Multiplicity and concentration of solutions for elliptic systems with vanishing potentials

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\textbf{A R T I C L E I N F O}

\textbf{Article history:}
Received 13 July 2009
Revised 30 July 2010
Available online 21 August 2010

\textbf{Keywords:}
Nonlinear Schrödinger systems
Positive solutions
Potential well

\textbf{A B S T R A C T}

In this article we use variational methods to study a strongly coupled elliptic system depending on a positive parameter \( \lambda \). We suppose that the potentials are nonnegative and the intersection of the sets where they vanish has positive measure. A technical condition, imposed on the product of the potentials, allows us to consider a setting where we do not assume any positive lower bound for the potentials. Considering the associated functional, defined on an appropriated subspace of \( \mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N) \), we are able to establish results on the existence and multiplicity of solutions for the system when the parameter \( \lambda \) is sufficiently large. We also study the asymptotic behavior of these solutions when \( \lambda \to \infty \).

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\section{1. Introduction}

The goal of this paper is to study the existence, multiplicity and asymptotic behavior of solutions for the coupled elliptic system

\[
\begin{aligned}
-\Delta u + \lambda a(x)u &= \frac{p}{p+q} |u|^{p-2}u|v|^q, \\
-\Delta v + \lambda b(x)v &= \frac{q}{p+q} |u|^{p} |v|^{q-2}v,
\end{aligned}
\]

\((S_{\lambda})\)

\(u, v \in \mathcal{D}^{1,2}(\mathbb{R}^N)\).

\textsuperscript{*} The first and second author were partially supported by CNPq/Brazil.

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0022-0396/$ – see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.jde.2010.08.002
where $N \geq 3$, $\lambda > 0$ is a parameter, $p, q > 1$ and $p + q < 2^* := 2N/(N - 2)$. Our hypotheses on the nonnegative potentials $a$ and $b$ are

$$(H_1) \quad a, b \in C(\mathbb{R}^N, (0, \infty)), \quad \Omega_a := \text{int } a^{-1}(0) \text{ and } \Omega_b := \text{int } b^{-1}(0) \text{ have smooth boundaries, } \overline{\Omega}_a = a^{-1}(0), \overline{\Omega}_b = b^{-1}(0), \text{ and } \Omega_a \cap \Omega_b \text{ is a nonempty set};$$

$$(H_2) \quad \text{there exists } M_0 > 0 \text{ such that the set } F := \{x \in \mathbb{R}^N : a(x)b(x) \leq M_0\} \text{ has finite Lebesgue measure.}$$

Note that we do not assume any positive lower bound for the potentials $a$ and $b$. Hence we do not expect to find solutions for $(S_\lambda)$ in the Sobolev space $H^1(\mathbb{R}^N)$. However, taking advantage of the strong coupling of the system and the hypothesis $(H_2)$, we are able to use variational methods to study $(S_\lambda)$ by considering the associated functional defined in a proper closed subspace of $D^{1.2}(\mathbb{R}^N) \times D^{1.2}(\mathbb{R}^N)$. We also observe that the sets $\Omega_a$ and $\Omega_b$ may be unbounded and that the fact that $\Omega_a \cap \Omega_b$ is a nonempty set is essential for our results.

As in the scalar case [4], the main results in this article show that the semilinear elliptic system

$$\begin{cases}
-\Delta u = \frac{p}{p+q}|u|^{p-2}u|v|^q & \text{in } \Omega_a, \\
-\Delta v = \frac{q}{p+q}|u|^{p-2}v & \text{in } \Omega_b, \\
u \in H^1_0(\Omega_a), & v \in H^1_0(\Omega_b),
\end{cases} \quad (L)$$

may be seen as a limit problem for $(S_\lambda)$ when $\lambda$ goes to infinity. It is worthwhile mentioning that, although $\Omega_a$ and $\Omega_b$ may be distinct open sets, the system $(L)$ is variational. We also note that condition $(H_2)$ implies that $\Omega_a$ and $\Omega_b$ have finite Lebesgue measure. So, we have the Sobolev compact imbedding $H^1(\Omega_a) \times H^1(\Omega_b) \hookrightarrow L^{r_1}(\Omega_a) \times L^{r_2}(\Omega_b)$, $1 \leq r_1, r_2 < 2^*$.

In order to state our results, we introduce the closed subspaces of $D^{1.2}(\mathbb{R}^N)$ associated with the potentials $a$ and $b$:

$$X_a := \left\{ u \in D^{1.2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} a(x)u^2 \, dx < \infty \right\},$$

and

$$X_b := \left\{ u \in D^{1.2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} b(x)u^2 \, dx < \infty \right\}.$$

For any given $\lambda > 0$, we consider the Hilbert space $X := X_a \times X_b$ endowed with the norm

$$\| (u, v) \|_\lambda^2 := \int_{\mathbb{R}^N} \left( |\nabla u|^2 + |\nabla v|^2 + \lambda a(x)u^2 + \lambda b(x)v^2 \right) \, dx.$$  

Notice that $\| \cdot \|_0$ is the usual norm of the space $D^{1.2}(\mathbb{R}^N) \times D^{1.2}(\mathbb{R}^N)$.

Associated with the problem $(S_\lambda)$ we have the energy functional $I_\lambda : X \to \mathbb{R}$ defined by

$$I_\lambda(u, v) := \frac{1}{2} \| (u, v) \|_\lambda^2 - \frac{1}{p+q} \int_{\mathbb{R}^N} |u|^p |v|^q \, dx, \quad (u, v) \in X.$$  

In view of the hypotheses $(H_1)$ and $(H_2)$, the functional $I_\lambda$ is well defined and of class $C^1$. Furthermore, standard regularity theory implies that the critical points of $I_\lambda$ are classical solutions of the problem $(S_\lambda)$ (see Section 2).
In our first result we consider the existence and behavior of least energy solutions of \((S_\lambda)\). We recall that a least energy solution of our problem is a critical point of \(I_\lambda\) associated with the lowest positive critical level of this functional. In our setting, once proved the existence of a least energy solution, we may always find a positive least energy solution. Here we observe that we call \(z = (u, v)\) a positive function if the functions \(u\) and \(v\) are positive almost everywhere in \(\mathbb{R}^N\).

**Theorem 1.1.** Suppose \((H_1)\) and \((H_2)\) hold. Then there is \(\Lambda > 0\) such that, for all \(\lambda \geq \Lambda\), the system \((S_\lambda)\) possesses a positive least energy solution \(z_\lambda\). Furthermore, if \((\lambda_n) \subset \mathbb{R}\) is such that \(\lambda_n \to \infty\) and \((z_{\lambda_n})\) is a sequence of positive least energy solution of \((S_{\lambda_n})\), then \((z_{\lambda_n})\) converges in \(D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)\) along a subsequence to a positive least energy solution of \((L)\).

In our next result we use the symmetry of our problem to establish multiplicity of solutions for large values of \(\lambda\). More specifically, we shall prove

**Theorem 1.2.** Suppose \((H_1)\) and \((H_2)\) hold. Then, for any given \(k \in \mathbb{N}\), there exists \(\Lambda_k > 0\) such that, for each \(\lambda \geq \Lambda_k\), the system \((S_\lambda)\) possesses at least \(k\) pairs of nonzero solutions.

As in the case of the least energy solutions found in Theorem 1.1, the solutions derived from Theorem 1.2 have uniformly-bounded energy with respect to \(\lambda\). This allows us to show that these solutions converge in \(D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)\) toward solutions of \((L)\) as \(\lambda \to \infty\). More generally, we have the following concentration result.

**Theorem 1.3.** Let \((\lambda_n) \subset \mathbb{R}\) be such that \(\lambda_n \to \infty\) and \((z_{\lambda_n})\) be a sequence of solutions of \((S_{\lambda_n})\) such that \(\liminf_{n \to \infty} I_{\lambda_n}(z_{\lambda_n}) < \infty\). Then \((z_{\lambda_n})\) converges in \(D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)\) along a subsequence to a solution of \((L)\).

