# On pairs of additive congruences of odd degree*

H. Godinho†(Brasília, DF), J.F. Lima Neto (Brasília, DF)
P.H.A. Rodrigues‡(Goiânia, GO)

**Abstract**

Here we present some conditions for pairs of additive congruences of odd degree to have non-trivial solutions and non-singular solutions modulo a prime $p$. For non-trivial solutions it is proved that pairs of additive congruences in $n$ variables and odd degree $d$ ($d \geqslant 5$ and $d \neq$(p-1/2)) have common non-trivial zeros provided $n \geqslant \frac{11}{6}d+1$. Later we consider pairs of additive forms of odd degree $d$ ($d \geqslant 5$ and $d \neq$(p-1/2)) in $n$ variables over $p$-adic fields and proved that one can guarantee $p$-adic common zeros provided $n \geqslant \frac{11}{6}d^2+d$.

## 1 Introduction

It follows from Chevalley [4] that the system

$$\begin{array}{rl} f = & a_{11}x_1^d + \cdots + a_{1N}x_N^d \equiv 0 \pmod p \\ g = & a_{21}x_1^d + \cdots + a_{2N}x_N^d \equiv 0 \pmod p \end{array} \tag{1}$$

has non-trivial solutions if $N \geq 2d+1$. It is also known that this result is best possible in the sense that there are examples of systems with degree d=p-1 and in N=2d variables with no non-trivial solutions. On the other hand it is expected that once we have the degree $d \not\equiv 0 \,(\mathrm{mod}\ p(p-1))$, one should need fewer variables to guarantee non-trivial solutions. The next theorem confirms this idea, for odd degrees $d \geqslant 5$ and d≠(p-1)/2.

**Theorem 1.1** *Let $d \geqslant 5$ be an odd integer and $p$ be a prime, $d \neq (p-1)/2$. If $N \geqslant \frac{11}{6}d+1$ then the system (1) has non-trivial solutions.*

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Our intent is also solve a similar system as in (1) for powers of $p$, or equivalently to find $p$-adic zeros for pairs of additive forms (this equivalence can be seen for example in Borevich and Shafarevich [3], ch. 1, sec. 5). A necessary step is to look for non-singular solutions for (1) (see definition 5.1). For general degrees, the best result was given by Davenport and Lewis [8]. They proved

**Theorem 1.2** *The system (1) has non-singular solutions, provided* $N \geqslant 2d+1$ *and any linear combination* $\lambda f + \mu g$ *has at least* $d+1$ *variables with non-zero coefficients modulo* $p$.

In the particular case of odd degrees, we were able to find an improvement for this result, stated on the next theorem

**Theorem 1.3** *Let* $d \in \mathbb{N}$, *an odd integer,* $d \geqslant 5$ *and* $p$ *be a prime,* $d \neq (p-1)/2$. *The system (1) has non-singular solutions, provided* $N \geqslant \frac{11}{6}d + 1$ *and any linear combination* $\lambda f + \mu g$ *has at least* $\frac{d+1}{2}$ *variables with coefficients not divisible by* $p$.

There is an extensive literature on pairs of $p$-adic forms, where the guideline is a special case of a conjecture made by E. Artin that says: *any pair of additive forms in* $n$ *variables, and of degree* $d$, *has* $p$*-adic zeros, provided* $n > 2d^2$. For odd degree this conjecture was verified by Davenport and Lewis[9], and for general degrees the best result is $n > 8d^2$, given by Brüdern and Godinho[2].

We would like to close this paper presenting a result on $p$-adic zeros for pairs of additive forms of odd degree, showing that the estimate given by Artin's conjecture can be improved.

**Theorem 1.4** *Any pair of additive forms of odd degree* $d \geqslant 5$, *in* $n$ *variables, has common* $p$*-adic zeros for all primes* $p$, *provided* $n \geqslant \frac{11}{6}d^2 + d$ *and* $d \neq (p-1)/2$.

In fact, for large values of $d$, this can be improved even more, as was proved by Davenport and Lewis[10]. They proved that, for pairs, $p$-adic zeros are guaranteed if $n > 36d \log 6d$. An interesting result, also in this direction, is presented in Atkinson and Cook[1].

# 2 Combinatorial Lemmas

In all that follows we are going to assume that $\lceil \alpha \rceil$ is the smallest integer bigger than or equal to $\alpha$, and $\lfloor \beta \rfloor$ is the biggest integer smaller than or equal to $\beta$.

**Lemma 2.1** *Let* $d, r \in \mathbb{N}$, $d \geqslant 3$ *and* $2 \leq r \leq d-1$. *And consider the set*

$$\mathcal{C} = \{t_0, t_1, \ldots, t_{d-1}\}$$

*of the triples $t_i = (i, i+1, i+r)$ modulo $d$. We say that $t_i$ and $t_j$ are disjoint if the sets $\{i, i+1, i+r\}$ and $\{j, j+1, j+r\}$ are disjoint.*

*Then, among the elements of $\mathcal{C}$ we can find at least $\lceil \frac{d-1}{6} \rceil$ parwise disjoint elements.*

*Proof*: Let $T_0 = t_0$ and $C_1 = \{T_0\}$. Observe that given $T_0 = (0, 1, r)$, there will be exactly two other $t_j$'s having 0 in one of the other two coordinates. And the same happens with 1 and $r$. In fact, the subset of $\mathcal{C}$ of all $t_j$'s that are not disjoint of $T_0$ is (including $t_0$)

$$E_1 = \{t_{d-1}, t_0, t_1, t_r, t_{r-1}, t_{d-r}, t_{d-r+1}\}, \quad \text{(thus } |E_1| = 7)$$

since $t_{d-r} = (d-r, d-r+1, 0)$ and $t_{d-r+1} = (d-r+1, d-r+2, 1)$ modulo $d$.

