Subnormality in Products

Maier (1977) Wielandt (1981)

Seminar Talk MAT - UnB

November 2003

The groups considered in the following are finite.

If we take in a group G two distinct Sylow p-subgroups A and B for some prime p what can we say about the embedding of $A \cap B$ in $\langle A, B \rangle$? This is an interesting and common, but also very difficult question. The difficulty is due to one of the problems in group theory which appears, if we try to substitute "normal embedding" by "subnormal embedding": Given a subgroup X, there is no analogue to the concept of the *normalizer* $\mathbf{N}_G(X)$. In general, there will be no *subnormalizer of* X *in* G, in the sense that there is no largest subgroup of G in which X is contained as a subnormal subgroup:

A) Remark.

If A, B, S *are three subgroups of a group* G, *such that* $S \leq A \cap B$ *with* $S \trianglelefteq\trianglelefteq A$ *and* $S \trianglelefteq\trianglelefteq B$, *then the conclusion that* $S \trianglelefteq\trianglelefteq \langle A, B \rangle$ *is not true in general.*

Proof: Consider for example the special projective linear group $G = \mathbf{PSL}(2, 17)$ and let A, B be two distinct Sylow 2-subgroups of G such that $S = A \cap B \neq 1$ (since $A, B <\cdot\, G$ are maximal subgroups and G is not a Frobenius group, such a pair exists!). We have $S \trianglelefteq\trianglelefteq A$ and $S \trianglelefteq\trianglelefteq B$, but S is not subnormal in $\langle A, B \rangle = G$.

A further example (solvable of Fitting length 2 - and order 72) is

$$G = \Big[C_3 \times C_3 \Big] A \text{ with } A = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \cong \mathbf{D}_4.$$

We have $A <\cdot\, G$ and once more there exists a conjugate $B = A^g \neq A$ of A with $1 \neq S = A \cap B$, as G is not a Frobenius group. Again $S \trianglelefteq\trianglelefteq A$, $S \trianglelefteq\trianglelefteq B$, but S is not subnormal in $G = \langle A, B \rangle$.

This makes the following result interesting:

Main Theorem

(Maier 1977, Wielandt 1982) [5], [10]

Let G *be a finite group,* $A, B, S \leq G$, *such that* $S \leq A \cap B$,

$$S \trianglelefteq\trianglelefteq A,\ S \trianglelefteq\trianglelefteq B \text{ and } AB = BA.$$

$$\textit{Then } S \trianglelefteq\trianglelefteq AB \ (= \langle A, B \rangle).$$

It seems that, the first time this phenomenon has been observed, was in connection with the study of the product of two nilpotent subgroups in MAIER [4] (1972):

B) Proposition.

Let $G = AB$ with nilpotent subgroups A and B. Then the following hold:

a) *G is solvable* (KEGEL-WIELANDT).

b) *If $A \neq B$, then $A^G < G$ or $B^G < G$* (ITÔ-KEGEL).

c) $A \cap B \trianglelefteq\trianglelefteq G$ (MAIER 1972) [4].

Proof: a) See [8], [3], [7].

b) (compare with the proof in ITÔ [2]) By a) we know that G is solvable. We induct on the order of G to show b): Let N be a minimal normal subgroup of G and suppose $|N|$ a power of the prime p. We have $G/N = (AN/N)(BN/N)$ with the nilpotent subgroups $AN/N \cong A/A \cap N$, $BN/N \cong B/B \cap N$. If $AN \neq BN$, we conclude by induction that $(AN)^G < G$ or $(BN)^G < G$. Thus also $A^G < G$ or $B^G < G$. If $AN = BN$, then $AN = BN = G$ and $|G:A| = |N:N \cap A|$ and $|G:B| = |N:B \cap N|$ are p-powers. Also $|G:D|$ is a p-power, when $D = A \cap B$. The p-complement K_p of D is also the p-complement of A and of B. Therefore, $K_p \trianglelefteq A$ and $K_p \trianglelefteq B$, by the nilpotency of A and B, whence $K_p \trianglelefteq G$. Now G/K_p is a p-group with $K_p \leq A < G$ or $K_p \leq B < G$. Clearly now $A^G < G$ or $B^G < G$.
c) We may assume $A \neq B$. By b) we have $M = A^G < G$, say. We have $M = A\bar{B}$ with $\bar{B} = M \cap B$. Moreover $D = A \cap \bar{B}$. By induction $D \trianglelefteq\trianglelefteq M$ and since $M \trianglelefteq G$, also $D \trianglelefteq\trianglelefteq G$.