The results presented in this article are motivated by that obtained in [4,5] (see also [3]) for the scalar case, where it is considered the potential \(c_\lambda(x) = \lambda c(x) + 1\) with \(c\) such that the set \(\{x \in \mathbb{R}^N\;|\;c(x) \leq M_0\}\) has finite Lebesgue measure, for some \(M_0 > 0\). Concerning our multiplicity result we follow a different approach from [5]. Instead of considering the Ljusternik-Schnirelmann category of some set related with the limit problem, here we use the symmetry of the nonlinearity to derive the existence of multiple solutions.

We observe that there exists an extensive bibliography in the study of elliptic systems on bounded domains (see [15,16,19,9,17,8,11] and references therein). In the case of gradient systems in the whole \(\mathbb{R}^N\), in [7] the author proves the existence of a nonzero solution for \((P)\) under the coercivity of the potentials \(a\) and \(b\), and a nonquadratic condition on the nonlinearity. A related result for noncoercive potentials is proved in [12] (see also [14] for the superlinear case). We should also mention the recent papers [13,11] where some existence results of positive solutions for weakly coupled system are established. We would like to emphasize that, instead of the aforementioned works, the coupling in our system \((S_\lambda)\) allows us to consider potentials which are not bounded from below by positive constants. We may have one of the potentials going to zero as \(|x| \to \infty\) provided the other one goes to infinity at an appropriated rate.

The paper is organized in the following way. In Section 2 we present technical results which will be used throughout the work. We also investigate the behavior of the Palais–Smale sequences when \(\lambda\) goes to infinity. We prove Theorem 1.1 in Section 3. The final Section 4 is devoted to the proof of Theorems 1.2 and 1.3.

2. Preliminaries

In this section we present some preliminaries for the proof of Theorem 1.1. In this paper, we denote by \(B_R\) the open ball in \(\mathbb{R}^N\) of radius \(R > 0\) and center at the origin. For any given set \(K \subset \mathbb{R}^N\), we set \(K^C := \mathbb{R}^N \setminus K\) and we write \(C(K)\) for the Lebesgue measure of \(K\) whenever this set is measurable. \(C_0^\infty(K)\) denotes the set of all functions \(u : K \to \mathbb{R}\) of class \(C^\infty\) with compact support contained in the
open set $K \subset \mathbb{R}^N$. If $u \in L^s(K)$, $s \geq 1$, we set $u_+ := \max\{u, 0\}$, $u_- := \max\{-u, 0\}$ and write $\|u\|_{L^s(K)}$ for the $L^s$-norm of $u$. In order to simplify the notation, we write $\int_K u$ instead of $\int_K u(x) \, dx$. We also omit the set $K$ whenever $K = \mathbb{R}^N$. Finally, we use the symbols $c_i$, $i \in \mathbb{N}$, to represent positive constants.

We start with two technical results.

**Lemma 2.1.** For any given measurable set $K \subset \mathbb{R}^N$ there exists a constant $c > 0$ such that

$$\int_K |u|^p |v|^q \leq c \|u, v\|_{L^0}^{p+q-2+2^*t/r} \left( \int_K |uv| \right)^{\beta}, \quad \text{for all } (u, v) \in X,$$

where $r := 2^*/(2^* - p - q + 2) > 1$, and $t \in (0, 1)$ satisfies $r = 2^*t/2 + (1 - t)$ and $\beta := (1 - t)/r$.

**Proof.** From the definition of $r > 1$ we have that

$$\frac{p - 1}{2^*} + \frac{q - 1}{2^*} + \frac{1}{r} = 1. \quad (2.1)$$

Hölder’s inequality and the imbedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ imply that

$$\int_K |u|^p |v|^q \leq \int_K |u|^{p-1} |v|^{q-1} |uv|$$

$$\leq \left( \int_K |u|^{2^*} \right)^{(p-1)/2^*} \left( \int_K |v|^{2^*} \right)^{(q-1)/2^*} \left( \int_K |uv| \right)^{1/r}$$

$$\leq c_1 \|u, v\|_{L^0}^{p+q-2} \left( \int_K |uv| \right)^{1/r}. \quad (2.2)$$

Since $1 < r < 2^*/2$ there exists $t \in (0, 1)$ such that $r = 2^*t/2 + (1 - t)$. By using Hölder’s inequality with exponents $1/t$, $1/(1 - t)$, and the imbedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ again, we obtain

$$\int_K |uv|^r = \int_K |uv|^{2^*t/2} |uv|^{(1-t)}$$

$$\leq \left( \int_K |uv|^{2^*/2} \right)^t \left( \int_K |uv| \right)^{1-t}$$

$$\leq \left( \frac{1}{2} \int_K (|u|^{2^*} + |v|^{2^*}) \right)^t \left( \int_K |uv| \right)^{1-t}$$

$$\leq c_2 \|u, v\|_{L^0}^{2^*t} \left( \int_K |uv| \right)^{1-t}. \quad (2.3)$$

Combining the last inequality and (2.2), we conclude the proof of the lemma. □
Lemma 2.2. There exists a constant $\hat{c} > 0$ such that
\[
\int |u|^p |v|^q \leq \hat{c} \| (u, v) \|_1^{p+q}, \quad \text{for all } (u, v) \in X.
\]

Proof. By Lemma 2.1 we have that
\[
\int |u|^p |v|^q \leq c \| (u, v) \|_0^{p+q-2+2^*t/r} \left( \int |u| \right)^{(1-t)/r}. \tag{2.4}
\]

We recall that the set $F$ given in $(H_2)$ has finite measure and $a(x)b(x) > M_0$ in $F^c$. Applying Hölder’s inequality with exponents $2^*$, $2^*$, $N/2$ and using the imbedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, we obtain
\[
\int \left| \left( \int_{F^c} |u| \right)^{1/2} \left( \int_{F^c} b(x) v^2 \right)^{1/2} \right| \leq c_2 \| (u, v) \|_1^2.
\]
The last inequality and (2.4) provide $\hat{c} > 0$ such that
\[
\int u_+^p v_+^q \leq \hat{c} \| (u, v) \|_1^{p+q-2+2^*t/r \tau} \| (u, v) \|_1^{(1-t) \tau/2} = \hat{c} \| (u, v) \|_1^{p+q-2+2^*t/(2^*+1-t)} = \hat{c} \| (u, v) \|_1^{p+q},
\]
where we have used that $r = 2^* t/2 + (1 - t)$. The lemma is proved. \qed

Since we are interested in positive solutions of $(S_\lambda)$ we will work with a functional slightly different from that defined in the introduction. More specifically, we consider $I_\lambda : X \to \mathbb{R}$ given by
\[
I_\lambda(u, v) := \frac{1}{2} \| (u, v) \|_\lambda^2 - \frac{1}{p+q} \int u_+^p v_+^q, \quad (u, v) \in X.
\]
In view of the above lemma, it is well defined. Moreover, we may use the above results and hypothesis $(H_2)$ to show that $I_\lambda \in C^1(X, \mathbb{R})$ for any $\lambda > 0$.

Let $E$ be a Banach space and $I \in C^1(E, \mathbb{R})$. We say that $(z_n) \subset E$ is a Palais–Smale sequence at level $c$ ($(PS)_c$ sequence for short) if $I(z_n) \to c$ and $I'(z_n) \to 0$. We say that $I$ satisfies $(PS)_c$ if any $(PS)_c$ sequence possesses a convergent subsequence.

Lemma 2.3. Let $\lambda \geq 1$ and $(z_n) \subset X$ be a $(PS)_c$ sequence for $I_\lambda$.

(i) $(z_n)$ is bounded in $X$;
(ii) $\lim_{n \to \infty} \| z_n \|_\lambda^2 = \lim_{n \to \infty} \int (u_n)^p (v_n)^q = c(\frac{1}{2} - \frac{1}{p+q})^{-1}$;
(iii) if $c \neq 0$, then $c \geq \gamma_0 > 0$, for some $\gamma_0$ independent of $\lambda$.  