Let $e_1 = |\mathcal{C} - E_1| = (d-1) - 6$. If $e_1 > 0$, we can choose $T_1 = t_{i_1}$ where $i_1$ is the smallest index such that $t_{i_1} \notin E_1$. Now let $C_2 = \{T_0, T_1\}$. The subset of $\mathcal{C}$ of all $t_j$'s that are not disjoint of $T_1$ is (including $T_1$)

$$E_2 = \{t_{i_1-1},\, t_{i_1},\, t_{i_1+1},\, t_{r+i_1-1},\, t_{r+i_1},\, t_{d-r+i_1},\, t_{d-r+i_1+1}\}.$$

Since $t_{i_1-1} \in E_1 \cap E_2$, we have $|\mathcal{C} - (E_1 \cup E_2)| \geqslant e_2 = (d-1) - 2 \times 6$. This argument can be repeated until we obtain

$$C_j = \{T_0, T_1, \ldots, T_{j-1}\}, \text{ where } e_j = (d-1) - j \times 6 \in \{0, 1, 2, 3, 4, 5\}$$

If $e_j = 0$ we have $|C_j| = \frac{d-1}{6} = \lceil \frac{d-1}{6} \rceil$. Otherwise we can choose another $T_j$ obtaining $|C_{j+1}| = j + 1 = \lfloor \frac{d-1}{6} \rfloor + 1 = \lceil \frac{d-1}{6} \rceil$. □

**Lemma 2.2** *If among $c_1, \cdots, c_s \in \mathbb{N}$ we can find exactly $t$ distinct elements, then we can form at least $\lfloor \frac{s-t+1}{2} \rfloor$ disjoints pairs of equal elements.*

*Proof.* If necessary, renumber the $c_i$'s such that $c_1, \cdots, c_t$ are pairwise distincts. Let $v_j$ be the number of elements among $c_1, \cdots, c_s$ that are aqual to $c_j$ for $j = 1, 2, \cdots, t$. Thus $s = v_1 + v_2 + \cdots + v_t$. Hence the number of disjoint pairs is equal to $N = \lfloor \frac{v_1}{2} \rfloor + \cdots + \lfloor \frac{v_t}{2} \rfloor$. And it is easy to see that

$$\frac{s}{2} \geqslant N \geqslant \frac{v_1 - 1}{2} + \cdots + \frac{v_t - 1}{2} = \frac{s-t}{2}.$$

If $s \equiv t \pmod 2$ then $\frac{s-t}{2} = \lfloor \frac{s-t+1}{2} \rfloor$. Thus suppose $s \not\equiv t \pmod 2$, but this implies that some $v_j$ must be even, say $v_1$. Thus

$$\left\lfloor \frac{v_1}{2} \right\rfloor + \cdots + \left\lfloor \frac{v_t}{2} \right\rfloor \geqslant \frac{v_1}{2} + \frac{v_2 - 1}{2} + \cdots + \frac{v_t - 1}{2} = \frac{s-t+1}{2} = \left\lfloor \frac{s-t+1}{2} \right\rfloor.$$

□

**Lemma 2.3** *Let $d \in \mathbb{N}$, $d \geqslant 5$, $d \neq 6$ and $p$ a prime, $p \equiv 1 \pmod{d}$. Let $\mathbb{F}_p^d$ be the subgroup of $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$ of all $d$-th powers. Then there exist $\delta \in \mathbb{F}_p$ and $r \in \{2, \dots, d-1\}$ such that*

$$\mathbb{F}_p^* = \mathbb{F}_p^d \cup \delta\mathbb{F}_p^d \cup \cdots \cup \delta^{d-1}\mathbb{F}_p^d \quad \textit{(disjoint union)}$$

*and for some $a, b \in \mathbb{F}_p^*$ we have in $\mathbb{F}_p$*

$$1 + \delta a^d + \delta^r b^d = 0. \tag{2}$$

*Proof.* Let $\tau$ be a primitive root of $\mathbb{F}_p^*$, hence we can write

$$\mathbb{F}_p^* = \mathbb{F}_p^d \cup \tau\mathbb{F}_p^d \cup \cdots \cup \tau^{d-1}\mathbb{F}_p^d.$$

Now define

$$X = \{a \in \mathbb{F}_p^* \,|\, a \in \tau^i\mathbb{F}_p^d \quad \text{and} \quad \gcd(i, d) = 1\}$$

and

$$W = \{-b^d - 1 \neq 0 \;\mid\; b \in \mathbb{F}_p^*\}.$$

Thus the set $V = \mathbb{F}_p^d \cup W \cup W^{-1}$ has at most $3\left(\dfrac{p-1}{d}\right)$ elements. Since $d \geqslant 5$, $d \neq 6$ then $\phi(d) > 3$ (the Euler function) and

$$|X| = \phi(d)\left(\frac{p-1}{d}\right) > 3\left(\frac{p-1}{d}\right) \geqslant |V|.$$

Hence we can find $\alpha \in X \setminus V$, thus $\alpha = \tau^i a^d$, $\gcd(i, d) = 1$ and $\alpha, \alpha^{-1} \notin \mathbb{F}_p^d$. Since $\alpha \notin W \cup W^{-1}$, we must have $-1 - \alpha \notin \mathbb{F}_p^d$ and $-1 - \alpha^{-1} \notin \mathbb{F}_p^d$ . Therefore $-1 - \alpha = \alpha(-1 - \alpha^{-1}) \notin \alpha\mathbb{F}_p^d = \tau^i\mathbb{F}_p^d$. Now, since $(i, d) = 1$, we also have

$$\mathbb{F}_p^* = \mathbb{F}_p^d \cup \tau^i\mathbb{F}_p^d \cup \cdots \cup (\tau^i)^{d-1}\mathbb{F}_p^d.$$

Hence $-1 - \alpha \in (\tau^i)^r\mathbb{F}_p^d$, for some $r \geqslant 2$ and thus $-1 - \tau^i a^d = (\tau^i)^r b^d$. This gives the result, taking $\delta = \tau^i$ and since $\gcd(i, d) = 1$. □

**Proposition 2.4** *Let $d \in \mathbb{N}$, $d \geqslant 5$, $d \neq 6$ and $p$ a prime, $p \equiv 1 \pmod{d}$. Let $\delta$ be the element given in lemma 2.3. If*

$$\mathbb{A} = \{a_1, \dots, a_s\} \;\subseteq\; \mathbb{S} = \{1, \delta, \dots, \delta^{d-1}\}$$

*then we can find at least $\lceil \frac{d-1}{6} \rceil - (d\text{-}s)$ disjoint subsets $\{a_i, a_j, a_k\}$ of $\mathbb{A}$, such that the congruences*

$$a_i x^d + a_j y^d + a_k z^d \equiv 0 \,(mod\, p)$$

*have non-trivial solutions.*

*Proof.* Lemma 2.3 gives us a $\delta$ such that

$$1 + \delta a^d + \delta^r b^d \equiv 0 \pmod p$$

with $a \cdot b \not\equiv 0 \pmod p$. Hence, for all $\delta^i \in \mathbb{S}$ we have

$$\delta^i + \delta^{i+1} a^d + \delta^{i+r} b^d \equiv 0 \pmod p. \tag{3}$$

Let us associate each of the forms (3) with the triple of their exponents modulo $d$, that is,

$$\delta^i + \delta^{i+1} a^d + \delta^{i+r} b^d \quad \longleftrightarrow \quad t_i = (i, i+1, i+r) \quad (\text{modulo } d).$$

