2^*}\right)^{(q-1)/2^*}\left(\int_K |uv|^r\right)^{1/r} \\
&\leqslant c_1\big\|(u,v)\big\|_0^{p+q-2}\left(\int_K |uv|^r\right)^{1/r}.
\end{aligned} \tag{2.2}$$

Since $1 < r < 2^*/2$ there exists $t \in (0,1)$ such that $r = 2^*t/2 + (1-t)$. By using Hölder's inequality with exponents $1/t$, $1/(1-t)$, and the imbedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ again, we obtain

$$\begin{aligned}
\int_K |uv|^r &= \int_K |uv|^{2^*t/2}|uv|^{(1-t)} \\
&\leqslant \left(\int_K |uv|^{2^*/2}\right)^t\left(\int_K |uv|\right)^{1-t} \\
&\leqslant \left(\frac{1}{2}\int_K \left(|u|^{2^*} + |v|^{2^*}\right)\right)^t\left(\int_K |uv|\right)^{1-t} \\
&\leqslant c_2\big\|(u,v)\big\|_0^{2^*t}\left(\int_K |uv|\right)^{1-t}.
\end{aligned} \tag{2.3}$$

Combining the last inequality and (2.2), we conclude the proof of the lemma. □

Lemma 2.2. *There exists a constant $\hat{c} > 0$ such that*

$$\int |u|^p |v|^q \leqslant \hat{c} \| (u,v) \|_1^{p+q}, \quad \text{for all } (u,v) \in X.$$

Proof. By Lemma 2.1 we have that

$$\int |u|^p |v|^q \leqslant c \| (u,v) \|_0^{p+q-2+2^* t/r} \left(\int |uv| \right)^{(1-t)/r}. \tag{2.4}$$

We recall that the set F given in (H_2) has finite measure and $a(x)b(x) > M_0$ in F^c. Applying Hölder's inequality with exponents 2^*, 2^*, $N/2$ and using the imbedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, we obtain

$$\begin{aligned} \int |uv| &= \int_F |uv| + \int_{F^c} |uv| \\ &\leqslant \|u\|_{L^{2^*}(F)} \|v\|_{L^{2^*}(F)} \mathcal{L}(F)^{2/N} + \frac{1}{\sqrt{M_0}} \int_{F^c} \sqrt{a(x)} |u| \sqrt{b(x)} |v| \\ &\leqslant c_1 \| (u,v) \|_0^2 + \frac{1}{\sqrt{M_0}} \left(\int_{F^c} a(x) u^2 \right)^{1/2} \left(\int_{F^c} b(x) v^2 \right)^{1/2} \\ &\leqslant c_2 \| (u,v) \|_1^2. \end{aligned}$$

The last inequality and (2.4) provide $\hat{c} > 0$ such that

$$\begin{aligned} \int u_+^p v_+^q &\leqslant \hat{c} \| (u,v) \|_1^{p+q-2+2^* t/r} \| (u,v) \|_1^{(1-t)\frac{2}{r}} \\ &= \hat{c} \| (u,v) \|_1^{p+q-2+\frac{2}{r}(2^* t/2 + (1-t))} \\ &= \hat{c} \| (u,v) \|_1^{p+q}, \end{aligned}$$

where we have used that $r = 2^* t/2 + (1-t)$. The lemma is proved. □

Since we are interested in positive solutions of (S_λ) we will work with a functional slightly different from that defined in the introduction. More specifically, we consider $I_\lambda : X \to \mathbb{R}$ given by

$$I_\lambda(u,v) := \frac{1}{2} \| (u,v) \|_\lambda^2 - \frac{1}{p+q} \int u_+^p v_+^q, \quad (u,v) \in X.$$

In view of the above lemma, it is well defined. Moreover, we may use the above results and hypothesis (H_2) to show that $I_\lambda \in C^1(X, \mathbb{R})$ for any $\lambda > 0$.

Let E be a Banach space and $I \in C^1(E, \mathbb{R})$. We say that $(z_n) \subset E$ is a Palais–Smale sequence at level c ($(\mathrm{PS})_c$ sequence for short) if $I(z_n) \to c$ and $I'(z_n) \to 0$. We say that I satisfies $(\mathrm{PS})_c$ if any $(\mathrm{PS})_c$ sequence possesses a convergent subsequence.

Lemma 2.3. *Let $\lambda \geqslant 1$ and $(z_n) \subset X$ be a $(\mathrm{PS})_c$ sequence for I_λ.*

(i) *(z_n) is bounded in X;*
(ii) $\lim_{n\to\infty} \|z_n\|_\lambda^2 = \lim_{n\to\infty} \int (u_n)_+^p (v_n)_+^q = c(\frac{1}{2} - \frac{1}{p+q})^{-1}$;
(iii) *if $c \neq 0$, then $c \geqslant \gamma_0 > 0$, for some γ_0 independent of λ.*

Proof. We have that

$$\left(\frac{1}{2}-\frac{1}{p+q}\right)\|z_n\|_\lambda^2 = I_\lambda(z_n) - \frac{1}{p+q}I'_\lambda(z_n)\cdot z_n = c + o(1)\|z_n\|_\lambda, \tag{2.5}$$

as $n \to \infty$, and therefore (i) holds. Moreover, as $n \to \infty$, we have that

$$\left(\frac{1}{2}-\frac{1}{p+q}\right)\|z_n\|_\lambda^2 = c + o(1)\|z_n\|_\lambda = I_\lambda(z_n) - \frac{1}{2}I'_\lambda(z_n)\cdot z_n$$
$$= \left(\frac{1}{2}-\frac{1}{p+q}\right)\int (u_n)_+^p (v_n)_+^q,$$

from which follows (ii). We now observe that, in view of Lemma 2.2 and $\lambda \geqslant 1$,

$$I'_\lambda(z)\cdot z = \|z\|_\lambda^2 - \int u_+^p v_+^q \geqslant \|z\|_\lambda^2 - \hat{c}\|z\|_\lambda^{p+q} \geqslant \frac{1}{2}\|z\|_\lambda^2,$$

whenever $\|z\|_\lambda \leqslant (2\hat{c})^{-1/(p+q-2)} := \sqrt{\delta}$. Suppose now that

$$c < \delta\left(\frac{1}{2}-\frac{1}{p+q}\right).$$

By (ii), there exists $n_0 \in \mathbb{N}$ such that $\|z_n\|_\lambda < \sqrt{\delta}$ for any $n \geqslant n_0$. Thus,

$$\frac{1}{2}\|z_n\|_\lambda^2 \leqslant I'_\lambda(z_n)\cdot z_n \leqslant o(1)\|z_n\|_\lambda \quad \text{as } n \to \infty,$$

and we conclude that $z_n \to 0$ in X. Hence, $I_\lambda(z_n) \to 0 = c$ and it follows that (iii) holds for $\gamma_0 := \delta(\frac{1}{2} - \frac{1}{p+q})$. $\square$