Proof. We have that
\[
\left( \frac{1}{2} - \frac{1}{p+q} \right) \|z_n\|_\lambda^2 = I_\lambda(z_n) - \frac{1}{p+q} I_\lambda'(z_n) \cdot z_n = c + o(1) \|z_n\|_\lambda, \tag{2.5}
\]
as \(n \to \infty\), and therefore (i) holds. Moreover, as \(n \to \infty\), we have that
\[
\left( \frac{1}{2} - \frac{1}{p+q} \right) \|z_n\|_\lambda^2 = c + o(1) \|z_n\|_\lambda = I_\lambda(z_n) - \frac{1}{2} I_\lambda'(z_n) \cdot z_n
\]
\[
= \left( \frac{1}{2} - \frac{1}{p+q} \right) \int (u_n)_+^p (v_n)_+^q,
\]
from which follows (ii). We now observe that, in view of Lemma 2.2 and \(\lambda \geq 1\),
\[
I_\lambda'(z) \cdot z = \|z\|_\lambda^2 - \int u_+^p v_+^q \geq \|z\|_\lambda^2 - \hat{c} \|z\|_\lambda^{p+q} \geq \frac{1}{2} \|z\|_\lambda^2,
\]
everywhere \(\|z\|_\lambda \leq (2\hat{c})^{-1/(p+q-2)} := \sqrt{\delta}\). Suppose now that
\[
c < \delta \left( \frac{1}{2} - \frac{1}{p+q} \right).
\]
By (ii), there exists \(n_0 \in \mathbb{N}\) such that \(\|z_n\|_\lambda < \sqrt{\delta}\) for any \(n \geq n_0\). Thus,
\[
\frac{1}{2} \|z_n\|_\lambda^2 \leq I_\lambda'(z_n) \cdot z_n \leq o(1) \|z_n\|_\lambda \quad \text{as} \quad n \to \infty,
\]
and we conclude that \(z_n \to 0\) in \(X\). Hence, \(I_\lambda(z_n) \to 0 = c\) and it follows that (iii) holds for \(\gamma_0 := \delta \left( \frac{1}{2} - \frac{1}{p+q} \right)\). \(\square\)

Lemma 2.4. Given \(\varepsilon > 0\) and \(C_0 > 0\), there exist \(A_\varepsilon = A(\varepsilon, C_0) > 0\) and \(R_\varepsilon = R(\varepsilon, C_0) > 0\) such that, if
\((u_n, v_n) \subset X\) is a (PS)\(_c\) sequence for \(I_\lambda\) with \(c \leq C_0\) and \(\lambda \geq \Lambda_\varepsilon\), then
\[
\limsup_{n \to \infty} \int_{B_{cR}} (u_n)_+^p (v_n)_+^q \leq \varepsilon.
\]
Proof. Since \(\|\cdot\|_0 \leq \|\cdot\|_\lambda\), we may use Lemma 2.1 and Lemma 2.3(i) to obtain
\[
\int_{B_{cR}} (u_n)_+^p (v_n)_+^q \leq c_1 \left( \int_{B_{cR}} |u_n v_n|^\beta \right), \tag{2.6}
\]
for any \(R > 0\). By Young and Hölder’s inequality, the imbedding \(\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)\) and Lemma 2.3(i), we get
\[
\int_{B_{cR} \cap F} |u_n v_n| \leq \frac{1}{2} \int_{B_{cR} \cap F} (|u_n|^2 + |v_n|^2)
\]
\[
\leq \frac{c_2 \mathcal{L}(B_{cR} \cap F)^{2/N}}{2} \left( \|u_n\|_{L^2}^2 + \|v_n\|_{L^2}^2 \right)
\]
\[
\leq c_2 \mathcal{L}(B_{cR} \cap F)^{2/N}. \tag{2.7}
\]
On the other hand, since \(((u_n, v_n))\) is bounded and \(a(x)b(x) > M_0\) in \(B_R^c \cap F^c\), we have
\[
\frac{1}{\lambda M_0} \int_{B_R^c \cap F^c} |u_n v_n| \leq \frac{1}{\lambda M_0} \int_{B_R^c \cap F^c} \sqrt{\lambda a(x)|u_n| \sqrt{\lambda b(x)|v_n|}}
\leq \frac{1}{2\lambda M_0} \int_{B_R^c \cap F^c} \left(\lambda a(x)u_n^2 + \lambda b(x)v_n^2\right) \leq c_3/\lambda.
\]

It follows from the above estimate, (2.7) and (2.6) that
\[
\int_{B_R} (u_n)^p(v_n)^q \leq c_2 \left(c_1 L(B_R^c \cap F)^{2/N} + c_3/\lambda\right)^{\beta}.
\]

Since \(F\) has finite Lebesgue measure, we have that \(L(B_R^c \cap F) \to 0\) as \(R \to \infty\). Hence, for \(R\) and \(\lambda\) sufficiently large, the right-hand side of the above expression is small. This concludes the proof.

In the next lemma we verify that \(I_\lambda\) satisfies the Mountain Pass geometry.

**Lemma 2.5.** There exist \(\alpha, \rho > 0\) and \(z_0 \in X\), all of them independent of \(\lambda \geq 1\), such that

(i) \(I_\lambda(z) \geq \alpha\) for all \(\|z\|_\lambda = \rho\),

(ii) \(I_\lambda(z_0) \leq I_\lambda(0) = 0\) and \(\|z_0\| > \rho\).

**Proof.** By Lemma 2.2, we have that
\[
I_\lambda(z) = \frac{1}{2} \|z\|_\lambda^2 - \frac{1}{p + q} \int u_+^p v_+^q \geq \frac{1}{2} \|z\|_\lambda^2 \frac{c}{p + q} \|z\|^{p+q}_\lambda \geq \frac{1}{4} \rho^2,
\]
whenever \(\|z\|_\lambda = \rho := ((p + q)/4\hat{c})^{1/(p+q-2)}\). Furthermore, if \(\varphi \in C_0^\infty(\Omega_\alpha \cap \Omega_\beta) \setminus \{0\}, \varphi_+ \not\equiv 0\), we have that \(a(x)\varphi \equiv b(x)\varphi \equiv 0\) on \(\mathbb{R}^N\). Hence,
\[
\lim_{t \to \infty} I_\lambda(t(\varphi, \varphi)) = \lim_{t \to \infty} \left(t^2 \int |\nabla \varphi|^2 - \frac{t^{p+q}}{p + q} \int \varphi_+^{p+q}\right) = -\infty,
\]
uniformly on \(\lambda\). It suffices to set \(z_0 := t_0(\varphi, \varphi)\) with \(t_0 > 0\) sufficiently large.

**Remark 2.6.** Let \(z_0\) be given by the above lemma. For each \(\lambda > 0\) we may define the Mountain Pass level of \(I_\lambda\) as
\[
c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I_\lambda(\gamma(t)),
\]
where
\[
\Gamma := \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = z_0\}.
\]

For future reference we observe that
\[
0 < \alpha \leq c_\lambda \leq \beta_0 := \max_{t \in [0, 1]} I_\lambda(tz_0).
\]
3. Least energy solutions

We devote this section to the proof of Theorem 1.1. Let \( \varepsilon > 0 \) to be chosen later, \( C_0 := \beta_0 \) given in (2.8), and consider \( A_\varepsilon, R_\varepsilon \) provided by Lemma 2.4. In view of Remark 2.6 we obtain, for any fixed \( \lambda \geq A_\varepsilon \), a sequence \( (z_k) \subset X \) such that

\[
I_\lambda(z_k) \to c_\lambda \geq \alpha \quad \text{and} \quad I'_\lambda(z_k) \to 0.
\]

By Lemma 2.3(i) \( (z_k) \) is bounded in \( X \) and therefore, up to a subsequence, we have that \( z_k \rightharpoonup z_\lambda := (u_\lambda, v_\lambda) \) weakly in \( X \).