And the question becomes, how many disjoint $t_i$'s can we have? Now lemma 2.1 tells us that when $\mathbb{A} = \mathbb{S}$ we can have at least $\left\lceil \frac{d-1}{6} \right\rceil$ disjoint triples. Now observe that if you take one element out of $\mathbb{S}$, you may lose one of these triples. If you take two elements out of $\mathbb{S}$ you may still lose the same triple (they have three coordinates), but in the worst case, you lose two triples. Now the result follows from this reasoning. $\square$

# 3 Additive congruences

In this section we want to determine conditions for the congruence

$$a_1 x_1^d + \cdots + a_n x_n^d \equiv 0 \pmod p \tag{4}$$

to have non-trivial solutions. It is quite easy to see that if $\mathbb{F}_p^d$ is the set of all d-th powers modulo $p$, then $\mathbb{F}_p^d = \mathbb{F}_p^\delta$, where $\delta = \gcd(d, p-1)$. Hence, with no loss in generality, we are going to assume throughout this paper that $p \equiv 1 \pmod d$. Let us start this section by recalling a famous result of Chowla, Mann and Straus [5].

**Theorem 3.1** *Let $d, p \in \mathbb{N}$, $p$ a prime number. If $d \neq (p-1)/2$, $n \geqslant (d+1)/2$ and $a_1 \cdots a_n \not\equiv 0$ (mod $p$) then for any $b$ there is always a solution for the congruence*

$$a_1 x_1^d + \cdots + a_n x_n^d \equiv b \ (mod\, p).$$

It is important to observe that theorem 3.1 does not guarantee the existence of non-trivial solutions for (4). For degrees $d = 3$ and $d = 5$, its is known that (4) has non-trivial solutions for $n \geqslant 3$ (proved by D. J. Lewis [12] , and Atkinson and Cook [1] respectively), hence we may assume $d \geqslant 7$. We present next a general proof of this result for odd degrees based on standard methods of exponential sums (a thorough study of additive congruences can be found in the paper of M. Dodson [11]).

Let $\mathcal{N}$ be the number of solutions of (4) and let $\zeta_p$ be a complex primitive $p$-th root of 1. Now define

$$T(\alpha) = \sum_{x=0}^{p-1} \zeta_p^{\alpha x^d}. \tag{5}$$

It is simple to see that (for details see [3] or [1])

$$\mathcal{N} \;=\; p^{-1} \sum_{u=0}^{p-1} T(a_i u) \cdots T(a_n u) \;=\; p^{n-1} + p^{-1} \sum_{u=1}^{p-1} T(a_i u) \cdots T(a_n u).$$

Let us now define

$$S_r = \sum_{u=1}^{p-1} |T(u)|^r. \tag{6}$$

By Hölder inequality we have

$$\sum_{u=1}^{p-1} |T(a_1 u) \cdots T(a_n u)| \leq (\sum_{u=1}^{p-1} |T(a_1 u)|^n)^{\frac{1}{n}} \cdots (\sum_{u=1}^{p-1} |T(a_n u)|^n)^{\frac{1}{n}}.$$

Since

$$\sum_{u=1}^{p-1} |T(au)|^n = \sum_{u=1}^{p-1} |T(u)|^n, \text{ if } a \not\equiv 0 \pmod{p},$$

we have that

$$\sum_{u=1}^{p-1} |T(a_1 u) \cdots T(a_n u)| \leq S_n.$$

Therefore, the congruence (4) above has non-trivial solutions whenever

$$\mathcal{N} \geqslant p^{n-1} - p^{-1} S_n > 1. \tag{7}$$

Now we can use the following classical results

$$S_2 \;=\; (d-1)p(p-1), \quad \text{(see [11], lemma 2.5.1)}, \tag{8}$$

$$|T(u)| \;\leq\; (d-1)p^{\frac{1}{2}}, \text{ for } u \not\equiv 0 \pmod{p} \quad \text{(see [6], lemma 12)}. \tag{9}$$

From (6), (8) and (9) follows that

$$S_n = \sum_{u=1}^{p-1} |T(u)|^n = \sum_{u=1}^{p-1} |T(u)|^{n-2} |T(u)|^2 \leq (d-1)^{n-1} p^{\frac{n}{2}-1} p(p-1).$$

Replacing this in (7) we have

$$N \geqslant p^{n-1} - (d-1)^{n-1} p^{\frac{n}{2}-1}(p-1).$$

Thus if

$$p^{\frac{n}{2}-1} > (d-1)^{n-1}, \tag{10}$$

then the congruence (4) has non-trivial solutions (the inequality (10) can also be found in Dodson[11]).

**Lemma 3.1** *Let $d$ be an odd integer and $p$ a prime. If $p < 2^n$ then the congruence (4) will have non-trivial solutions.*

*Proof.* This is essentially lemma 2.2.1 in Dodson[11].

**Lemma 3.2** *Let $d$ be an odd integer $d \geqslant 21$, and $p$ a prime. Then the congruence (4) has non-trivial solutions provided $n \geqslant \dfrac{d+1}{2}$*

*Proof.* Let us assume $n = \frac{d+1}{2}$. Since we are also assuming $d \geqslant 21$, then $2(d-1)^2 < 2^{\frac{d+1}{2}}$. Hence it follows from lemma 3.1 that (4) has non-trivial solutions for all primes $p < 2(d-1)^2$. So let us assume

$$p > 2(d-1)^2. \tag{11}$$

Since $d \geqslant 21$ then

$$2^{\frac{d-3}{4}} \geqslant (d-1). \tag{12}$$

Now follows from (11) and (12) that

$$p^{\frac{d-3}{4}} > 2^{\frac{d-3}{4}}(d-1)^{\frac{d-3}{2}} > (d-1)^{\frac{d-1}{2}}.$$

Hence when $n = (d+1)/2$ we have non-trivial solutions for (4), by (10). □

Now let us considerer the remaining degrees $7 \leq d \leq 19$. It follows from lemma 3.1 and (10) that we need to consider only the primes in the interval (since $n = (d+1)/2$)

$$2^{\frac{d+1}{2}} < \ p \ < (d-1)^{2+\frac{4}{d-3}}.$$

And it is easy to verify that there are no primes $p \equiv 1 \pmod{d}$, in the interval above for $d = 19$ and 17. Thus we are left with the list below

| degree: $d$ | number of variables: $n = (d+1)/2$ | primes: $p \equiv 1 \pmod{d}$ |
|---|---|---|
| 7 | 4 | $29, 43, 71, 113, 127, 197, 211$ |
| 9 | 5 | $37, 73, 109, 127, 163, 181, 199$ |
| 11 | 6 | $67, 89, 199$ |
| 13 | 7 | $131, 157, 313$ |
| 15 | 8 | $271, 331, 421$ |