Lemma 2.4. *Given $\varepsilon > 0$ and $C_0 > 0$, there exist $\Lambda_\varepsilon = \Lambda(\varepsilon, C_0) > 0$ and $R_\varepsilon = R(\varepsilon, C_0) > 0$ such that, if $((u_n, v_n)) \subset X$ is a $(\mathrm{PS})_c$ sequence for I_λ with $c \leqslant C_0$ and $\lambda \geqslant \Lambda_\varepsilon$, then*

$$\limsup_{n\to\infty} \int_{B^c_{R_\varepsilon}} (u_n)_+^p (v_n)_+^q \leqslant \varepsilon.$$

Proof. Since $\|\cdot\|_0 \leqslant \|\cdot\|_\lambda$, we may use Lemma 2.1 and Lemma 2.3(i) to obtain

$$\int_{B_R^c} (u_n)_+^p (v_n)_+^q \leqslant c_1\left(\int_{B_R^c} |u_n v_n|\right)^\beta, \tag{2.6}$$

for any $R > 0$. By Young and Hölder's inequality, the imbedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and Lemma 2.3(i), we get

$$\int_{B_R^c\cap F} |u_n v_n| \leqslant \frac{1}{2}\int_{B_R^c\cap F} \left(|u_n|^2 + |v_n|^2\right)$$
$$\leqslant \frac{\mathcal{L}(B_R^c\cap F)^{2/N}}{2}\left(\|u_n\|_{L^{2^*}}^2 + \|v_n\|_{L^{2^*}}^2\right)$$
$$\leqslant c_2\mathcal{L}\left(B_R^c\cap F\right)^{2/N}. \tag{2.7}$$

On the other hand, since $((u_n, v_n))$ is bounded and $a(x)b(x) > M_0$ in $B_R^c \cap F^c$, we have

$$\int\limits_{B_R^c \cap F^c} |u_n v_n| \leqslant \frac{1}{\lambda M_0} \int\limits_{B_R^c \cap F^c} \sqrt{\lambda a(x)}|u_n|\sqrt{\lambda b(x)}|v_n|$$
$$\leqslant \frac{1}{2\lambda M_0} \int \big(\lambda a(x)u_n^2 + \lambda b(x)v_n^2\big) \leqslant c_3/\lambda.$$

It follows from the above estimate, (2.7) and (2.6) that

$$\int\limits_{B_R^c} (u_n)_+^p (v_n)_+^q \leqslant c_2\big(c_1\mathcal{L}\big(B_R^c \cap F\big)^{2/N} + c_3/\lambda\big)^{\beta}.$$

Since F has finite Lebesgue measure, we have that $\mathcal{L}(B_R^c \cap F) \to 0$ as $R \to \infty$. Hence, for R and λ sufficiently large, the right-hand side of the above expression is small. This concludes the proof. □

In the next lemma we verify that I_λ satisfies the Mountain Pass geometry.

Lemma 2.5. *There exist $\alpha, \rho > 0$ and $z_0 \in X$, all of them independent of $\lambda \geqslant 1$, such that*

(i) $I_\lambda(z) \geqslant \alpha$ *for all* $\|z\|_\lambda = \rho$,
(ii) $I_\lambda(z_0) \leqslant I_\lambda(0) = 0$ *and* $\|z_0\| > \rho$.

Proof. By Lemma 2.2, we have that

$$I_\lambda(z) = \frac{1}{2}\|z\|_\lambda^2 - \frac{1}{p+q}\int u_+^p v_+^q \geqslant \frac{1}{2}\|z\|_\lambda^2 - \frac{\hat{c}}{p+q}\|z\|_\lambda^{p+q} \geqslant \frac{1}{4}\rho^2,$$

whenever $\|z\|_\lambda = \rho := ((p+q)/4\hat{c})^{1/(p+q-2)}$. Furthermore, if $\varphi \in C_0^\infty(\Omega_a \cap \Omega_b) \setminus \{0\}$, $\varphi_+ \not\equiv 0$, we have that $a(x)\varphi \equiv b(x)\varphi \equiv 0$ on $\mathbb{R}^N$. Hence,

$$\lim_{t\to\infty} I_\lambda\big(t(\varphi,\varphi)\big) = \lim_{t\to\infty}\left(t^2\int |\nabla\varphi|^2 - \frac{t^{p+q}}{p+q}\int \varphi_+^{p+q}\right) = -\infty,$$

uniformly on λ. It suffices to set $z_0 := t_0(\varphi,\varphi)$ with $t_0 > 0$ sufficiently large. □

Remark 2.6. Let z_0 be given by the above lemma. For each $\lambda > 0$ we may define the Mountain Pass level of I_λ as

$$c_\lambda := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda\big(\gamma(t)\big),$$

where

$$\Gamma := \big\{\gamma \in C\big([0,1], X\big) : \gamma(0) = 0,\ \gamma(1) = z_0\big\}.$$

For future reference we observe that

$$0 < \alpha \leqslant c_\lambda \leqslant \beta_0 := \max_{t\in[0,1]} I_\lambda(tz_0). \tag{2.8}$$

3. Least energy solutions

We devote this section to the proof of Theorem 1.1. Let $\varepsilon > 0$ to be chosen later, $C_0 := \beta_0$ given in (2.8), and consider Λ_ε, R_ε provided by Lemma 2.4. In view of Remark 2.6 we obtain, for any fixed $\lambda \geqslant \Lambda_\varepsilon$, a sequence $(z_k) \subset X$ such that

$$I_\lambda(z_k) \to c_\lambda \geqslant \alpha \quad \text{and} \quad I'_\lambda(z_k) \to 0.$$

By Lemma 2.3(i) (z_k) is bounded in X and therefore, up to a subsequence, we have that $z_k \rightharpoonup z_\lambda := (u_\lambda, v_\lambda)$ weakly in X.