We shall prove that \( I'_\lambda(z_\lambda) = 0 \). Let \( \varphi \in C_0^\infty(\mathbb{R}^N) \) and denote by \( K \) the support of \( \varphi \). Since \( \mathcal{D}^{1,2}(\mathbb{R}^N) \) is compactly embedded in \( L_{loc}^{p+q-1}(\mathbb{R}^N) \), up to a subsequence, we have that

\[
(u_k, v_k) \to (u_\lambda, v_\lambda) \quad \text{strongly in} \quad L^{p+q-1}(K) \times L^{p+q-1}(K),
\]

\[
(u_k(x), v_k(x)) \to (u_\lambda(x), v_\lambda(x)) \quad \text{a.e. in} \quad K,
\]

\[
|u_k(x)|, |v_k(x)| \leq h_K(x) \in L^{p+q-1}(K) \quad \text{a.e. in} \quad K.
\]

Hence, almost everywhere in \( K \),

\[
(u_k)_+^{p-1}(v_k)_+^q |\varphi| \leq |u_k|^{p-1}|v_k|^q|\varphi| \leq h_K^{p+q-1}|\varphi| \in L^1(K).
\]

It follows from the above convergences and the Lebesgue Dominated Convergence Theorem that

\[
\lim_{k \to \infty} \int (u_k)_+^{p-1}(v_k)_+^q \varphi = \int (u_\lambda)_+^{p-1}(v_\lambda)_+^q \varphi, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).
\]

(3.1)

Analogously, we obtain

\[
\lim_{k \to \infty} \int (u_k)_+^p(v_k)_+^{q-1} \psi = \int (u_\lambda)_+^p(v_\lambda)_+^{q-1} \psi, \quad \forall \psi \in C_0^\infty(\mathbb{R}^N).
\]

The two above limits and the weak convergence of \( (z_k) \) imply that, for each \( (\varphi, \psi) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N) \), there hold

\[
0 = \lim_{k \to \infty} I'_\lambda(z_k) \cdot (\varphi, \psi) = I'_\lambda(z_\lambda) \cdot (\varphi, \psi)
\]

and therefore \( z_\lambda \) is a critical point of \( I_\lambda \).

Suppose that \( z_\lambda \equiv 0 \). Since \( u_k, v_k \rightharpoonup 0 \) in \( L^2(B_{R_\varepsilon}) \) we may use Lemma 2.1, the boundedness of \( (z_k) \) in \( X \) and Young’s inequality, to obtain

\[
\int_{B_{R_\varepsilon}} (u_k)_+^p(v_k)_+^q \leq c_1 \left( \int_{B_{R_\varepsilon}} |u_k v_k| \right)^{\beta} \leq c_2 \left( \int_{B_{R_\varepsilon}} |u_k|^2 + |v_k|^2 \right)^{\beta} \to 0,
\]

as \( k \to \infty \). So, it follows from Lemma 2.3(ii) and Lemma 2.4 that, for \( \lambda \geq A_\varepsilon \),
\[ c_\lambda \left( \frac{1}{2} - \frac{1}{p+q} \right)^{-1} \lim_{k \to \infty} \int_{\Omega} (u_k)_+^p (v_k)_-^q = \lim_{k \to \infty} \left( \int_{\Omega} (u_k)_+^p (v_k)_-^q + \int_{\Omega_x} (u_k)_+^p (v_k)_+^q \right) \leq \varepsilon. \]

If we choose \( \varepsilon > 0 \) sufficiently small, we conclude that \( c_\lambda = 0 \), contradicting (2.8). This shows that \( z_\lambda \neq 0 \).

Applying Fatou’s Lemma we get

\[ c_\lambda = \lim_{k \to \infty} \left( I_\lambda(z_k) - \frac{1}{2} I'_\lambda(z_k) \cdot z_k \right) = \lim_{k \to \infty} \left( \frac{1}{2} - \frac{1}{p+q} \right) \int_{\Omega} (u_k)_+^p (v_k)_-^q \]

\[ \geq \left( \frac{1}{2} - \frac{1}{p+q} \right) \int_{\Omega} u_\lambda^p v_\lambda^q = I_\lambda(z_\lambda) \geq c_\lambda, \]

from which follows that \( I_\lambda(z_\lambda) = c_\lambda \). Hence, \( z_\lambda \) is a least energy solution.

Since \( I'(z_\lambda) \cdot ((u_\lambda)_-, (v_\lambda)_-) = \|((u_\lambda)_-, (v_\lambda)_-)\|_{L^2(\Omega_x)}^2 = 0 \), we have that \( u_\lambda, v_\lambda > 0 \) in \( \mathbb{R}^N \). Furthermore, by applying the Strong Maximum Principle in each equation of \((S_\lambda)\) we conclude that \( u_\lambda, v_\lambda > 0 \) in \( \mathbb{R}^N \). This proves the first part of Theorem 1.1.

We now consider the concentration behavior of the solutions. Suppose that \( (\lambda_n) \subset \mathbb{R} \) is such that \( \lambda_n \to \infty \) and let \( z_{\lambda_n} = (u_{\lambda_n}, v_{\lambda_n}) \) be the associated solution of \((S_{\lambda_n})\) such that \( I_{\lambda_n}(z_{\lambda_n}) = c_{\lambda_n} \). In what follows we write only \( z_n, u_n \) and \( v_n \) to denote \( z_{\lambda_n}, u_{\lambda_n} \) and \( v_{\lambda_n} \) respectively.

First note that, in view of (2.8),

\[ \left( \frac{1}{2} - \frac{1}{p+q} \right) \|z_n\|_{L^2(\Omega)}^2 = I_{\lambda_n}(z_n) = c_{\lambda_n} \leq \beta_0. \]

Thus, up to a subsequence, we have that \( z_n \rightharpoonup \tilde{z} := (\tilde{u}, \tilde{v}) \) weakly in \( D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \) and \( z_n(x) \to \tilde{z}(x) \) almost everywhere in \( \mathbb{R}^N \). Given \( \varphi \in C_0^\infty(\Omega_a) \), recalling that \( a \equiv 0 \) in \( \Omega_a \) and using \((\varphi, 0)\) as a test function we get

\[ \int_{\Omega} \nabla u_n \nabla \varphi = \frac{p}{p+q} \int_{\Omega} (u_n)_+^{p-1} (v_n)_-^q \varphi. \]

Since \( \varphi \) has compact support, we may take the limit in the above expression and argue as in the proof of (3.1) to get

\[ \int_{\Omega_a \cup \Omega_b} \nabla \tilde{u} \nabla \varphi = \frac{p}{p+q} \int_{\Omega_a \cup \Omega_b} \tilde{u}_+^{p-1} \tilde{v}_-^q \varphi, \quad \forall \varphi \in C_0^\infty(\Omega_a). \]

(3.4)

Analogously, we have

\[ \int_{\Omega_a \cup \Omega_b} \nabla \tilde{v} \nabla \psi = \frac{q}{p+q} \int_{\Omega_a \cup \Omega_b} \tilde{u}_+^{p} \tilde{v}_-^{q-1} \psi, \quad \forall \psi \in C_0^\infty(\Omega_b). \]

(3.5)

We claim that \( \tilde{u} \equiv 0 \) in \( \Omega_a^\varepsilon \). In order to see this we take \( j \in \mathbb{N} \), set

\[ C_j := \left\{ x \in B_j(0): a(x) > \frac{1}{j} \right\} \]
and notice that, by (3.3),

\[ 0 \leq \int_{C_j} u_n^2 \leq \frac{j}{\lambda_n} \int_{C_j} \lambda_n a(x)u_n^2 \leq \frac{j}{\lambda_n} \|z_n\|^2 \rightarrow 0, \quad \text{as } n \rightarrow \infty. \]

Since \( C_j \) is bounded and \( u_n \rightarrow \bar{u} \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \), we conclude that \( \int_{C_j} \bar{u}^2 = 0 \) for all \( j \in \mathbb{N} \). Thus \( \bar{u} \equiv 0 \) almost everywhere in \( \Omega_a = \bigcup_{j=1}^n C_j \). Recalling that \( \Omega_a \) has smooth boundary we conclude that \( \bar{u} \in H^1_0(\Omega_a) \). Analogously, \( \bar{v} \in H^1_0(\Omega_b) \). Thus, \( (\bar{u}, \bar{v}) \) is a solution of the limit problem (L).