Let $\tau \in \mathbb{F}_p^*$ be a primitive root. Then $\mathbb{F}_p^d = \langle \tau^d \rangle$ and

$$\mathbb{F}_p^* = \mathbb{F}_p^d \cup \tau\mathbb{F}_p^d \cup \cdots \cup \tau^{d-1}\mathbb{F}_p^d. \tag{13}$$

Let us define

$$T_i = |\sum_{x=0}^{p-1} \zeta_p^{x^d \tau^{i-1}}| \ \text{ for } i = 1, \ldots, d.$$

Hence for every $u \in \tau^i\mathbb{F}_p^d$, we have $|T(u)| = T_{i+1}$ (see(5)). Thus we can write (see (6))

$$S_n = \frac{p-1}{d} \sum_{i=1}^{d} T_i^n.$$

Using this formula we can calcule the values of $S_n$ for every value of $d$ and $p$ listed in the table above ($n = (d+1)/2$). And we have found that (7) holds for all cases but $d = 7$ and $p = 29, 43$, $d = 9$ and $p = 37$, and $d = 11$ and $p = 67$ (the results are listed in the table below).

| $d$ | $p$ | $n$ | $\tau$ | $S_n$ | $p^{n-1} - p^{-1}S_n$ |
|---|---|---|---|---|---|
| 7 | 71 | 4 | 7 | 20824299.99 | 64611.0001 |
| 7 | 113 | 4 | 3 | 137013855.8 | 230385.002 |
| 7 | 127 | 4 | 3 | 158451804.2 | 800730.998 |
| 7 | 197 | 4 | 2 | 567210279.3 | 4766133.004 |
| 7 | 211 | 4 | 2 | 818405699.9 | 5515231.0 |
| 9 | 73 | 5 | 5 | 1939984892.0 | 1823105.49 |
| 9 | 109 | 5 | 6 | 6477144852.0 | 81734813.73 |
| 9 | 127 | 5 | 3 | 20872678790.0 | 95792839.5 |
| 9 | 163 | 5 | 2 | 28247342190.0 | 532615183.1 |
| 9 | 181 | 5 | 2 | 52937667830.0 | 780809818.1 |
| 9 | 199 | 5 | 3 | 144175667500.0 | 843738359.3 |
| 11 | 89 | 6 | 3 | 403009884500.0 | 1055858500.0 |
| 11 | 199 | 6 | 3 | 10783376170000.0 | 257891781000.0 |
| 13 | 131 | 7 | 2 | 212641952900000.0 | 3430692129000.0 |
| 13 | 157 | 7 | 5 | 613405793900000.0 | 11069028560000.0 |
| 13 | 313 | 7 | 10 | $1.243236294 \times 10^{16}$ | $9.005791011 \times 10^{14}$ |
| 15 | 271 | 8 | 6 | $6.265491168 \times 10^{17}$ | $1.050338055 \times 10^{17}$ |
| 15 | 331 | 8 | 3 | $8.787668395 \times 10^{18}$ | $4.087584591 \times 10^{17}$ |
| 15 | 421 | 8 | 2 | $3.599570724 \times 10^{19}$ | $2.258591606 \times 10^{18}$ |

For the remaining four cases we observe that the coefficients $a_i$'s of (4) can be written in the form $a_j = \tau^i \alpha_i^d$ (see (13)), and using the substitutions

$$x_i \to \alpha_i x_i, \quad \text{for } \ i = 1, \dots, n \tag{14}$$

we can then make the extra hypothesis that $a_i \in \{1, \tau, \tau^2, \dots, \tau^{d-1}\}$ for $\ i = 1, 2, \dots, n$. Since the degree is odd, we can exclude the case of forms with equal coefficients. Hence we can assume that the coefficients satisfy

$$1 = a_1 < a_2 < \cdots < a_n \quad \text{with} \quad a_2, a_3, \dots, a_n \in \{1, \tau, \tau^2, \dots, \tau^{d-1}\}.$$

And with the help of the computer, it was verified that each of these forms has non-trivial solutions.

Putting all these results together, we have the following theorem

**Theorem 3.2** *Let* $d, p \in \mathbb{N}$, $d \geqslant 5$ *and odd,* $p$ *a prime number. If* $d \neq (p-1)/2$, $\ n \geqslant (d+1)/2$ *and* $\ a_1 \cdots a_n \not\equiv 0 \ (mod\, p)$ *then for any* *b there is always a solution for the congruence*

$$a_1x_1^d + \cdots + a_nx_n^d \equiv b \ (mod\, p),$$

*and if* $b \equiv 0\,(mod p)$ *the solution is non-trivial.*

# 4 Proof of Theorem 1.1

Let $\mathcal{M} = (a_{ij})$ be the $2 \times N$ coefficient matrix of the system (1), and let $r$ be the maximum number of columns of $\mathcal{M}$ which lie in an one dimensional linear subspace of $\mathbb{F}_p \times \mathbb{F}_p$ . Thus this system can be considered in the form

$$\begin{cases} f = & a_1x_1^d + \cdots + a_rx_r^d \quad + \quad b_1y_1^d + \cdots + b_sy_s^d \ \equiv \ 0\,(\text{mod}\, p) \\ g = & \qquad\qquad\qquad\qquad\quad c_1y_1^d + \cdots + c_sy_s^d \ \equiv \ 0\,(\text{mod}\, p) \end{cases} \tag{15}$$

From lemma 2.3 we can find $\delta$ with the property given in (2), and such that

$$\mathbb{F}_p^* = \mathbb{F}_p^d \cup \delta\mathbb{F}_p^d \cup \cdots \cup \delta^{d-1}\mathbb{F}_p^d \quad \text{(disjoint union)}.$$

Thus, repeating the arguments given in (14) we can assume that, for all coefficients $c_j$'s,

$$c_j \in \mathbb{S} = \{1, \delta, \delta^2, \ldots, \delta^{d-1}\} \quad \text{for} \quad i = 1, 2, \ldots, s. \tag{16}$$

And, from now on, we are going to also assume (see theorem 1.1) that

$$d \geqslant 5, \text{ odd}, \quad d \neq (p-1)/2, \quad p \equiv 1\,(\text{mod}\, d) \quad \text{and} \quad r + s \geqslant \frac{11}{6}d + 1.$$

The next definition presents an important and useful technique called contraction of variables.