We shall prove that $I'_\lambda(z_\lambda) = 0$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ and denote by K the support of φ. Since $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is compactly embedded in $L^{p+q-1}_{\mathrm{loc}}(\mathbb{R}^N)$, up to a subsequence, we have that

$$(u_k, v_k) \to (u_\lambda, v_\lambda) \quad \text{strongly in } L^{p+q-1}(K) \times L^{p+q-1}(K),$$
$$\big(u_k(x), v_k(x)\big) \to \big(u_\lambda(x), v_\lambda(x)\big) \quad \text{a.e. in } K,$$
$$\big|u_k(x)\big|, \big|v_k(x)\big| \leqslant h_K(x) \in L^{p+q-1}(K) \quad \text{a.e. in } K.$$

Hence, almost everywhere in K,

$$(u_k)_+^{p-1}(v_k)_+^q|\varphi| \leqslant |u_k|^{p-1}|v_k|^q|\varphi| \leqslant h_K^{p+q-1}|\varphi| \in L^1(K).$$

It follows from the above convergences and the Lebesgue Dominated Convergence Theorem that

$$\lim_{k\to\infty} \int (u_k)_+^{p-1}(v_k)_+^q\varphi = \int (u_\lambda)_+^{p-1}(v_\lambda)_+^q\varphi, \quad \forall \varphi \in C_0^\infty\big(\mathbb{R}^N\big). \tag{3.1}$$

Analogously, we obtain

$$\lim_{k\to\infty} \int (u_k)_+^{p}(v_k)_+^{q-1}\psi = \int (u_\lambda)_+^{p}(v_\lambda)_+^{q-1}\psi, \quad \forall \psi \in C_0^\infty\big(\mathbb{R}^N\big).$$

The two above limits and the weak convergence of (z_k) imply that, for each $(\varphi, \psi) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$, there hold

$$0 = \lim_{k\to\infty} I'_\lambda(z_k)\cdot(\varphi, \psi) = I'_\lambda(z_\lambda)\cdot(\varphi, \psi)$$

and therefore z_λ is a critical point of I_λ.

Suppose that $z_\lambda \equiv 0$. Since $u_k, v_k \to 0$ in $L^2(B_{R_\varepsilon})$ we may use Lemma 2.1, the boundedness of (z_k) in X and Young's inequality, to obtain

$$\int\limits_{B_{R_\varepsilon}} (u_k)_+^p(v_k)_+^q \leqslant c_1\Bigg(\int\limits_{B_{R_\varepsilon}} |u_k v_k|\Bigg)^\beta \leqslant c_2\Bigg(\int\limits_{B_{R_\varepsilon}} |u_k|^2 + |v_k|^2\Bigg)^\beta \to 0, \tag{3.2}$$

as $k \to \infty$. So, it follows from Lemma 2.3(ii) and Lemma 2.4 that, for $\lambda \geqslant \Lambda_\varepsilon$,

$$c_\lambda\left(\frac{1}{2}-\frac{1}{p+q}\right)^{-1}=\lim_{k\to\infty}\int (u_k)_+^p(v_k)_+^q$$
$$=\lim_{k\to\infty}\left(\int\limits_{B_{R_\varepsilon}}(u_k)_+^p(v_k)_+^q+\int\limits_{B_{R_\varepsilon}^c}(u_k)_+^p(v_k)_+^q\right)\leqslant\varepsilon.$$

If we choose $\varepsilon>0$ sufficiently small, we conclude that $c_\lambda=0$, contradicting (2.8). This shows that $z_\lambda\not\equiv 0$.

Applying Fatou's Lemma we get

$$c_\lambda=\lim_{k\to\infty}\left(I_\lambda(z_k)-\frac{1}{2}I_\lambda'(z_k)\cdot z_k\right)=\lim_{k\to\infty}\left(\frac{1}{2}-\frac{1}{p+q}\right)\int (u_k)_+^p(v_k)_+^q$$
$$\geqslant\left(\frac{1}{2}-\frac{1}{p+q}\right)\int u_\lambda^p v_\lambda^q=I_\lambda(z_\lambda)\geqslant c_\lambda,$$

from which follows that $I_\lambda(z_\lambda)=c_\lambda$. Hence, z_λ is a least energy solution.

Since $I'(z_\lambda)\cdot((u_\lambda)_-,(v_\lambda)_-)=\|((u_\lambda)_-,(v_\lambda)_-)\|_\lambda^2=0$, we have that $u_\lambda, v_\lambda\geqslant 0$ in $\mathbb{R}^N$. Furthermore, by applying the Strong Maximum Principle in each equation of (S_λ) we conclude that u_λ, $v_\lambda>0$ in $\mathbb{R}^N$. This proves the first part of Theorem 1.1.

We now consider the concentration behavior of the solutions. Suppose that $(\lambda_n)\subset\mathbb{R}$ is such that $\lambda_n\to\infty$ and let $z_{\lambda_n}=(u_{\lambda_n},v_{\lambda_n})$ be the associated solution of (S_{λ_n}) such that $I_{\lambda_n}(z_{\lambda_n})=c_{\lambda_n}$. In what follows we write only z_n, u_n and v_n to denote z_{λ_n}, u_{λ_n} and v_{λ_n} respectively.

First note that, in view of (2.8),

$$\left(\frac{1}{2}-\frac{1}{p+q}\right)\|z_n\|_{\lambda_n}^2=I_{\lambda_n}(z_n)=c_{\lambda_n}\leqslant\beta_0. \tag{3.3}$$

Thus, up to a subsequence, we have that $z_n\rightharpoonup\bar z:=(\bar u,\bar v)$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)\times\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $z_n(x)\to\bar z(x)$ almost everywhere in $\mathbb{R}^N$. Given $\varphi\in C_0^\infty(\Omega_a)$, recalling that $a\equiv 0$ in Ω_a and using $(\varphi,0)$ as a test function we get

$$\int\nabla u_n\nabla\varphi=\frac{p}{p+q}\int (u_n)_+^{p-1}(v_n)_+^q\varphi.$$

Since φ has compact support, we may take the limit in the above expression and argue as in the proof of (3.1) to get

$$\int\limits_{\Omega_a\cup\Omega_b}\nabla\bar u\nabla\varphi=\frac{p}{p+q}\int\limits_{\Omega_a\cup\Omega_b}\bar u_+^{p-1}\bar v_+^q\varphi,\quad\forall\varphi\in C_0^\infty(\Omega_a). \tag{3.4}$$

Analogously, we have

$$\int\limits_{\Omega_a\cup\Omega_b}\nabla\bar v\nabla\psi=\frac{q}{p+q}\int\limits_{\Omega_a\cup\Omega_b}\bar u_+^{p}\bar v_+^{q-1}\psi,\quad\forall\psi\in C_0^\infty(\Omega_b). \tag{3.5}$$