In order to verify that \( z \neq 0 \) we define

\[ m := \inf_{z \in \mathcal{N}} J(z), \]

where \( J : H^1_0(\Omega_a) \times H^1_0(\Omega_b) \rightarrow \mathbb{R} \) is given by

\[ J(u, v) := \frac{1}{2} \int_{\Omega_a \cup \Omega_b} (|\nabla u|^2 + |\nabla v|^2) - \frac{1}{p + q} \int_{\Omega_a \cup \Omega_b} u^p v^q \]

and \( \mathcal{N} \) is the Nehari manifold of \( J \), namely:

\[ \mathcal{N} := \{ (u, v) \in H^1_0(\Omega_a) \times H^1_0(\Omega_b) : (u, v) \neq (0, 0), \quad J'(u, v) \cdot (u, v) = 0 \}. \]

Since \( H^1_0(\Omega_a) \) and \( H^1_0(\Omega_b) \) can be viewed as a subspace of \( X \), we have that \( c_\lambda \leq m \), for all \( \lambda \). On the other hand

\[ m \geq c_{\lambda_n} = I_{\lambda_n}(z_n) - \frac{1}{2} I'_{\lambda_n}(z_n) \cdot z_n = \left( \frac{1}{2} - \frac{1}{p + q} \right) \int_{\Omega_a \cup \Omega_b} (u_n)^p (v_n)^q. \]

Taking \( n \rightarrow \infty \), using Fatou’s Lemma and \( J'(\bar{u}, \bar{v}) = 0 \) we obtain

\[ m \geq \lim_{n \rightarrow \infty} \left( \frac{1}{2} - \frac{1}{p + q} \right) \int_{\Omega_a \cup \Omega_b} (u_n)^p (v_n)^q \geq \left( \frac{1}{2} - \frac{1}{p + q} \right) \int_{\Omega_a \cup \Omega_b} \bar{u}^p \bar{v}^q = J(\bar{u}, \bar{v}) \geq m. \]

Hence \( J(\bar{u}, \bar{v}) = m \) and therefore \( z \neq 0 \) is a least energy solution of \( (L) \). By using (3.4) and (3.5) we obtain \( \|(\bar{u}_-, \bar{v}_-)|_0 = 0 \). Thus, \( \bar{u}, \bar{v} > 0 \) and it follows from the Strong Maximum Principle and (3.4)-(3.5) that \( \bar{u} > 0 \) in \( \Omega_a \) and \( \bar{v} > 0 \) in \( \Omega_b \).

In order to finish the proof we use the weak convergence of \( (z_n) \), the fact that \( z_n \) is a solution of \( (S_{\lambda_n}) \), (3.6) and \( (\bar{u}, \bar{v}) \in \mathcal{N} \) to get

\[ \|z_n - z\|^2_{\lambda_n} = \int (|\nabla u_n|^2 + |\nabla v_n|^2 + \lambda_n a(x)u_n^2 + \lambda_n b(x)v_n^2) - \int (|\nabla \bar{u}|^2 + |\nabla \bar{v}|^2) + o(1) \]

\[ = \int (u_n)^p (v_n)^q - \int (|\nabla \bar{u}|^2 + |\nabla \bar{v}|^2) + o(1) \]

\[ = \int \bar{u}^p \bar{v}^q - \int (|\nabla \bar{u}|^2 + |\nabla \bar{v}|^2) + o(1) = o(1). \]
as \( n \to \infty \). Since \( \| \cdot \|_0 \leq \| \cdot \|_{\lambda_n} \) it follows that \( z_n \to z \) in \( D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \). This concludes the proof of Theorem 1.1.

4. Multiplicity of bound state solutions

In this section we present the proofs of Theorems 1.2 and 1.3. Since we are not interested in the sign of the solutions, we redefine the functional \( I_\lambda \) by setting

\[
I_\lambda(u, v) := \frac{1}{2} \| (u, v) \|^2 - \frac{1}{p + q} \int |u|^p |v|^q, \quad (u, v) \in X.
\]

As in Section 2, the functional is of class \( C^1 \) and its critical points are the weak solutions of \( (S_\lambda) \). For future reference we notice that, arguing as in the proof of Lemma 2.1 and Lemma 2.2, we obtain \( c, \hat{c} > 0 \) such that

\[
\int_{B^c_R} |u|^{p-1}|v|^{q-1} |\phi \eta| \leq c \| u \|^{p-1}_{L^2(B^c_R)} \| v \|^{q-1}_{L^2(B^c_R)} \| (\phi, \eta) \|_0^{2^* \alpha / r} \left( \int_{B^c_R} |\phi \eta| \right)^{\beta},
\]

and

\[
\int_{B^c_R} |\phi \eta| \leq \hat{c} \| (\phi, \eta) \|_0^2 L(B^c_R \cap F)^{2/N} + \frac{1}{\sqrt{M_0}} \left( \int_{B^c_R \cap F} a(x)^2 \right)^{1/2} \left( \int_{B^c_R \cap F} b(x)^2 \right)^{1/2},
\]

for any \( R > 0 \) and \( (u, v), (\phi, \eta) \in X \). Here \( r > 1, t \in (0, 1) \) and \( \beta > 0 \) are given by Lemma 2.1.

In order to obtain multiple critical points for \( I_\lambda \), we shall use the following version of the Symmetric Mountain Pass Theorem [2] (see also [18, Theorem 2.1]).

**Theorem 4.1.** Let \( E \) be a real Banach space and \( W \subset E \) a finite dimensional subspace. Suppose that \( I \in C^1(E, \mathbb{R}) \) is an even functional satisfying \( I(0) = 0 \) and

(i) there exists a constant \( \rho > 0 \) such that \( I|_{\partial B_\rho(0)} > 0 \);

(ii) there exists \( M > 0 \) such that \( \sup_{z \in W} I(z) < M \).

If \( I \) satisfies (PS)\(_c\) for any \( 0 < c < M \), then \( I \) possesses at least \( \dim W \) pairs of nontrivial critical points.

Our first goal is to prove a local compactness condition for \( I_\lambda \). We start with the following version of Brezis–Lieb Lemma [6] (see also [10]).

**Lemma 4.2.** Let \( ((u_n, v_n)) \subset X \) be such that \( (u_n, v_n) \rightharpoonup (u, v) \) weakly in \( X \). Then

\[
\lim_{n \to \infty} \int (|u_n|^p |v_n|^q - |u_n - u|^p |v_n - v|^q) = \int |u|^p |v|^q.
\]

**Proof.** Let \( A_n \) be the integral on the left-hand side of the above expression and notice that

\[
A_n = - \int_0^1 \frac{d}{dt} \left( |u_n - tu|^p |v_n|^q + |u_n - u|^p |v_n - tv|^q \right) dt dx
= p \int_0^1 f_n(t, x) u dt dx + q \int_0^1 g_n(t, x) v dt dx.
\]
with
\[ f_n(t, x) := |u_n - tu|^{p-2}(u_n - tu)|v_n|^q \]

and
\[ g_n(t, x) := |u_n - u|^p|v_n - tv|^{q-2}(v_n - tv). \]

Since \((u_n, v_n)\) is bounded in \(X\) and \(p + q < 2^*\), taking a subsequence if necessary, we may suppose that
\[ (u_n(x), v_n(x)) \to (u(x), v(x)) \quad \text{strongly in } L^{p+q}(\mathbb{R}^N) \times L^{p+q}(\mathbb{R}^N), \]
\[ |u(x)|, |v(x)|, |u_n(x)|, |v_n(x)| \leq h_R(x) \in L^{p+q}(B_R) \quad \text{a.e. in } B_R, \]
\[ (4.4) \]

for any \(R > 0\).