**Definition 4.1** *Let*

$$\begin{array}{c} a_1x_1^d + \cdots + a_\mu x_\mu^d \\ b_1x_1^d + \cdots + b_\mu x_\mu^d \end{array}$$

*be a pair of subforms found among the variables of* $f, g$ *(see (1)). Let* $\xi = (\xi_1, \ldots, \xi_\mu)$ *be an non-trivial solution for*

$$b_1x_1^d + \cdots + b_\mu x_\mu^d \equiv 0\,(mod p).$$

*Multiply* $\xi$ *by a new variable* $T$*, and substitute it in* $a_1x_1^d + \cdots + a_\mu x_\mu^d$ *to have*

$$(a_1\xi_1^d + \cdots + a_\mu\xi_\mu^d)T^d \equiv \alpha T^d\,(mod p).$$

*The replacement of* $(x_1, \ldots, x_\mu)$ *by* $(T\xi_1, \ldots, T\xi_\mu)$ *will be called a* **contraction** *of* $\mu$ *variables to a new variable* $T$.

**Lemma 4.2** *If we have $r \geqslant \frac{d+1}{2}$ then the system (15) has non-trivial solutions.*

*Proof.* By Theorem 3.2 there exists an non-trivial solution $(\omega_1, \ldots, \omega_r)$ for $a_1 x_1^d + \cdots + a_r x_r^d \equiv 0$ $(mod\ p)$. Then $(\omega_1, \ldots, \omega_r, 0, \ldots, 0)$ is a non-trivial solution for the system (15).

$\square$

From now we may assume $1 \leq r \leq (d-1)/2$, and then

$$s \geqslant \frac{11}{6}d + 1 - \frac{d-1}{2} = \frac{8d+9}{6} > d.$$

As seen above, we are assuming $c_1, \ldots, c_s \in \mathbb{S}$, and since $s > d$, we must have equal coefficients. Let us suppose we have exactly $\pi$ disjoint pairs of equal coefficients, and after renumbering them, we have

$$c_1 = c_2,\ c_3 = c_4, \cdots, c_{2\pi-1} = c_{2\pi}. \tag{17}$$

Hence we can find $\pi$ disjoint subforms of $c_1 y_1^d + \cdots + c_s y_s^d$ with two variables and equal coefficients, such that the following congruences have solutions $(1, -1)$, for $i = 1, \ldots, \pi$,

$$c_{2i-1}\,(y_{2i-1}^d + y_{2i}^d) \equiv 0 \,(\mathrm{mod}\, p). \tag{18}$$

Let $t$ be the number of pairwise distinct coefficients among $c_1, \ldots, c_s$ (thus $1 \leq t \leq d$).

**Lemma 4.3** *If we have $t \leq \frac{5}{6}d + 1$ then the system (15) has an non-trivial solution.*

*Proof.* From (18) follows that we can contract each of these pairs of variables to new variables $T_1, T_2, \ldots, T_\pi$ (see definition 4.1). And now we can form the following congruence

$$a_1 x_1^d + \cdots + a_r x_r^d + \alpha_1 T_1^d + \cdots + \alpha_\pi T_\pi^d \equiv 0 \,(\mathrm{mod}\, p). \tag{19}$$

If one of the $\alpha_i$'s is zero modulo $p$, let us say $\alpha_1 \not\equiv 0 \,(\mathrm{mod}\, p)$, then $(0, \ldots, 0, 1, -1, 0, \ldots, 0)$ is an non-trivial solution for (15). Thus let us assume that (19) has $r + \pi$ variables. Now since $t \leq \frac{5}{6}d + 1$ and by lemma 2.2 we have

$$r + \pi \geqslant r + \frac{s-t}{2} \geqslant \frac{r+s}{2} + \frac{r}{2} - \frac{t}{2}.$$

Since $r + s \geqslant \frac{11}{6}d + 1$ and $r \geqslant 1$, we have

$$r + \pi \geqslant \frac{11}{12}d + \frac{1}{2} + \frac{1}{2} - \frac{5}{12}d - \frac{1}{2} = \frac{d+1}{2}.$$

By theorem 3.2 there exists an non-trivial solution $(\omega_1, \ldots \omega_{r+\pi})$ for (19). Then

$$(\omega_1, \ldots, \omega_r, \omega_{r+1}, -\omega_{r+1}, \ldots, \omega_{r+\pi}, -\omega_{r+\pi})$$

is an non-trivial solution for the system (15) $\square$

**Lemma 4.4** *If we have* $\frac{5}{6}d + 2 \leq t \leq d$ *then system (15) has an non-trivial solution.*

*Proof.* By lemma 2.2 we can form $\kappa = \lfloor \frac{s-t+1}{2} \rfloor$ pairs of subforms as in (18) and

$$\kappa = \left\lfloor \frac{s-t+1}{2} \right\rfloor = \begin{cases} \frac{s-t}{2} & \text{if } s \equiv t \pmod 2 \\ \frac{s-t+1}{2} & \text{otherwise.} \end{cases} \tag{20}$$

This procedure still leaves

$$c_{2\kappa+1} y_{2\kappa+1}^d + \cdots + c_s y_s^d \tag{21}$$

with $\mu = s - 2\kappa$ variables, where

$$\mu = s - 2\kappa = \begin{cases} t & \text{if } s \equiv t \pmod 2 \\ t-1 & \text{otherwise.} \end{cases} \tag{22}$$

Let $\tau$ be the number of pairwise distinct coefficients among $c_{2\kappa+1}, \ldots, c_s$ (thus $\tau \leq t$, for many coefficients were used to form the binary subforms (18)).

If $\tau < \frac{5}{6}d+2 \leq t$ then we could form some extra binary subforms, at least as much as $\lfloor \frac{(s-2\kappa)-\tau+1}{2} \rfloor$ (see lemma 2.2 and (21)). Thus we would have by (20) and (22)

$$\theta = \left\lfloor \frac{s-t+1}{2} \right\rfloor + \left\lfloor \frac{(s-2\kappa)-\tau+1}{2} \right\rfloor \geqslant \frac{s-\tau}{2}.$$

Contracting each of these $\theta$ binary subforms to new variables $T_1, \cdots, T_\theta$, we could form the congruence

$$a_1 x_1^d + \cdots + a_r x_r^d + \alpha_1 T_1^d + \cdots + \alpha_\theta T_\theta^d \equiv 0 \pmod p \tag{23}$$

with $r+\theta$ variables (the $\alpha_i$'s can be considered non-zero modulo $p$, by the reasons given in the proof of lemma 4.3, see (19)), and

$$r + \theta \geqslant r + \frac{s-\tau}{2}$$

with $\tau \leq \frac{5}{6}d + 1$. Now we are in the same situation as in the lemma 4.3 above, which gives us an non-trivial for (15). Thus, with no loss in generality, let us assume

$$\tau = t \geqslant \frac{5}{6}d + 2,$$

and then we have all coefficients in (21) pairwise distinct.