We claim that $\bar u\equiv 0$ in Ω_a^c. In order to see this we take $j\in\mathbb{N}$, set

$$C_j:=\left\{x\in B_j(0)\colon a(x)>\frac{1}{j}\right\}$$

and notice that, by (3.3),

$$0 \leqslant \int_{C_j} u_n^2 \leqslant \frac{j}{\lambda_n} \int_{C_j} \lambda_n a(x) u_n^2 \leqslant \frac{j}{\lambda_n} \|z_n\|_{\lambda_n}^2 \to 0, \quad \text{as } n \to \infty.$$

Since C_j is bounded and $u_n \to \bar{u}$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, we conclude that $\int_{C_j} \bar{u}^2 = 0$ for all $j \in \mathbb{N}$. Thus $\bar{u} \equiv 0$ almost everywhere in $\Omega_a^c = \bigcup_{j=1}^n C_j$. Recalling that Ω_a has smooth boundary we conclude that $\bar{u} \in H_0^1(\Omega_a)$. Analogously, $\bar{v} \in H_0^1(\Omega_b)$. Thus, $(\bar{u}, \bar{v})$ is a solution of the limit problem (L).

In order to verify that $\bar{z} \neq 0$ we define

$$m := \inf_{z \in \mathcal{N}} J(z),$$

where $J : H_0^1(\Omega_a) \times H_0^1(\Omega_b) \to \mathbb{R}$ is given by

$$J(u, v) := \frac{1}{2} \int_{\Omega_a \cup \Omega_b} \left(|\nabla u|^2 + |\nabla v|^2\right) - \frac{1}{p+q} \int_{\Omega_a \cup \Omega_b} u_+^p v_+^q$$

and $\mathcal{N}$ is the Nehari manifold of J, namely:

$$\mathcal{N} := \left\{(u, v) \in H_0^1(\Omega_a) \times H_0^1(\Omega_b)\colon (u, v) \neq (0, 0),\ J'(u, v) \cdot (u, v) = 0\right\}.$$

Since $H_0^1(\Omega_a) \times H_0^1(\Omega_b)$ can be viewed as a subspace of X, we have that $c_\lambda \leqslant m$, for all λ. On the other hand

$$m \geqslant c_{\lambda_n} = I_{\lambda_n}(z_n) - \frac{1}{2} I'_{\lambda_n}(z_n) \cdot z_n = \left(\frac{1}{2} - \frac{1}{p+q}\right) \int (u_n)_+^p (v_n)_+^q.$$

Taking $n \to \infty$, using Fatou's Lemma and $J'(\bar{u}, \bar{v}) = 0$ we obtain

$$\begin{aligned} m &\geqslant \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{p+q}\right) \int (u_n)_+^p (v_n)_+^q \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p+q}\right) \int_{\Omega_a \cup \Omega_b} \bar{u}_+^p \bar{v}_+^q = J(\bar{u}, \bar{v}) \geqslant m. \end{aligned} \tag{3.6}$$

Hence $J(\bar{u}, \bar{v}) = m$ and therefore $\bar{z} \neq 0$ is a least energy solution of (L). By using (3.4) and (3.5) we obtain $\|(\bar{u}_-, \bar{v}_-)\|_0 = 0$. Thus, $\bar{u}, \bar{v} \geqslant 0$ and it follows from the Strong Maximum Principle and (3.4)–(3.5) that $\bar{u} > 0$ in Ω_a and $\bar{v} > 0$ in Ω_b.

In order to finish the proof we use the weak convergence of (z_n), the fact that z_n is a solution of (S_{λ_n}), (3.6) and $(\bar{u}, \bar{v}) \in \mathcal{N}$ to get

$$\begin{aligned} \|z_n - z\|_{\lambda_n}^2 &= \int \left(|\nabla u_n|^2 + |\nabla v_n|^2 + \lambda_n a(x) u_n^2 + \lambda_n b(x) v_n^2\right) - \int \left(|\nabla \bar{u}|^2 + |\nabla \bar{v}|^2\right) + o(1) \\ &= \int (u_n)_+^p (v_n)_+^q - \int \left(|\nabla \bar{u}|^2 + |\nabla \bar{v}|^2\right) + o(1) \\ &= \int \bar{u}_+^p \bar{v}_+^q - \int \left(|\nabla \bar{u}|^2 + |\nabla \bar{v}|^2\right) + o(1) = o(1), \end{aligned}$$

as $n \to \infty$. Since $\| \cdot \|_0 \leqslant \| \cdot \|_{\lambda_n}$ it follows that $z_n \to \bar{z}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$. This concludes the proof of Theorem 1.1. □

4. Multiplicity of bound state solutions

In this section we present the proofs of Theorems 1.2 and 1.3. Since we are not interested in the sign of the solutions, we redefine the functional I_λ by setting

$$I_\lambda(u,v) := \frac{1}{2}\|(u,v)\|_\lambda^2 - \frac{1}{p+q}\int |u|^p|v|^q, \quad (u,v) \in X.$$

As in Section 2, the functional is of class C^1 and its critical points are the weak solutions of (S_λ). For future reference we notice that, arguing as in the proof of Lemma 2.1 and Lemma 2.2, we obtain $c, \hat{c} > 0$ such that

$$\int_{B_R^c} |u|^{p-1}|v|^{q-1}|\phi\eta| \leqslant c\|u\|_{L^{2^*}(B_R^c)}^{p-1}\|v\|_{L^{2^*}(B_R^c)}^{q-1}\|(\phi,\eta)\|_0^{2^*t/r}\left(\int_{B_R^c}|\phi\eta|\right)^\beta \tag{4.1}$$

and

$$\int_{B_R^c}|\phi\eta| \leqslant \hat{c}\|(\phi,\eta)\|_0^2\mathcal{L}(B_R^c \cap F)^{2/N} + \frac{1}{\sqrt{M_0}}\left(\int_{B_R^c\cap F^c} a(x)\phi^2\right)^{1/2}\left(\int_{B_R^c\cap F^c} b(x)\eta^2\right)^{1/2}, \tag{4.2}$$

for any $R > 0$ and $(u,v), (\phi,\eta) \in X$. Here $r > 1$, $t \in (0,1)$ and $\beta > 0$ are given by Lemma 2.1.

In order to obtain multiple critical points for I_λ we shall use the following version of the Symmetric Mountain Pass Theorem [2] (see also [18, Theorem 2.1]).

Theorem 4.1. *Let E be a real Banach space and $W \subset E$ a finite dimensional subspace. Suppose that $I \in C^1(E,\mathbb{R})$ is an even functional satisfying $I(0) = 0$ and*

(i) *there exists a constant $\rho > 0$ such that $I|_{\partial B_\rho(0)} \geqslant 0$;*
(ii) *there exists $M > 0$ such that $\sup_{z\in W} I(z) < M$.*

If I satisfies $(\mathrm{PS})_c$ for any $0 < c < M$, then I possesses at least $\dim W$ pairs of nontrivial critical points.