The pointwise convergence implies that, for almost every \((t, x) \in (0, 1) \times \mathbb{R}^N\),
\[ f_n(t, x) \to f(t, x) := (1 - t)^{p-1}|u|^{p-2}u|v|^q, \quad g_n(t, x) \to g(t, x) \equiv 0. \] (4.5)

We claim that
\[ \lim_{n \to \infty} \int_0^1 \int_0^1 f_n(t, x) u \, dt \, dx = \int_0^1 \int_0^1 f(t, x) u \, dt \, dx \] (4.6)

and
\[ \lim_{n \to \infty} \int_0^1 \int_0^1 g_n(t, x) v \, dt \, dx = \int_0^1 \int_0^1 g(t, x) v \, dt \, dx = 0. \] (4.7)

Assuming the claim, noticing that for any measurable set \(K \subset \mathbb{R}^N\), we have
\[ \int_K \int_0^1 f u \, dt \, dx = \frac{1}{p} \int_K |u|^p|v|^q \, dx, \]
and taking the limit in (4.3), we obtain
\[ \lim_{n \to \infty} A_n = p \int_0^1 \int_0^1 f(t, x) u \, dt \, dx = \int |u|^p|v|^q \, dx. \]

So, in order to prove the lemma, it suffices to verify (4.6) and (4.7).

In view of \((H_2)\), for any given \(0 < \varepsilon < 1\) we may choose \(R = R(\varepsilon) > 0\) such that
\[ \max \left\{ \frac{1}{p} \int_{B_R^c} |u|^p|v|^q, \mathcal{L}(B_R^c \cap F)^{2\beta/N}, \left( \int_{B_R^c} au^2 \right)^{\beta/2} \right\} < \varepsilon, \] (4.8)

where \(\beta > 0\) comes from Lemma 2.1. So, we have that
\[
\left| \int_0^1 \left( f_n - f \right) u \, dt \, dx \right| \leq \left| \int_0^1 (f_n - f) u \, dt \, dx \right| + \left| \int_0^1 f_n u \, dt \, dx \right| + \varepsilon. \tag{4.9}
\]

In view of (4.4) we have that
\[
\left| (f_n - f) u \right| \leq \left( |u_n - tu|^{p-1} |v_n|^q + |u|^{p-1} |v|^q \right) |u| \leq c_1 h_R(x)^{p+q} \in L^1(B_R),
\]
for almost every \( x \in B_R \). Hence, we can use (4.5) and the Lebesgue Dominated Convergence Theorem to get
\[
\lim_{n \to \infty} \left| \int_0^1 (f_n - f) u \, dt \, dx \right| = 0. \tag{4.10}
\]

On the other hand,
\[
\left| \int_{B_R^c}^1 f_n u \, dt \, dx \right| \leq \int_{B_R^c}^1 |u_n - tu|^{p-1} |v_n|^q |u| \, dt \, dx \leq c_2 \left( \int_{B_R^c}^1 \left( |u_n|^{p-1} |v_n|^q |u| + |u|^{p-1} |v_n|^q \right) \right). \tag{4.11}
\]

Since \((u_n, v_n)\) is bounded in \( X \), we may use (4.1), (4.2) and (4.8) to conclude that
\[
\int_{B_R^c}^1 |u_n|^{p-1} |v_n|^{q-1} |uv_n| \leq c_3 \left( \int_{B_R^c}^1 |uv_n| \right)^\beta \leq c_4 \varepsilon
\]
and
\[
\int_{B_R^c}^1 |u|^{p-1} |v_n|^q \, dt \leq c_5 \varepsilon.
\]

By replacing these expressions in (4.11) we get
\[
\left| \int_{B_R^c}^1 f_n u \, dt \, dx \right| \leq c_6 \varepsilon.
\]

The above estimate, (4.10) and (4.9) imply that
\[
\limsup_{n \to \infty} \left| \int_0^1 (f_n - f) u \, dt \, dx \right| \leq c_7 \varepsilon.
\]

Since \( 0 < \varepsilon < 1 \) is arbitrary, we conclude that (4.6) holds. The proof of (4.7) is analogous and it will be omitted. The lemma is proved. \( \square \)
Lemma 4.3. Let \((z_n) = (u_n, v_n) \subset X\) be a \((PS)_{\lambda}\) sequence for \(I_\lambda\). Then, up to a subsequence, \(z_n \rightharpoonup z := (u, v)\) weakly in \(X\), where \(z\) is a critical point of \(I_\lambda\). Furthermore, \((\tilde{z}_n) := (z_n - z)\) is a \((PS)_{\lambda'}\) sequence for \(I_{\lambda'}\), with \(\lambda' = \lambda - I_\lambda(z)\).

Proof. Since \((z_n)\) is bounded in \(X\), up to a subsequence, \(z_n \rightharpoonup z := (u, v)\) weakly in \(X\). Arguing as in the proof of Theorem 1.1 we may show that \(I'(z) = 0\). The weak convergence of \((z_n)\) and Lemma 4.2 imply that

\[
I_\lambda(z_n - z) = \frac{1}{2}\|z_n\|_{\lambda}^2 - \frac{1}{2}\|z\|_{\lambda}^2 - \frac{1}{p+q} \int |u_n|^p |v_n|^q + \frac{1}{p+q} \int |u|^p |v|^q + o(1)
\]

as \(n \to \infty\).

It remains to show that \(I'(z_n - z) \to 0\). We first notice that, for any given \((\varphi, \psi) \in X\) such that \(\|\varphi, \psi\| \leq 1\),

\[
I'(z_n - z) \cdot (\varphi, \psi) = I'(z_n) \cdot (\varphi, \psi) - I'(z) \cdot (\varphi, \psi) - \frac{p}{p+q} \int f_n \varphi - \frac{q}{p+q} \int g_n \psi,
\]

where

\[
f_n(x) := |u_n - u|^{p-2}(u_n - u) v_n - v|^q - |u_n|^{p-2} u_n v_n|^q + |u|^{p-2} u |v|^q
\]

and

\[
g_n(x) := |u_n - u|^p |v_n - v|^q - |u_n|^p |v_n|^q - |u|^p v |v|^q.
\]

Since \(I'(z_n) \to 0\) and \(I'(z) = 0\), it suffices to show that

\[
\lim \sup_{n \to \infty} \|f_n\|_{X_{\lambda}} = 0 = \lim \sup_{n \to \infty} \|g_n\|_{X_{\lambda}}
\]

where we are denoting

\[
\|\varphi\|_{X_{\lambda}}^2 := \int (|\nabla \varphi|^2 + \lambda a(x) \varphi^2), \quad \|\psi\|_{X_{\lambda}}^2 := \int (|\nabla \psi|^2 + \lambda b(x) \psi^2).
\]

Given \(0 < \varepsilon < 1\), we may choose \(R = R(\varepsilon) > 0\) such that

\[
\max \left\{ \|u\|_{L^{2^*}(B_R)}, \mathcal{L}(B_R \cap F)^{2^*/N}, \left( \int_{B_R} b v^2 \right)^{\beta/2} \right\} < \varepsilon,
\]

with \(\beta > 0\) given by Lemma 2.1. Using Hölder’s inequality, the imbedding \(\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)\) and \(\|\varphi\|_{X_{\lambda}} \leq 1\), we get

\[
\int_{B_R} |f_n| \varphi \leq \left( \int_{B_R} |f_n|^{2N/(N+2)} \right)^{(N+2)/2N} \left( \int_{B_R} |\varphi|^{2^*} \right)^{1/2^*} \leq c_1 \left( \int_{B_R} |f_n|^{2N/(N+2)} \right)^{(N+2)/2N}.
\]
Setting $\theta := (p + q - 1)2N/(N + 2) < 2^s$, we may suppose that

$$(u_n, v_n) \to (u, v) \quad \text{strongly in } L^\theta(B_R) \times L^\theta(B_R),$$

$$(u_n(x), v_n(x)) \to (u(x), v(x)) \quad \text{a.e. in } B_R,$$

$$|u(x)|, |v(x)|, |u_n(x)|, |v_n(x)| \leq h_R(x) \in L^\theta(B_R) \quad \text{a.e. in } B_R(0).$$