According to lemma 2.4, we can find at least

$$\lambda = \left\lceil \frac{d-1}{6} \right\rceil - (d-\mu) \tag{24}$$

disjoint subforms of (21) with three variables, all having non-trivial solutions. Let us rewrite the form (21) as

$$c_{11}y_{11}^d + c_{12}y_{12}^d + c_{13}y_{13}^d + \cdots + c_{\lambda 1}y_{\lambda 1}^d + c_{\lambda 2}y_{\lambda 2}^d + c_{\lambda 3}y_{\lambda 3}^d$$

putting all the remaining variables $y_j$'s equal to zero. Now let $(\rho_{j1}, \rho_{j2}, \rho_{j3})$, for $j = 1, \ldots, \lambda$, be the non-trivial solutions of the subforms $c_{j1}y_{j1}^d + c_{j2}y_{j2}^d + c_{j3}y_{j3}^d$ respectively. Thus we can contract each of these variables to new variables $S_1, \ldots, S_\lambda$. Hence we can add to the congruence (23) $\lambda$ new contracted variables, having the following congruence

$$a_1x_1^d + \cdots + a_rx_r^d + \alpha_1T_1^d + \ldots + \alpha_\kappa T_\kappa^d + \gamma_1S_1^d + \cdots + \gamma_\lambda S_\lambda^d \equiv 0 \pmod{p}, \tag{25}$$

with $r+\kappa+\lambda$ variables. By (20), (22) and (24) we have

$$r+\kappa+\lambda \geqslant r + \frac{s-t}{2} + \frac{d-1}{6} - d + t - 1.$$

We know that $r \geqslant 1$, $r+s \geqslant \frac{11}{6}d+1$ and $t \geqslant \frac{5}{6}d+2$, hence

$$\begin{aligned} r+\kappa+\lambda \geqslant\ & \tfrac{r+s}{2} + \tfrac{r}{2} - \tfrac{t}{2} + \tfrac{d}{6} - \tfrac{1}{6} - d + t - 1 \\ \geqslant\ & \tfrac{r+s}{2} + \tfrac{1}{2} + \tfrac{t}{2} - \tfrac{5}{6}d - \tfrac{7}{6} \\ \geqslant\ & \tfrac{11}{12}d + \tfrac{1}{2} + \tfrac{5}{12}d + 1 - \tfrac{10}{12}d - \tfrac{4}{6} \ > \ \tfrac{d+1}{2}. \end{aligned}$$

Now we can use proposition 3.2 to find a solution $(\omega_1, \ldots, \omega_{r+\kappa+\lambda})$ for (25), and then, writing $u = r+\kappa$,

$$\begin{aligned} (\ &\omega_1, \ldots, \omega_r,\ \ \omega_{r+1}, -\omega_{r+1},\ \ldots\ , \omega_u, -\omega_u, \\ &\omega_{u+1}\rho_{11},\ \omega_{u+1}\rho_{21},\ \omega_{u+1}\rho_{31},\ \ldots,\ \omega_{u+\lambda}\rho_{1\lambda},\ \omega_{u+\lambda}\rho_{2\lambda},\ \omega_{u+\lambda}\rho_{3\lambda}\ ) \end{aligned} \tag{26}$$

will be an non-trivial solution for the system (15). $\square$

# 5 Proof of Theorem 1.3

As in section 4. let $\mathcal{M}$ be the coefficient matrix of the system (1).

**Definition 5.1** *A solution $\xi = (\xi_1, \cdots, \xi_n)$ for the congruence system above will be called an* **non-singular solution** *if the matrix $\mathcal{M} \cdot \xi^t = (a_{ij}\xi_j)$ has rank 2 modulo p.*

From the hypothesis of the theorem 1.3, we have that $N \geqslant \frac{11}{6}d+1$. Hence, follows from theorem 1.1 that there exists an non-trivial solution for (1). Let us then assume that this solution is a singular solution, and let $\xi$ be a singular solution of (1) with the maximum number of coordinates different from zero modulo $p$. By permuting the variables if necessary, we may consider

$$\xi = (\xi_1, \ldots, \xi_R, 0, \ldots, 0). \tag{27}$$

This implies that the linear subspace of $\mathbb{F}_p \times \mathbb{F}_p$ generated by the first $R$ columns of the coefficient matrix $\mathcal{M}$ has dimension one. Now, if there are any other columns in the matrix $\mathcal{M}$, that belong to this one-dimensional subspace, renumber them as $R+1, \ldots, r$. Hence any column with index $j > r$ is linearly independent of the first $r$ columns of the coefficient matrix $\mathcal{M}$. Thus this system is equivalent to

$$\begin{cases} a_1x_1^d + \cdots + a_rx_r^d + & b_1y_1^d + \ldots + b_sy_s^d \equiv 0 \text{ (mod } p) \\ & c_1y_1^d + \ldots + c_sy_s^d \equiv 0 \text{ (mod } p), \end{cases} \tag{28}$$

with all the coefficients $c_j$'s non-zero, $r+s \geqslant \frac{11}{6}d+1$, $s \geqslant (d+1)/2$ and $r \geqslant 2$. Thus, repeating the arguments given in section 4., we can also assume that, for all coefficients $c_j$'s, (see (16))

$$c_j \in \mathbb{S} = \{1, \delta, \delta^2, \ldots, \delta^{d-1}\} \quad \text{for} \quad i = 1, 2, \ldots, s.$$

where $\delta$ has all the properties stated in lemma 2.3. And, from now on, we are going to also assume (see theorem 1.3) that $d, p \in \mathbb{N}$, $d$ odd and $p$ a prime with

$$d \geqslant 5, \quad d \neq (p-1)/2, \quad p \equiv 1 \,(\operatorname{mod} d), \quad r+s \geqslant \frac{11}{6}d+1, \quad \text{and} \quad s \geqslant \frac{d+1}{2}.$$

**Lemma 5.2** *If in the system (28) we have $r \geqslant (d+1)/2$ then this system has an non-singular solution.*

*Proof.* Since $s \geqslant (d+1)/2$, we can use proposition 3.2 and find an non-trivial solution $(\rho_1, \ldots, \rho_s)$ for $c_1y_1^d + \ldots + c_sy_s^d \equiv 0 \,(\operatorname{mod} p)$. Now write $b_1\rho_1^d + \ldots + b_s\rho_s^d \equiv \beta \,(\operatorname{mod} p)$. If $\beta \equiv 0 \,(\operatorname{mod} p)$ then

$$(\xi_1, \ldots, \xi_R, 0, \ldots, 0, \rho_1, \ldots, \rho_s)$$

is an non-singular solution for (28) (see (27)). If $\beta \not\equiv 0 \,(\operatorname{mod} p)$, we can now use proposition 3.2 and find an non-trivial solution $(\tau_1, \ldots, \tau_r)$ for

$$a_1x_1^d + \cdots + a_rx_r^d + \beta \equiv 0 \,(\operatorname{mod} p),$$

and now $(\tau_1, \ldots, \tau_r, \rho_1, \ldots, \rho_s)$ is an non-singular solution for (28). $\square$

Let us then assume $r \leq (d-1)/2$, and then

$$s \geqslant \frac{11}{6}d + 1 - (d-1)/2 = \frac{8d+9}{6} \ > \ d.$$

We are now in the same situation described just after lemma 4.2 (see (17) and (18)) Again let $t$ be the number of pairwise distinct coefficients among $c_1, \ldots, c_s$.