Our first goal is to prove a local compactness condition for I_λ. We start with the following version of Brezis–Lieb Lemma [6] (see also [10]).

Lemma 4.2. *Let $((u_n, v_n)) \subset X$ be such that $(u_n, v_n) \rightharpoonup (u,v)$ weakly in X. Then*

$$\lim_{n\to\infty}\int\left(|u_n|^p|v_n|^q - |u_n - u|^p|v_n - v|^q\right) = \int |u|^p|v|^q.$$

Proof. Let A_n be the integral on the left-hand side of the above expression and notice that

$$\begin{aligned} A_n &= -\int\int_0^1 \frac{d}{dt}\left(|u_n - tu|^p|v_n|^q + |u_n - u|^p|v_n - tv|^q\right)dt\,dx \\ &= p\int\int_0^1 f_n(t,x)u\,dt\,dx + q\int\int_0^1 g_n(t,x)v\,dt\,dx, \end{aligned} \tag{4.3}$$

with

$$f_n(t,x) := |u_n - tu|^{p-2}(u_n - tu)|v_n|^q$$

and

$$g_n(t,x) := |u_n - u|^p |v_n - tv|^{q-2}(v_n - tv).$$

Since $((u_n, v_n))$ is bounded in X and $p + q < 2^*$, taking a subsequence if necessary, we may suppose that

$$\begin{gathered}
(u_n, v_n) \to (u, v) \quad \text{strongly in } L^{p+q}_{\text{loc}}(\mathbb{R}^N) \times L^{p+q}_{\text{loc}}(\mathbb{R}^N), \\
\big(u_n(x), v_n(x)\big) \to \big(u(x), v(x)\big) \quad \text{a.e. in } \mathbb{R}^N, \\
\big|u(x)\big|, \big|v(x)\big|, \big|u_n(x)\big|, \big|v_n(x)\big| \leqslant h_R(x) \in L^{p+q}(B_R) \quad \text{a.e. in } B_R,
\end{gathered} \tag{4.4}$$

for any $R > 0$.

The pointwise convergence implies that, for almost every $(t,x) \in (0,1) \times \mathbb{R}^N$,

$$f_n(t,x) \to f(t,x) := (1-t)^{p-1}|u|^{p-2}u|v|^q, \quad g_n(t,x) \to g(t,x) \equiv 0. \tag{4.5}$$

We claim that

$$\lim_{n\to\infty} \int \int_0^1 f_n(t,x)u\,dt\,dx = \int \int_0^1 f(t,x)u\,dt\,dx \tag{4.6}$$

and

$$\lim_{n\to\infty} \int \int_0^1 g_n(t,x)v\,dt\,dx = \int \int_0^1 g(t,x)v\,dt\,dx = 0. \tag{4.7}$$

Assuming the claim, noticing that for any measurable set $K \subset \mathbb{R}^N$, we have $\int_K \int_0^1 fu\,dt\,dx = \frac{1}{p}\int_K |u|^p|v|^q\,dx$, and taking the limit in (4.3), we obtain

$$\lim_{n\to\infty} A_n = p \int \int_0^1 f(t,x)u\,dt\,dx = \int |u|^p|v|^q\,dx.$$

So, in order to prove the lemma, it suffices to verify (4.6) and (4.7).

In view of (H_2), for any given $0 < \varepsilon < 1$ we may choose $R = R(\varepsilon) > 0$ such that

$$\max\left\{\frac{1}{p}\int_{B_R^c} |u|^p|v|^q, \mathcal{L}\big(B_R^c \cap F\big)^{2\beta/N}, \left(\int_{B_R^c} au^2\right)^{\beta/2}\right\} < \varepsilon, \tag{4.8}$$

where $\beta > 0$ comes from Lemma 2.1. So, we have that

$$\left|\int\int_0^1 (f_n - f)u\,dt\,dx\right| \leqslant \left|\int_{B_R}\int_0^1 (f_n - f)u\,dt\,dx\right| + \left|\int_{B_R^c}\int_0^1 f_n u\,dt\,dx\right| + \varepsilon. \tag{4.9}$$

In view of (4.4) we have that

$$\left|(f_n - f)u\right| \leqslant \left(|u_n - tu|^{p-1}|v_n|^q + |u|^{p-1}|v|^q\right)|u| \leqslant c_1 h_R(x)^{p+q} \in L^1(B_R),$$

for almost every $x \in B_R$. Hence, we can use (4.5) and the Lebesgue Dominated Convergence Theorem to get

$$\lim_{n\to\infty}\left|\int_{B_R}\int_0^1 (f_n - f)u\,dt\,dx\right| = 0. \tag{4.10}$$

On the other hand,

$$\begin{aligned}\left|\int_{B_R^c}\int_0^1 f_n u\,dt\,dx\right| &\leqslant \int_{B_R^c}\int_0^1 |u_n - tu|^{p-1}|v_n|^q|u|\,dt\,dx \\ &\leqslant c_2\left(\int_{B_R^c}\left(|u_n|^{p-1}|v_n|^q|u| + |u|^p|v_n|^q\right)\right).\end{aligned} \tag{4.11}$$

Since $((u_n, v_n))$ is bounded in X, we may use (4.1), (4.2) and (4.8) to conclude that

$$\int_{B_R^c} |u_n|^{p-1}|v_n|^{q-1}|uv_n| \leqslant c_3\left(\int_{B_R^c}|uv_n|\right)^{\beta} \leqslant c_4\varepsilon$$

and

$$\int_{B_R^c} |u|^p|v_n|^q = \int_{B_R^c} |u|^{p-1}|v_n|^{q-1}|uv_n| \leqslant c_5\varepsilon.$$

By replacing these expressions in (4.11) we get

$$\left|\int_{B_R^c}\int_0^1 f_n u\,dt\,dx\right| \leqslant c_6\varepsilon.$$

The above estimate, (4.10) and (4.9) imply that

$$\limsup_{n\to\infty}\left|\int\int_0^1 (f_n - f)u\,dt\,dx\right| \leqslant c_7\varepsilon.$$

Since $0 < \varepsilon < 1$ is arbitrary, we conclude that (4.6) holds. The proof of (4.7) is analogous and it will be omitted. The lemma is proved. □