■

In 1977, the Main Theorem was formulated and published as a conjecture together with its proof in the case where the subgroup S is solvable:

C) Theorem. (MAIER 1977) [5]

Let G be a finite group, $A, B, S \leq G$ such that $S \trianglelefteq\trianglelefteq A$, $S \trianglelefteq\trianglelefteq B$ and $AB = BA$.

If S is a solvable subgroup, then $S \trianglelefteq\trianglelefteq AB$.

D) Proposition. (WIELANDT's "Zipper-Lemma" 1974) [9]
Let $T \leq G$ such that T is not subnormal in G but $T \trianglelefteq\trianglelefteq U$ whenever $T \leq U < G$.

Then there is a **unique** *maximal subgroup $M <\cdot\, G$ with $T \leq M$.*

E) Consequence. (WIELANDT's subnormality criterion, 1974) [9]
Let T be a subgroup of G. Then

$$T \trianglelefteq\trianglelefteq G \iff \Big(\forall\, g \in G: \;\; g \in \langle T, T^g\rangle \implies g \in T\Big) \;.$$

Proof: " $\implies$ ": Let $T \trianglelefteq\trianglelefteq G$. If $g \in G$ is such that $T \neq T^g$, then there is $R \lhd \langle T, T^g\rangle$ with $T \leq R$. If $g \in \langle T, T^g\rangle$, it follows $T^g \leq R$ and we have the contradiction $\langle T, T^g\rangle \leq R < \langle T, T^g\rangle$. Therefore $g \in \langle T, T^g\rangle$ is only possible if $g \in T$.

" $\impliedby$ ": Suppose, a subgroup $T \leq G$ satisfies

$$"\forall\, g \in G: \;\; g \in \langle T, T^g\rangle \implies g \in T" .$$

This property of T is inherited by subgroups which contain T. If G is a minimal counterexample against the subnormality of T in G, then $T \trianglelefteq\trianglelefteq U$ whenever $T \leq U < G$. By D), there is a unique $M <\cdot\, G$ with $T \leq M$. Obviously, $\mathbf{N}_G(M) = M$. We pick any $g \in G \setminus M$. If we had $\langle T, T^g\rangle < G$, then $\langle T, T^g\rangle \leq M$ and $T^g \leq M$. Therefore $T \leq M^{g^{-1}} \neq M$, against the uniqueness of M. This shows $\langle T, T^g\rangle = G$ and we have the contradiction $g \in \langle T, T^g\rangle \setminus T$. Therefore $T \trianglelefteq\trianglelefteq G$.

∎

F) Subconsequence. (BAER's Lemma 1957) [1]
Let p be a prime and $T \leq G$ a p-subgroup of G. Then

$$T \trianglelefteq\trianglelefteq G \iff T^G \text{ is a } p\text{-group} \iff \langle T, T^g\rangle \text{ is a } p\text{-group for all } g \in G.$$

Proof: The first "$\iff$" is well known. Of the second "$\iff$" only "$\impliedby$" needs a proof: Suppose $\langle T, T^g\rangle$ is a p-group for every $g \in G$. If $T \neq T^g$, there will be $R \lhd \langle T, T^g\rangle$ with $T \leq R$ as $\langle T, T^g\rangle$ is a p-group. If $g \in \langle T, T^g\rangle$, we get the contradiction $\langle T, T^g\rangle \leq R$. Thus, "$\forall\, g \in G: \;\; g \in \langle T, T^g\rangle \implies g \in T$" is satisfied and we conclude $T \trianglelefteq\trianglelefteq G$ by E).

∎

G) Proposition. (WIELANDT 1939) [6]

Let $X \trianglelefteq\trianglelefteq G$ with X simple nonabelian. Then

$$X \leq \mathbf{N}_G(Y) \text{ for all } Y \trianglelefteq\trianglelefteq G\,.$$

■

H) Proposition. (WIELANDT 1951) [7]

Let $G = AB$ with subgroups A, B of G and let p be a prime. Then there exist an $A_p \in Syl_pA$ and a $B_p \in Syl_pB$ such that $A_pB_p = B_pA_p = G_p \in Syl_pG$.