**Lemma 5.3** *If we have* $\ t \leq \frac{d+1}{2}$ *then system (28) has an non-singular solution.*

*Proof.* Repeating the arguments given in the proof of lemma 4.3, we can produce the congruence (see (19))

$$a_1 x_1^d + \cdots + a_r x_r^d + \alpha_1 T_1^d + \ldots + \alpha_\pi T_\pi^d \equiv \ 0 \,(\bmod\, p),$$

where $\alpha_i \ = \ b_{2i-1} - b_{2i}$. If any $\alpha_i \equiv \ 0 \,(\bmod\, p)$, say $\alpha_1 \equiv \ 0 \,(\bmod\, p)$, then

$$(\xi_1, \ldots, \xi_R, 0, \ldots, 0, 1, -1, 0 \ldots, 0)$$

is an non-singular solution for (28) (for any column with index $j > r$ is linearly independent of the first $r$ columns of $\mathcal{M}$). So let us assume otherwise, and set $T_\pi = 1$. Consider the congruence,

$$a_1 x_1^d + \cdots + a_r x_r^d + \alpha_1 T_1^d + \ldots + \alpha_{\pi-1} T_{\pi-1}^d + \alpha_\pi \equiv \ 0 \,(\bmod\, p). \tag{29}$$

Now (see proof of lemma 4.3)

$$r + (\pi - 1) \ \geqslant \ r + \frac{s-t}{2} - 1 \ \geqslant \ \frac{r+s}{2} + \frac{r}{2} - 1 - \frac{t}{2},$$

Hence, since $r \geqslant 2$, $\ r + s \geqslant \frac{11}{6}d + 1\ $ and $\ t \leq \frac{d+1}{2}$, we have

$$r + (\pi - 1) \ \geqslant \ \frac{11d}{12} + \frac{1}{2} - \frac{d+1}{4} \ = \ \frac{8d+3}{12} \ > \ \frac{d+1}{2}.$$

Then we can find an non-trivial solution $(\rho_1, \ldots, \rho_{r+\pi-1})$ for (29), by proposition 3.2. And

$$(\rho_1, \ldots, \rho_{r+\pi-1}, 1, -1, 0 \ldots, 0)$$

is an non-singular solution for (28), since $b_{2\pi-1}c_\pi - b_{2\pi}c_{2\pi-1} \not\equiv \ 0 \,(\bmod\, p)$. $\square$

**Lemma 5.4** *If we have* $\ \frac{d+3}{2} \leq t \leq \frac{5}{6}d + 1$ *then system (28) has an non-singular solution.*

*Proof.* Repeating the arguments given in the beginning of the proof of lemma 4.4 (see (20)), we can have the congruence

$$a_1x_1^d + \cdots + a_rx_r^d + \alpha_1T_1^d + \ldots + \alpha_{\kappa-1}T_{\kappa-1}^d + \alpha_\kappa \equiv 0 \,(\mathrm{mod}\, p), \quad (30)$$

where $\alpha_1, \ldots, \alpha_\kappa$ are all non-zero modulo $p$, by the reasons given in the proof of lemma 5.3. And by (21) and (22) we still can consider the congruence

$$c_{2\kappa+1}y_{2\kappa+1}^d + \ldots + c_sy_s^d \equiv 0 \,(\mathrm{mod}\, p),$$

with at least $\mu \geqslant t-1 \geqslant (d+3)/2-1 = (d+1)/2$ variables. Thus we can find an non-trivial solution $(\theta_{2\kappa+1}, \ldots, \theta_s)$ for this congruence. After contracting them to a new variable $S$, we could have the congruence (see (30))

$$a_1x_1^d + \cdots + a_rx_r^d + \alpha_1T_1^d + \ldots + \alpha_{\kappa-1}T_{\kappa-1}^d + \alpha_\kappa + \gamma S^d \equiv 0 \,(\mathrm{mod}\, p) \quad (31)$$

with $\gamma \not\equiv 0 \,(\mathrm{mod}\, p)$ by the same reason each $\alpha_i \not\equiv 0 \,(\mathrm{mod}\, p)$. And this congruence has $r + (\kappa - 1) + 1$ variables. Now (see (20))

$$r + \kappa \geqslant r + \frac{s-t}{2} = \frac{r+s}{2} + \frac{r}{2} - \frac{t}{2},$$

Hence, since $r \geqslant 2$ , $r + s \geqslant \frac{11}{6}d + 1$ and $t \geqslant \frac{5}{6}d + 1$, we have

$$r + \kappa \geqslant \frac{11d}{12} + \frac{3}{2} - \frac{5d}{12} - \frac{1}{2} > \frac{d+1}{2}.$$

Then we can find an non-trivial solution $(\rho_1, \ldots, \rho_{r+\kappa-1}, \rho_{r+\kappa})$ for (31), by proposition 3.2. And

$$(\rho_1, \ldots, \rho_{r+\pi-1}, \ 1, -1, \ \rho_{r+\pi}\theta_{2\pi+1}, \ \ldots, \ \rho_{r+\pi}\theta_s)$$

is an non-singular solution for (28), since $b_{2\kappa-1}c_\kappa - b_{2\kappa}c_{2\kappa-1} \not\equiv 0 \,(\mathrm{mod}\, p)$. □

**Lemma 5.5** *If we have* $\frac{5}{6}d + 2 \leq t \leq d$ *then system (28) has an non-singular solution.*

*Proof.* This proof follows closely the proof of lemma 4.4 up to the congruence given in (25), using lemmas 5.3 and 5.4 instead of lemma 4.3, whenever this lemma is necessary. Now let us set $T_\kappa = 1$ in (25) having

$$a_1x_1^d + \cdots + a_rx_r^d + \alpha_1T_1^d + \ldots + \alpha_\kappa T_\kappa^d + \gamma_1S_1^d + \cdots + \gamma_\lambda S_\lambda^d \equiv 0 \,(\mathrm{mod}\, p),$$

with $r + (\kappa - 1) + \lambda$ variables and all coefficients non-zero modulo $p$.