Lemma 4.3. *Let $(z_n) = ((u_n, v_n)) \subset X$ be a $(\mathrm{PS})_c$ sequence for I_λ. Then, up to a subsequence, $z_n \rightharpoonup z := (u, v)$ weakly in X, where z is a critical point of I_λ. Furthermore, $(\tilde{z}_n) := (z_n - z)$ is a $(\mathrm{PS})_{c'}$ sequence for I_λ, with $c' = c - I_\lambda(z)$.*

Proof. Since (z_n) is bounded in X, up to a subsequence, $z_n \rightharpoonup z := (u, v)$ weakly in X. Arguing as in the proof of Theorem 1.1 we may show that $I'_\lambda(z) = 0$. The weak convergence of (z_n) and Lemma 4.2 imply that

$$\begin{aligned} I_\lambda(z_n - z) &= \frac{1}{2}\|z_n\|_\lambda^2 - \frac{1}{2}\|z\|_\lambda^2 - \frac{1}{p+q}\int |u_n|^p|v_n|^q + \frac{1}{p+q}\int |u|^p|v|^q + o(1) \\ &= I_\lambda(z_n) - I_\lambda(z) + o(1) = c - I(z) + o(1), \end{aligned}$$

as $n \to \infty$.

It remains to show that $I'(z_n - z) \to 0$. We first notice that, for any given $(\varphi, \psi) \in X$ such that $\|(\varphi, \psi)\|_\lambda \leqslant 1$,

$$I'_\lambda(z_n - z)\cdot(\varphi, \psi) = I'_\lambda(z_n)\cdot(\varphi, \psi) - I'_\lambda(z)\cdot(\varphi, \psi) - \frac{p}{p+q}\int f_n\varphi - \frac{q}{p+q}\int g_n\psi,$$

where

$$f_n(x) := |u_n - u|^{p-2}(u_n - u)|v_n - v|^q - |u_n|^{p-2}u_n|v_n|^q + |u|^{p-2}u|v|^q$$

and

$$g_n(x) := |u_n - u|^p|v_n - v|^{q-2}(v_n - v) - |u_n|^p|v_n|^{q-2}v_n + |u|^p v|v|^{q-2}.$$

Since $I'(z_n) \to 0$ and $I'(z) = 0$, it suffices to show that

$$\lim_{n\to\infty} \sup_{\|\varphi\|_{X_a}\leqslant 1} \int |f_n||\varphi| = 0 = \lim_{n\to\infty} \sup_{\|\psi\|_{X_b}\leqslant 1} \int |g_n||\psi|, \tag{4.12}$$

where we are denoting

$$\|\varphi\|_{X_a}^2 := \int \big(|\nabla\varphi|^2 + \lambda a(x)\varphi^2\big), \qquad \|\psi\|_{X_b}^2 := \int \big(|\nabla\psi|^2 + \lambda b(x)\psi^2\big).$$

Given $0 < \varepsilon < 1$, we may choose $R = R(\varepsilon) > 0$ such that

$$\max\left\{\|u\|_{L^{2^*}(B_R^c)}, \mathcal{L}\big(B_R^c \cap F\big)^{2\beta/N}, \bigg(\int_{B_R^c} bv^2\bigg)^{\beta/2}\right\} < \varepsilon, \tag{4.13}$$

with $\beta > 0$ given by Lemma 2.1. Using Hölder's inequality, the imbedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and $\|\varphi\|_{X_a} \leqslant 1$, we get

$$\begin{aligned} \int_{B_R} |f_n||\varphi| &\leqslant \bigg(\int_{B_R} |f_n|^{2N/(N+2)}\bigg)^{(N+2)/2N} \bigg(\int_{B_R} |\varphi|^{2^*}\bigg)^{1/2^*} \\ &\leqslant c_1\bigg(\int_{B_R} |f_n|^{2N/(N+2)}\bigg)^{(N+2)/2N}. \end{aligned} \tag{4.14}$$

Setting $\theta := (p+q-1)2N/(N+2) < 2^*$, we may suppose that

$$(u_n, v_n) \to (u, v) \quad \text{strongly in } L^\theta(B_R) \times L^\theta(B_R),$$
$$\big(u_n(x), v_n(x)\big) \to \big(u(x), v(x)\big) \quad \text{a.e. in } B_R,$$
$$\big|u(x)\big|, \big|v(x)\big|, \big|u_n(x)\big|, \big|v_n(x)\big| \leqslant h_R(x) \in L^\theta(B_R) \quad \text{a.e. in } B_R(0).$$

Hence,

$$|f_n| \leqslant |u_n - u|^{p-1}|v_n - v|^q + |u_n|^{p-1}|v_n|^q + |u|^{p-1}|v|^q \leqslant c_2 h_R^{p+q-1},$$

and therefore $|f_n(x)|^{2N/(N+2)} \leqslant c_2 h_R^\theta(x) \in L^1(B_R)$, almost everywhere in B_R. Since $f_n(x) \to 0$ almost everywhere in B_R, the Lebesgue Dominated Convergence Theorem and (4.14) imply that

$$\lim_{n\to\infty} \int_{B_R} |f_n||\varphi| = 0 \quad \text{uniformly for } \|\varphi\|_{X_a} \leqslant 1. \tag{4.15}$$

On the other hand, by adding and subtracting the term $|u_n - u|^{p-2}(u_n - u)|v_n|^q$ to f_n, we have

$$\int_{B_R^c} |f_n||\varphi| \leqslant \int_{B_R^c} s_n|\varphi| + \int_{B_R^c} t_n|\varphi| + \int_{B_R^c} |u|^{p-1}|v|^{q-1}|\varphi v| \tag{4.16}$$

with

$$s_n := |u_n - u|^{p-1}\big||v_n - v|^q - |v_n|^q\big|$$

and

$$t_n := \big||u_n - u|^{p-2}(u_n - u) - |u_n|^{p-2}u_n\big||v_n|^q.$$

Using (4.1), (4.2) and (4.13), we may estimate the last term in (4.16) as follows

$$\int_{B_R^c} |u|^{p-1}|v|^{q-1}|\varphi v| \leqslant c_3\varepsilon^{p-1} \quad \text{for any } \|\varphi\|_{X_a} \leqslant 1. \tag{4.17}$$

Now, we proceed with the estimate of $\int_{B_R^c} s_n|\varphi|$. Setting $w(t) := |v_n - tv|^q$, recalling that $q > 1$ and using the Mean Value Theorem we obtain

$$\big|w(1) - w(0)\big| = \big||v_n - v|^q - |v_n|^q\big| \leqslant q|v_n - t_0 v|^{q-1}|v|$$

for some $t_0 \in [0, 1]$. The boundedness of $((u_n, v_n))$ in X, $t_0 \in [0, 1]$, (4.1), (4.2) and (4.13) imply that

$$\int_{B_R^c} s_n|\varphi| \leqslant q\int_{B_R^c} |u_n - u|^{p-1}|v_n - t_0 v|^{q-1}|\varphi v| \leqslant c_4\varepsilon \quad \text{for any } \|\varphi\|_{X_a} \leqslant 1. \tag{4.18}$$

The estimates for $\int_{B_R^c} t_n|\varphi|$ are more involved since we may have $p - 1 < 1$. We consider two possible cases:

Case 1. $p \geqslant 2$.