■

Proof of the Main Theorem: We may assume that $AB = BA = G$. Let G be a group of minimal order in which the theorem is not true. In G we choose the subgroups A, B and S in such a way that A and S have maximal orders with respect to $S \leq A \cap B$, $S \trianglelefteq\trianglelefteq A$ and $S \trianglelefteq\trianglelefteq B$, but S not subnormal in G.

(**Remark**: The proof we give here is a variation of the original proofs in [5] and [10], where A is chosen of *maximal* and S of *minimal* order)

In this situation we can conclude:

i) $A <\cdot\, G$:

Let $A \leq M <\cdot\, G$. We have $M = A\bar{B}$ where $\bar{B} = M \cap B$, $S \trianglelefteq\trianglelefteq A$ and $S \trianglelefteq\trianglelefteq \bar{B}$ and $|M| < |G|$. By the minimality of $|G|$ we conclude $S \trianglelefteq\trianglelefteq M$. Now, $G = MB$, $S \trianglelefteq\trianglelefteq M$ and $S \trianglelefteq\trianglelefteq B$. If $A < M$, we conclude by the maximality of $|A|$ that $S \trianglelefteq\trianglelefteq G$. Therefore, $A = M$ is a maximal subgroup of G.

ii) $A_G = \bigcap\limits_{g \in G} A^g = 1$:

Suppose, $1 \neq N \trianglelefteq G$ with $N \leq A$. We have $\frac{G}{N} = \frac{A}{N}\frac{BN}{N}$, $\frac{SN}{N} \leq \frac{A}{N} \cap \frac{BN}{N}$, $\frac{SN}{N} \trianglelefteq\trianglelefteq \frac{A}{N}$ and $\frac{SN}{N} \trianglelefteq\trianglelefteq \frac{BN}{N}$. Since $|G/N| < |G|$, we conclude $\frac{SN}{N} \trianglelefteq\trianglelefteq \frac{G}{N}$, whence $SN \trianglelefteq\trianglelefteq G$. But $S \trianglelefteq\trianglelefteq SN$ as $SN \leq A$. Consequently $S \trianglelefteq\trianglelefteq G$.

iii) *If $T \trianglelefteq\trianglelefteq G$ and $T \leq A$, then $T = 1$*:

If $1 \neq T \trianglelefteq\trianglelefteq G$ with $T \leq A$, we have $T^A \trianglelefteq\trianglelefteq G$ and $T^A \leq A$ and $T^G \not\leq A$. Therefore, $AT^G = G$ and $T^A < T^G$. It follows the contradiction $T^G = T^{AT^G} = (T^A)^{T^G} < T^G$, as $T^A \trianglelefteq\trianglelefteq T^G$.

iv) $\mathbf{O}_p(S) = 1$ *for all primes* p:

If $\mathbf{O}_p(S) \neq 1$ for some p, we put $T = \mathbf{O}_p(S)$. We have $T \trianglelefteq\trianglelefteq A$ and $T \trianglelefteq\trianglelefteq B$ and therefore $T \leq \mathbf{O}_p(A) \cap \mathbf{O}_p(B)$. Now $\mathbf{O}_p(A) \leq A_p$ for any $A_p \in Syl_pA$ and $\mathbf{O}_p(B) \leq B_p$ for all $B_p \in Syl_pB$. By H), there exist $A_p \in Syl_pA$ and $B_p \in Syl_pB$ such that $A_pB_p = B_pA_p = G_p \in Syl_pG$. Therefore $\Big\langle \mathbf{O}_p(A), \mathbf{O}_p(B) \Big\rangle \leq G_p$ and we see that $\Big\langle \mathbf{O}_p(A), \mathbf{O}_p(B) \Big\rangle$ is a p-group. Let $g \in G$ and write $g = ab$ with $a \in A$, $b \in B$. We have

$$\Big\langle T, T^g \Big\rangle = \Big\langle T, T^{ab} \Big\rangle = \langle T^{b^{-1}}, T^a \rangle^b \leq \Big\langle \mathbf{O}_p(A), \mathbf{O}_p(B) \Big\rangle^b \leq G_p^b$$

and therefore, $\langle T, T^g \rangle$ is a p-subgroup for every $g \in G$. By F), T^G is a p-group, i.e. $T \trianglelefteq\trianglelefteq G$. This is impossible by item iii).