Hence,

$$|f_n| \leq |u_n - u|^p - 1|v_n - v|^q + |u_n|^p - 1|v_n|^q + |u|^{p-1}|v|^q \leq c_2 h_R^{p+q-1},$$

and therefore $|f_n(x)|^{2N/(N+2)} \leq c_2 h_R^\theta(x) \in L^1(B_R)$, almost everywhere in $B_R$. Since $f_n(x) \to 0$ almost everywhere in $B_R$, the Lebesgue Dominated Convergence Theorem and (4.14) imply that

$$\lim_{n \to \infty} \int_{B_R} |f_n||\varphi| = 0 \quad \text{uniformly for } \|\varphi\|_{X_a} \leq 1. \quad (4.15)$$

On the other hand, by adding and subtracting the term $|u_n - u|^p - 2(u_n - u)|v_n|^q$ to $f_n$, we have

$$\int_{B_R^c} |f_n||\varphi| \leq \int_{B_R^c} s_n|\varphi| + \int_{B_R^c} t_n|\varphi| + \int_{B_R^c} |u|^{p-1}|v|^{q-1}|\varphi v| \quad (4.16)$$

with

$$s_n := |u_n - u|^p - 1|v_n - v|^q - |v_n|^q,$$

and

$$t_n := \left||u_n - u|^p - 2(u_n - u) - |u_n|^p - 2u_n\right||v_n|^q.$$

Using (4.1), (4.2) and (4.13), we may estimate the last term in (4.16) as follows

$$\int_{B_R^c} |u|^{p-1}|v|^{q-1}|\varphi v| \leq c_3 e^{p-1} \quad \text{for any } \|\varphi\|_{X_a} \leq 1. \quad (4.17)$$

Now, we proceed with the estimate of $\int_{B_R^c} s_n|\varphi|$. Setting $w(t) := |v_n - tv|^q$, recalling that $q > 1$ and using the Mean Value Theorem we obtain

$$|w(1) - w(0)| = |\left||v_n - v|^q - |v_n|^q\right| \leq q|v_n - t_0 v|^q - 1|v|$$

for some $t_0 \in [0, 1]$. The boundedness of $(u_n, v_n)$ in $X$, $t_0 \in [0, 1]$, (4.1), (4.2) and (4.13) imply that

$$\int_{B_R^c} s_n|\varphi| \leq q \int_{B_R^c} |u_n - u|^p - 1|v_n - t_0 v|^q - 1|\varphi v| \leq c_4 e \quad \text{for any } \|\varphi\|_{X_a} \leq 1. \quad (4.18)$$

The estimates for $\int_{B_R^c} t_n|\varphi|$ are more involved since we may have $p - 1 < 1$. We consider two possible cases:
Case 1.  \( p \geq 2 \).

Suppose first \( p > 2 \) and define \( w(t) := |u_n - tu|^p - 2 |u_n - tu| \). Applying the Mean Value Theorem and proceeding as in (4.17) we obtain

\[
\int_{B_c^R} t_n |\varphi| \leq (p - 1) \int_{B_c^R} |u_n - t_0 u|^p - 2 |u||v_n|^{q - 1} |\varphi v_n|
\]

\[
\leq c_5 \left( \int_{B_c^R} |u|^{p - 1} |v_n|^{q - 1} |\varphi v_n| + \int_{B_c^R} |u_n|^{p - 2} |u||v_n|^{q - 1} |\varphi v_n| \right)
\]

\[
\leq c_6 \left( \varepsilon^{p - 1} + \int_{B_c^R} |u_n|^{p - 2} |u||v_n|^{q - 1} |\varphi v_n| \right). \quad (4.19)
\]

In order to estimate the last integral we apply Hölder’s inequality with

\[
\frac{p - 2}{2^*} + \frac{1}{2^*} + \frac{q - 1}{2^*} + \frac{1}{r} = 1,
\]

to get

\[
\int_{B_c^R} |u_n|^{p - 2} |u||v_n|^{q - 1} |\varphi v_n| \leq c_7 \|u\|_{L^{2^*}(B_c^R)} \left( \int_{B_c^R} |\varphi v_n|^r \right)^{1/r} \leq c_8 \varepsilon,
\]

where we have used, in the last inequality, (4.13), \( \|\varphi\|_{X_a} \leq 1 \), the boundedness of \( (v_n) \) in \( X_b \), the same calculation performed in (2.3) and (4.2). So,

\[
\int_{B_c^R} t_n |\varphi| \leq c_9 \varepsilon \quad \text{for any } \|\varphi\|_{X_a} \leq 1.
\]

If \( p = 2 \) the second integral in the second line of (4.19) does not appear and therefore the above estimate holds in this case too.

Case 2.  \( 1 < p < 2 \).

In this case the derivative of the function \( w \) defined in the first case can be singular, and we may not apply the Mean Value Theorem directly. In order to overcome this difficult, we first set

\[
h_n(x) := |u_n - u|^p - 2 (u_n - u) - |u_n|^{p - 2} u_n.
\]

As before, we have that

\[
\int_{B_c^R} t_n |\varphi| = \int_{B_c^R} |h_n||v_n|^{q - 1} |\varphi v_n|
\]

\[
\leq c_{10} \left( \int_{B_c^R} |h_n|^{2^*/(p - 1)} \right)^{(p - 1)/2^*} . \quad (4.20)
\]
We claim that the last integral in the above inequality is small. Indeed, first note that

$$|h_n| \leq c_{11}|u|^{p-1} \quad \text{a.e. in the set } \{ |u_n| \leq 2|u| \} \cup \{ |u| = 0 \}. \quad (4.21)$$

On the other hand, in the set \( \{ |u_n| > 2|u| > 0 \} \), as in the first case, we may apply the Mean Value Theorem for \( w(t) := |u_n - tu|^{p-2}(u_n - tu) \) to get

$$|h_n| = |w(1) - w(0)| \leq c_{12}|u_n - t_0 u|^{p-2}|u| \leq c_{13}|u|^{p-1},$$

for some \( t_0 \in [0, 1] \), since for \( |u_n| > 2|u| \) we have that \( |u_n - t_0 u| \geq |u_n| - |t_0||u| \geq |u| \). This, (4.21) and (4.13) imply that

$$\int_{B_R} |h_n|^{2^*/(p-1)} \leq c_{14} \int_{B_R} |u|^{2^*} \leq c_{15} \varepsilon^{2^*}.$$

It follows from (4.20) that

$$\int_{B_R} t_n |\varphi| \leq c_{16} \varepsilon^{p-1} \quad \text{for any } \|\varphi\|_{X_a} \leq 1.$$

All together, the two cases provide

$$\int_{B_R} t_n |\varphi| \leq c_{17} \varepsilon^{\min\{1, p-1\}} \quad \text{for any } \|\varphi\|_{X_a} \leq 1.$$

Thus, we may use (4.15)–(4.18) and the above estimate to conclude that

$$\int_{B_R} |f_n| |\varphi| = \int_{B_R} |f_n| |\varphi| + \int_{B_R^c} |f_n| |\varphi| \leq c_{18} \varepsilon^{\min\{1, p-1\}},$$

for any \( \|\varphi\|_{X_a} \leq 1 \) and \( n \geq n_0 \). Since \( \varepsilon > 0 \) is arbitrary we conclude that the first equality (4.12) holds. The second one may be verified in a similar way and this concludes the proof of Lemma 4.3. \( \square \)

In the sequel we follow [5] in order to obtain a local compactness property for the functional \( I_{\lambda} \).