Suppose first $p > 2$ and define $w(t) := |u_n - tu|^{p-2}(u_n - tu)$. Applying the Mean Value Theorem and proceeding as in (4.17) we obtain

$$\begin{aligned}\int_{B_R^c} t_n|\varphi| &\leqslant (p-1)\int_{B_R^c} |u_n - t_0 u|^{p-2}|u||v_n|^{q-1}|\varphi v_n| \\ &\leqslant c_5\left(\int_{B_R^c} |u|^{p-1}|v_n|^{q-1}|\varphi v_n| + \int_{B_R^c} |u_n|^{p-2}|u||v_n|^{q-1}|\varphi v_n|\right) \\ &\leqslant c_6\left(\varepsilon^{p-1} + \int_{B_R^c} |u_n|^{p-2}|u||v_n|^{q-1}|\varphi v_n|\right). \end{aligned} \tag{4.19}$$

In order to estimate the last integral we apply Hölder's inequality with

$$\frac{p-2}{2^*} + \frac{1}{2^*} + \frac{q-1}{2^*} + \frac{1}{r} = 1,$$

to get

$$\int_{B_R^c} |u_n|^{p-2}|u||v_n|^{q-1}|\varphi v_n| \leqslant c_7\|u\|_{L^{2^*}(B_R^c)}\left(\int_{B_R^c} |\varphi v_n|^r\right)^{1/r} \leqslant c_8\varepsilon,$$

where we have used, in the last inequality, (4.13), $\|\varphi\|_{X_a} \leqslant 1$, the boundedness of (v_n) in X_b, the same calculation performed in (2.3) and (4.2). So,

$$\int_{B_R^c} t_n|\varphi| \leqslant c_9\varepsilon \quad \text{for any } \|\varphi\|_{X_a} \leqslant 1.$$

If $p = 2$ the second integral in the second line of (4.19) does not appear and therefore the above estimate holds in this case too.

Case 2. $1 < p < 2$.

In this case the derivative of the function w defined in the first case can be singular, and we may not apply the Mean Value Theorem directly. In order to overcome this difficult, we first set

$$h_n(x) := |u_n - u|^{p-2}(u_n - u) - |u_n|^{p-2}u_n.$$

As before, we have that

$$\begin{aligned}\int_{B_R^c} t_n|\varphi| &= \int_{B_R^c} |h_n||v_n|^{q-1}|\varphi v_n| \\ &\leqslant c_{10}\left(\int_{B_R^c} |h_n|^{2^*/(p-1)}\right)^{(p-1)/2^*}. \end{aligned} \tag{4.20}$$

We claim that the last integral in the above inequality is small. Indeed, first note that

$$|h_n| \leqslant c_{11}|u|^{p-1} \quad \text{a.e. in the set } \big\{|u_n| \leqslant 2|u|\big\} \cup \big\{|u| = 0\big\}. \tag{4.21}$$

On the other hand, in the set $\{|u_n| > 2|u| > 0\}$, as in the first case, we may apply the Mean Value Theorem for $w(t) := |u_n - tu|^{p-2}(u_n - tu)$ to get

$$|h_n| = \big|w(1) - w(0)\big| \leqslant c_{12}|u_n - t_0 u|^{p-2}|u| \leqslant c_{13}|u|^{p-1},$$

for some $t_0 \in [0,1]$, since for $|u_n| > 2|u|$ we have that $|u_n - t_0 u| \geqslant |u_n| - |t_0||u| \geqslant |u|$. This, (4.21) and (4.13) imply that

$$\int_{B_R^c} |h_n|^{2^*/(p-1)} \leqslant c_{14} \int_{B_R^c} |u|^{2^*} \leqslant c_{15}\varepsilon^{2^*}.$$

It follows from (4.20) that

$$\int_{B_R^c} t_n|\varphi| \leqslant c_{16}\varepsilon^{p-1} \quad \text{for any } \|\varphi\|_{X_a} \leqslant 1.$$

All together, the two cases provide

$$\int_{B_R^c} t_n|\varphi| \leqslant c_{17}\varepsilon^{\min\{1,p-1\}} \quad \text{for any } \|\varphi\|_{X_a} \leqslant 1.$$

Thus, we may use (4.15)–(4.18) and the above estimate to conclude that

$$\int |f_n||\varphi| = \int_{B_R} |f_n||\varphi| + \int_{B_R^c} |f_n||\varphi| \leqslant c_{18}\varepsilon^{\min\{1,p-1\}},$$

for any $\|\varphi\|_{X_a} \leqslant 1$ and $n \geqslant n_0$. Since $\varepsilon > 0$ is arbitrary we conclude that the first equality (4.12) holds. The second one may be verified in a similar way and this concludes the proof of Lemma 4.3. □

In the sequel we follow [5] in order to obtain a local compactness property for the functional I_λ.

Proposition 4.4. *For any given $C_0 > 0$ there exists $\Lambda = \Lambda(p,q,C_0) > 0$ such that I_λ satisfies* (PS)$_c$ *for any $c \leqslant C_0$ and $\lambda \geqslant \Lambda$.*

Proof. Let γ_0 be given by Lemma 2.3(iii) and fix $\varepsilon > 0$ such that

$$\varepsilon < \frac{\gamma_0}{2}\left(\frac{1}{2} - \frac{1}{p+q}\right)^{-1}.$$

Fixed $C_0 > 0$, let Λ_ε and R_ε be given by Lemma 2.4. We will prove that the proposition holds for $\Lambda := \Lambda_\varepsilon$. Let $(z_n) = ((u_n, v_n)) \subset X$ be a (PS)$_c$ sequence for I_λ with $c \leqslant C_0$ and $\lambda \geqslant \Lambda$. In view of Lemma 4.3 we may suppose that $(u_n, v_n) \rightharpoonup z := (u, v)$ weakly in X and $\tilde{z}_n := (u_n - u, v_n - v)$ is a $(PS)_{c'}$ sequence for I_λ, with $c' = c - I_\lambda(z)$. We claim that $c' = 0$. If this is true, it follows from Lemma 2.3(ii) that

$$\lim_{n\to\infty} \|\tilde{z}_n\|_\lambda^2 = c'\left(\frac{1}{2} - \frac{1}{p+q}\right)^{-1} = 0,$$

that is, $z_n \to z$ in X.