So far, our proof follows the proof presented in Maier [5].
iv) furnishes a contradiction when S is solvable (i.e. finishes the case of Theorem C)). The rest of our proof, is a variation of Wielandt [10].

Let now $T = \langle X \mid X \trianglelefteq\trianglelefteq S \text{ and } X \text{ minimal (simple)} \rangle$. Clearly $T \neq 1$.

v) T *is normalized by every* A*-conjugate and by every* B*-conjugate of* T:

By iv), all the minimal subnormal subgroups X of S are nonabelian. We apply proposition G).

vi) *The contradiction*:

Again it suffices to show, by iii), that $T \trianglelefteq\trianglelefteq G$. If T is not subnormal in G, there will be, by consequence E), a $g \in G$ such that $g \in \langle T, T^g \rangle \setminus T$. If $g = ab$ with $a \in A$, $b \in B$, then $ab \in \langle T, T^{ab} \rangle$, whence $ba = (ab)^{b^{-1}} \in \langle T, T^{ab} \rangle^{b^{-1}} = \langle T^{b^{-1}}, T^a \rangle \leq \mathbf{N}_G(T)$ by v). Consequently, $T^{ba} = T$ and it follows $W = T^b = T^{a^{-1}} \leq A \cap B$ with $W \trianglelefteq\trianglelefteq A$ and $W \trianglelefteq\trianglelefteq B$. If $W \not\leq S$, then $S < WS \leq A \cap B$ with $SW \trianglelefteq\trianglelefteq A$ and $SW \trianglelefteq\trianglelefteq B$. By the maximal choice of S we conclude $SW \trianglelefteq\trianglelefteq G$ and then also $T \trianglelefteq\trianglelefteq G$. Therefore, $W \leq S$. So W is the join of (some) simple subnormal subgroups of A (of B) contained in S. By the meaning of T it follows now $W \leq T$ and, since $|W| = |T|$, then $W = T = T^{b^{-1}} = T^a$. So we have individually $a, b \in \mathbf{N}_G(T)$ and therefore also $g = ab \in \mathbf{N}_G(T)$. This gives the final contradiction $g \in \langle T, T^g \rangle \setminus T = T \setminus T = \emptyset$.

■

References

[1] **Baer, R.** *Engelsche Elemente Noetherscher Gruppen.* Math. Annalen **133**, 256-270 (1957).

[2] **Itô, N.** *Produkt von zwei abelschen Gruppen.* Mathem. Zeitschrift, **62**, 400 - 401 (1955).

[3] **Kegel, O.H.** *Produkte nilpotenter Gruppen.* Archiv der Mathematik, **12**, 90 - 93 (1961).

[4] **Maier, R.** *Endliche metanilpotente Gruppen.* Archiv der Mathematik **23**, 139-144, (1972).

[5] **Maier, R.** *Um problema da teoria dos subgrupos subnormais.* Boletim da Soc. Brasileira de Matemática, 127-130 (1977).

[6] **Wielandt, H.** *Eine Verallgemeinerung der invarianten Untergruppen.* Math. Zeitschrift **45**, 209-244, (1939).

[7] **Wielandt, H.** *Über das Produkt paarweise vertauschbarer nilpotenter Gruppen.* Math. Zeitschrift **55**, 1-7 (1951).

[8] **Wielandt, H.** *Über Produkte von nilpotenten Gruppen.* Illinois J. Math. **2**, 611 - 618 (1958).

[9] **Wielandt, H.** *Kriterien für Subnormalität in endlichen Gruppen.* Math. Zeitschrift **138**, 199-203 (1974).

[10] **Wielandt, H.** *Subnormalität in faktorisierten endlichen Gruppen.* J. Algebra **69**, 305 - 311 (1982).

Brasília, December 08, 2003.

Rudolf Maier