Now we want to prove that $r + (\kappa - 1) + \lambda > \frac{d+1}{2}$, for this will imply that the congruence above has an non-trivial solution

$$\omega = (\omega_1 \cdots, \omega_{r+(\kappa-1)+\lambda})$$

by proposition 3.2, and a combination of all the contracted solutions with this solution $\omega$ (see (26)) gives an non-singular solution for (28), since $\alpha_\kappa \not\equiv 0 \,(\mathrm{mod}\, p)$, and consequently $b_{2\kappa-1}c_\kappa - b_{2\kappa}c_{2\kappa-1} \not\equiv 0 \,(\mathrm{mod}\, p)$. Thus let us prove this fact, considering two cases:

let us recall that: $r \geqslant 2, \quad r+s \geqslant \dfrac{11}{6}d+1 \quad$ and $\quad t \geqslant \dfrac{5}{6}d+2$

(i) $t \equiv s \,(\mathrm{mod}\, p)$. Hence (see (20), (22) and (24))

$$\begin{aligned} r+(\kappa-1)+\lambda &\geqslant r+\tfrac{s-t}{2}-1+\tfrac{d-1}{6}-d+t. \\ &\geqslant \tfrac{r+s}{2}+\tfrac{r}{2}-\tfrac{t}{2}-1+\tfrac{d}{6}-\tfrac{1}{6}-d+t \\ &\geqslant \tfrac{r+s}{2}+1-1-\tfrac{5}{6}d-\tfrac{1}{6}+\tfrac{t}{2} \\ &\geqslant \tfrac{11}{12}d+\tfrac{1}{2}-\tfrac{5}{6}d-\tfrac{1}{6}+\tfrac{5}{12}d+1 \quad > \quad \tfrac{d+1}{2}. \end{aligned}$$

(ii) $t \not\equiv s \,(\mathrm{mod}\, p)$. Hence

$$\begin{aligned} r+(\kappa-1)+\lambda &\geqslant r+\tfrac{s-t+1}{2}-1+\tfrac{d-1}{6}-d+t-1 \\ &\geqslant \tfrac{r+s}{2}+\tfrac{r}{2}+\tfrac{1}{2}-\tfrac{t}{2}-1+\tfrac{d}{6}-\tfrac{1}{6}-d+t-1 \\ &\geqslant \tfrac{r+s}{2}+\tfrac{3}{2}-1-\tfrac{5}{6}d-\tfrac{1}{6}+\tfrac{t}{2}-1 \\ &\geqslant \tfrac{11}{12}d+\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{5}{6}d-\tfrac{1}{6}+\tfrac{5}{12}d+1-1 \quad > \quad \tfrac{d+1}{2}. \end{aligned}$$

□

From these three lemmas follows the proof of theorem 1.3.

# 6 Proof of Theorem 1.4

As introduced by Davenport and Lewis[9], we can associate to each pair of forms $f, g$ as in (1) a parameter

$$\vartheta(f,g) = \prod_{i\neq j}(a_{1i}a_{2j} - a_{1j}a_{2i}).$$

And, when $\vartheta(f,g) \neq 0$, we can find a pair of forms $f^*, g^*$ $p$-equivalent to $f, g$, called a $p$-**normalized pair**, with the following properties (this is lemma 9 in [9])

**Lemma 6.1** *Any $p$-normalized pair $f, g$ can be written in the form*

$$\begin{aligned} f &= f_0(x_1,\dots,x_{m_0}) + pf_1(x_{m_0+1},\dots,x_N) \\ g &= g_0(x_1,\dots,x_{m_0}) + pg_1(x_{m_0+1},\dots,x_N) \end{aligned}$$

*with*

$$m_0 \geqslant \frac{N}{d},$$

*and each of the variables $x_1, \ldots, x_n$ occurs in one at least of $f_0, g_0$ with a coefficient not divisible by $p$. Moreover, if $q_0$ denotes the minimum number of variables occuring in any linear combination $\lambda f_0 + \mu g_0$ (with $\lambda$ and $\mu$ not both congruent to zero modulo $p$ with a coefficient not divisible by $p$, then*

$$q_0 \geqslant \frac{N}{2d}.$$

The importance of the $p$-normalization lies on the fact that if a $p$-normalized pair has $p$-adic non-trivial zeros, then any $p$-equivalent pair will also have $p$-adic non-trivial zeros. And by a compactness argument given in Davenport and Lewis [9], it is enough to prove theorem 1.4 under the hypothesis that $\vartheta(f, g) \neq 0$.

The next lemma is an adaptation of lemma 7 in [9]

**Lemma 6.2** *$d \in \mathbb{N}$ be an odd integer, and $p$ an odd prime such that $gcd(p, d) = 1$. If the system*

$$\begin{aligned} f_0 \equiv \;& a_1 x_1^d + \cdots + a_{m_0} x_{m_0}^d \equiv 0 \,(mod\, p) \\ g_0 \equiv \;& b_1 x_1^d + \cdots + b_{m_0} x_{m_0}^d \equiv 0 \,(mod\, p) \end{aligned}$$

*has an non-singular solution modulo $p$, then the pair $f_0, g_0$ has non-trivial $p$-adic zeros.*

Now we are ready to prove theorem 1.4. With no loss in generality, we may assume $f, g$ to be $p$-normalized. Hence, by lemma 6.1, we have

$$\begin{aligned} f &= f_0(x_1, \ldots, x_{m_0}) + p f_1(x_{m_0+1}, \ldots, x_N) \\ g &= g_0(x_1, \ldots, x_{m_0}) + p g_1(x_{m_0+1}, \ldots, x_N) \end{aligned}$$

with

$$m_0 \geqslant \frac{11}{6} d + 1 \quad \text{and} \quad q_0 \geqslant \frac{11}{12} d + \frac{1}{2},$$

since we are assuming $n \geqslant \frac{11}{6} d^2 + d$. Now, by theorem 1.3, we have that the congruence system

$$\begin{aligned} f_0 &\equiv 0 \,(\mathrm{mod}\, p) \\ g_0 &\equiv 0 \,(\mathrm{mod}\, p) \end{aligned}$$

has an non-singular solution modulo $p$. □

Thus the pair $f, g$ has non-trivial p-adic zeros, by lemma 6.2, concluding the proof of theorem 1.4.

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Departamento de Matemática
Universidade de Brasília
Brasília, DF, Brasil, 70910-900
hemar@mat.unb.br and jorge@mat.unb.br

Instituto de Matemática e Estatística
Universidade Federal de Goiás
Goiânia,GO, Brasil, 74001-970
paulo@mat.ufg.br