Suppose, by contradiction, that $c' \neq 0$. Lemma 2.3(iii) implies that $c' \geqslant \gamma_0 > 0$. Since $\tilde{u}_n, \tilde{v}_n \to 0$ in $L^2(B_{R_\varepsilon})$, we may use Lemma 2.3(ii), Lemma 2.4, the same calculation of (3.2) and the choice of $\varepsilon > 0$, to get

$$\begin{aligned} \gamma_0\left(\frac{1}{2} - \frac{1}{p+q}\right)^{-1} &\leqslant c'\left(\frac{1}{2} - \frac{1}{p+q}\right)^{-1} = \lim_{n\to\infty} \int |\tilde{u}_n|^p |\tilde{v}_n|^q \\ &= \lim_{n\to\infty}\left(\int_{B_{R_\varepsilon}} |\tilde{u}_n|^p |\tilde{v}_n|^q + \int_{B_{R_\varepsilon}^c} |\tilde{u}_n|^p |\tilde{v}_n|^q\right) \\ &\leqslant \frac{\gamma_0}{2}\left(\frac{1}{2} - \frac{1}{p+q}\right)^{-1}, \end{aligned} \tag{4.22}$$

which contradicts $\gamma_0 > 0$. This contradiction finishes the proof. □

We are now ready to prove Theorems 1.2 and 1.3 as follows.

Proof of Theorem 1.2. We first take a bounded open smooth set $\Omega \subset \Omega_a \cap \Omega_b$. Given $k \in \mathbb{N}$ we set $W := \operatorname{span}\{(\varphi_1, \varphi_1), \ldots, (\varphi_k, \varphi_k)\}$, where φ_i is an eigenfunction corresponding to the i-th eigenvalue of $(-\Delta, H_0^1(\Omega))$. For each $i = 1, \ldots, k$ we have that

$$\lim_{t\to\infty} I_\lambda\big(t(\varphi_i, \varphi_i)\big) = \lim_{t\to\infty}\left(t^2 \int |\nabla\varphi_i|^2 - \frac{t^{p+q}}{p+q} \int |\varphi_i|^{p+q}\right) = -\infty,$$

uniformly on λ. Since W has finite dimension we obtain $M_k > 0$, independent of $\lambda > 0$, such that

$$\sup_{z\in W} I_\lambda(z) < M_k.$$

Moreover, as in the proof of Lemma 2.5 we may obtain $\rho > 0$, independent of $\lambda > 0$, such that

$$I_\lambda(z) \geqslant 0 \quad \text{for any } \|z\|_\lambda = \rho.$$

In view of Proposition 4.4 there exists $\Lambda_k > 0$ such that I_λ satisfies $(PS)_c$ for any $c \leqslant M_k$ and $\lambda \geqslant \Lambda_k$. Thus, for any fixed $\lambda \geqslant \Lambda_k$ we may apply Theorem 4.1 to obtain k pairs of nontrivial solutions. The theorem is proved. □

Proof of Theorem 1.3. We first notice that

$$\left(\frac{1}{2} - \frac{1}{p+q}\right)\|z_{\lambda_n}\|_{\lambda_n}^2 = I_{\lambda_n}(z_{\lambda_n}) - \frac{1}{2} I'_{\lambda_n}(z_{\lambda_n}) = I_{\lambda_n}(z_{\lambda_n}).$$

Since $\liminf_{n\to\infty} I_\lambda(z_{\lambda_n}) < \infty$ we may suppose, taking a subsequence if necessary, that (z_{λ_n}) is bounded. Thus, up to a subsequence, we have that

$$z_{\lambda_n} \rightharpoonup \bar{z} := (\bar{u}, \bar{v}) \quad \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N),$$
$$(u_n, v_n) \to (\bar{u}, \bar{v}) \quad \text{strongly in } L^p_{\mathrm{loc}}(\mathbb{R}^N) \times L^q_{\mathrm{loc}}(\mathbb{R}^N),$$
$$\big(u_n(x), v_n(x)\big) \to (\bar{u}, \bar{v}) \quad \text{a.e. in } \mathbb{R}^N. \tag{4.23}$$

As in the proof of the last statement of Theorem 1.1 we can show that $\bar{u} \in H^1_0(\Omega_a)$, $\bar{v} \in H^1_0(\Omega_b)$ and $\bar{z}$ is a solution of (L).

Given $\varepsilon > 0$ we can argue as in the proof of Lemma 2.4 to conclude that, for some $R > 0$ large, there holds

$$\limsup_{n\to\infty} \int\limits_{B_R(0)^c} |u_n|^p |v_n|^q \leqslant \varepsilon.$$

By taking R larger if necessary, we may suppose that $\int_{B_R(0)^c} |\bar{u}|^p |\bar{v}|^q \leqslant \varepsilon$. Moreover, the local convergence in (4.23) and the Lebesgue Dominated Convergence Theorem imply that $\int_{B_R(0)} |u_n|^p |v_n|^q \to \int_{B_R(0)} |\bar{u}|^p |\bar{v}|^q$ as $n \to \infty$. Since

$$\left| \int \big(|u_n|^p |v_n|^q - |\bar{u}|^p |\bar{v}|^q\big) \right| \leqslant \int\limits_{B_R(0)^c} |u_n|^p |v_n|^q + \int\limits_{B_R(0)^c} |\bar{u}|^p |\bar{v}|^q + \left| \int\limits_{B_R(0)} \big(|u_n|^p |v_n|^q - |\bar{u}|^p |\bar{v}|^q\big) \right|,$$

it follows from the above estimates and convergences that

$$\limsup_{n\to\infty} \left| \int \big(|u_n|^p |v_n|^q - |\bar{u}|^p |\bar{v}|^q\big) \right| \leqslant 2\varepsilon,$$

and therefore

$$\lim_{n\to\infty} \int |u_n|^p |v_n|^q = \int |\bar{u}|^p |\bar{v}|^q.$$

Thus, we can argue as in the final of the proof of Theorem 1.1 to conclude that $\|z_{\lambda_n} - \bar{z}\|_0 \leqslant \|z_{\lambda_n} - \bar{z}\|_{\lambda_n} \to 0$ as $n \to \infty$. Hence, $z_n \to \bar{z}$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ and the theorem is proved. □

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