**Proposition 4.4.** For any given \( C_0 > 0 \) there exists \( \Lambda = \Lambda(p, q, C_0) > 0 \) such that \( I_{\lambda} \) satisfies \((PS)_e\) for any \( c \leq C_0 \) and \( \lambda \geq \Lambda \).

**Proof.** Let \( \gamma_0 \) be given by Lemma 2.3(iii) and fix \( \varepsilon > 0 \) such that

$$\varepsilon < \frac{\gamma_0}{2} \left( \frac{1}{2} - \frac{1}{p+q} \right)^{-1}.$$

Fixed \( C_0 > 0 \), let \( \Lambda_\varepsilon \) and \( R_\varepsilon \) be given by Lemma 2.4. We will prove that the proposition holds for \( \Lambda := \Lambda_\varepsilon \). Let \( (z_n) = ((u_n, v_n)) \subset X \) be a \((PS)_e\) sequence for \( I_{\lambda} \) with \( c \leq C_0 \) and \( \lambda \geq \Lambda \). In view of Lemma 4.3 we may suppose that \( (u_n, v_n) \rightharpoonup z := (u, v) \) weakly in \( X \) and \( z := (u_n - u, v_n - v) \) is
a (PS)$_c'$ sequence for $I_\lambda$, with $c' = c - I_\lambda(z)$. We claim that $c' = 0$. If this is true, it follows from Lemma 2.3(ii) that
\[ \lim_{n \to \infty} \|z_n\|^2_\lambda = c' \left( \frac{1}{2} - \frac{1}{p + q} \right)^{-1} = 0, \]
that is, $z_n \to z$ in $X$.

Suppose, by contradiction, that $c' \neq 0$. Lemma 2.3(iii) implies that $c' \geq \gamma_0 > 0$. Since $\tilde{u}_n, \tilde{v}_n \to 0$ in $L^2(B_{R_\varepsilon})$, we may use Lemma 2.3(ii), Lemma 2.4, the same calculation of (3.2) and the choice of $\varepsilon > 0$, to get
\[ \gamma_0 \left( \frac{1}{2} - \frac{1}{p + q} \right)^{-1} \leq c' \left( \frac{1}{2} - \frac{1}{p + q} \right)^{-1} = \lim_{n \to \infty} \int_{B_{R_\varepsilon}} |\tilde{u}_n|^p |\tilde{v}_n|^q + \int_{B_{R_\varepsilon}^c} |\tilde{u}_n|^p |\tilde{v}_n|^q \]
\[ \leq \gamma_0 \left( \frac{1}{2} - \frac{1}{p + q} \right)^{-1}, \quad (4.22) \]
which contradicts $\gamma_0 > 0$. This contradiction finishes the proof. $\square$

We are now ready to prove Theorems 1.2 and 1.3 as follows.

**Proof of Theorem 1.2.** We first take a bounded open smooth set $\Omega \subset \Omega_a \cap \Omega_b$. Given $k \in \mathbb{N}$ we set $W := \text{span}\{(\varphi_1, \varphi_1), \ldots, (\varphi_k, \varphi_k)\}$, where $\varphi_i$ is an eigenfunction corresponding to the $i$-th eigenvalue of $(-\Delta, H^1_0(\Omega))$. For each $i = 1, \ldots, k$ we have that
\[ \lim_{t \to \infty} I_\lambda(t(\varphi_i, \varphi_i)) = \lim_{t \to \infty} \left( \frac{1}{2} - \frac{t^{p+q}}{p + q} \int |\varphi_i|^2 \frac{1}{p + q} \int |\varphi_i|^{p+q} \right) = -\infty, \]
uniformly on $\lambda$. Since $W$ has finite dimension we obtain $M_k > 0$, independent of $\lambda > 0$, such that
\[ \sup_{z \in W} I_\lambda(z) < M_k. \]

Moreover, as in the proof of Lemma 2.5 we may obtain $\rho > 0$, independent of $\lambda > 0$, such that
\[ I_\lambda(z) \geq 0 \quad \text{for any } \|z\|_\lambda = \rho. \]

In view of Proposition 4.4 there exists $\Lambda_k > 0$ such that $I_\lambda$ satisfies (PS)$_c$ for any $c \leq M_k$ and $\lambda \geq \Lambda_k$. Thus, for any fixed $\lambda \geq \Lambda_k$ we may apply Theorem 4.1 to obtain $k$ pairs of nontrivial solutions. The theorem is proved. $\square$

**Proof of Theorem 1.3.** We first notice that
\[ \left( \frac{1}{2} - \frac{1}{p + q} \right) \|z_{\lambda_n}\|^2_{\lambda_n} = I_{\lambda_n}(z_{\lambda_n}) - \frac{1}{2} I'_{\lambda_n}(z_{\lambda_n}) = I_{\lambda_n}(z_{\lambda_n}). \]
Since $\liminf_{n \to \infty} I_\lambda(z_{\lambda_n}) < \infty$ we may suppose, taking a subsequence if necessary, that $(z_{\lambda_n})$ is bounded. Thus, up to a subsequence, we have that
\[
\begin{align*}
  z_{\lambda_n} \to \bar{z} := (\bar{u}, \bar{v}) & \text{ weakly in } \mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N), \\
  (u_n, v_n) \to (\bar{u}, \bar{v}) & \text{ strongly in } L^p_{\text{loc}}(\mathbb{R}^N) \times L^q_{\text{loc}}(\mathbb{R}^N), \\
  (u_n(x), v_n(x)) \to (\bar{u}(x), \bar{v}) & \text{ a.e. in } \mathbb{R}^N.
\end{align*}
\]

(4.23)

As in the proof of the last statement of Theorem 1.1 we can show that \( \bar{u} \in H^1_0(\Omega_a), \bar{v} \in H^1_0(\Omega_b) \) and \( \bar{z} \) is a solution of \( (L) \).

Given \( \varepsilon > 0 \) we can argue as in the proof of Lemma 2.4 to conclude that, for some \( R > 0 \) large, there holds

\[
\limsup_{n \to \infty} \int_{B_R(0)^c} |u_n|^p |v_n|^q \leq \varepsilon.
\]

By taking \( R \) larger if necessary, we may suppose that \( \int_{B_R(0)^c} |\bar{u}|^p |\bar{v}|^q \leq \varepsilon \). Moreover, the local convergence in (4.23) and the Lebesgue Dominated Convergence Theorem imply that \( \int_{B_R(0)} |u_n|^p |v_n|^q \to \int_{B_R(0)} |\bar{u}|^p |\bar{v}|^q \) as \( n \to \infty \). Since

\[
\left| \int_{B_R(0)} |u_n|^p |v_n|^q - |\bar{u}|^p |\bar{v}|^q \right| \leq \int_{B_R(0)^c} |u_n|^p |v_n|^q + \int_{B_R(0)^c} |\bar{u}|^p |\bar{v}|^q \\
+ \int_{B_R(0)} \left( |u_n|^p |v_n|^q - |\bar{u}|^p |\bar{v}|^q \right),
\]

it follows from the above estimates and convergences that

\[
\limsup_{n \to \infty} \left| \int_{B_R(0)} |u_n|^p |v_n|^q - |\bar{u}|^p |\bar{v}|^q \right| \leq 2\varepsilon,
\]

and therefore

\[
\lim_{n \to \infty} \int_{B_R(0)} |u_n|^p |v_n|^q = \int_{B_R(0)} |\bar{u}|^p |\bar{v}|^q.
\]

Thus, we can argue as in the final of the proof of Theorem 1.1 to conclude that \( \|z_{\lambda_n} - z\|_0 \leq \|z_{\lambda_n} - \bar{z}\|_{\lambda_n} \to 0 \) as \( n \to \infty \). Hence, \( z_n \to z \) strongly in \( \mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N) \) and the theorem is proved. \( \Box \)

References