ON THE 2D NAVIER-STOKES EQUATION WITH SINGULAR INITIAL DATA AND FORCING TERM

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Abstract

We consider the Cauchy problem for the Navier-Stokes equation for an incompressible fluid in $I\!R^2$ with a forcing term and the associated vorticity equation. The initial data and the forcing terms are singular in a suitable sense. We get results for the existence and uniqueness of strong global solutions, as well as L^q estimates of the solutions down to $t = 0$.

Resumo

Consideramos o problema de Cauchy para a equação de Navier-Stokes para um fluido incompressível em $I\!R^2$ com um termo forçante e para a equação de vorticidade associada. O dado inicial e o termo forçante são singulares num certo sentido. Obtemos resultados de existência e unicidade de soluções globais fortes, assim como estimativas L^q de soluções até o plano $t = 0$.

1. Introduction

This paper is concerned with the Cauchy problem for the Navier-Stokes equation for an incompressible fluid in $I\!R^2$ with an external forcing term and the associated vorticity equation. The initial data and the forcing terms are singular in a suitable sense.

We write the Navier-Stokes (NS) equation with a forcing term in the form

$$\partial_t u - \nu \triangle u + \Pi\partial(u \otimes u) = G, \quad t > 0,\, x \in I\!R^2,\; u = \Pi u,\; G = \Pi G, \qquad (NS)$$

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where $u=(u_1,u_2)$ stands for the velocity field, Π is the projection onto the solenoidal vectors along gradients, $u\otimes u$ is a tensor with $jk-$component $u_k u_j$, $1\le j,k\le 2$ and the vector $\partial(u\otimes u)$ has as its $j-$th component $\partial_k(u_k u_j)=u_k\partial_k u_j$ (the summation convention), $j=1,2$, $\nu>0$ stands for the viscosity.

The associated vorticity equation for the scalar vorticity $\zeta=\partial_1 u_2-\partial_2 u_1$ with the corresponding forcing term $F=\partial_1 G_2-\partial_2 G_1$ is written as follows

$$\partial_t\zeta-\nu\triangle\zeta+\partial\cdot(\zeta S*\zeta)=F,\quad S(x)=(2\pi)^{-1}|x|^{-2}(x_2,-x_1), \tag{V}$$

where $*$ denotes convolution and $S*$ is a linear operator such that $u=S*\zeta$ solves the equations $\partial\cdot u=\partial_1u_1+\partial_2u_2=0$ and $\partial_1u_2-\partial_2u_1=\zeta$ and S satisfies the Hardy-Littlewood-Sobolev inequality

$$\|S*\phi\|_p\le\sigma_q\|\phi\|_q,\quad \phi\in L^q(I\!\!R^2),\ \sigma_q>0,\ \frac{1}{p}=\frac{1}{q}-\frac{1}{2},\ 1<q<2. \tag{1.1}$$

Setting $\nu=0$ we get the Euler equation.

The case of the free (NS) and (V) in $I\!\!R^2$ with initial vorticity $\omega=\zeta(0,\cdot)$ finite Radon measures has been settled by Y. Giga, T. Miyakawa and H. Osada [7] requiring smallness of the atomic part of ω (provided the total variation of ω is bounded) for the uniqueness (see also previous results in [2], [6]). Later on T. Kato [9] simplified their proof and gave, in particular, explicit bounds on the atomic part which imply the uniqueness. The recent paper of H. Kozono and M. Yamazaki [13] seems to be the first one dealing with initial velocity $a=u(0,\cdot)$ for the free (NS) in $I\!\!R^n$, $n\ge 2$ which could be in a certain sense more singular than Radon measures. There appears a smallness condition on the initial data which in particular recovers the previous results for the free (NS) and (V) in $I\!\!R^2$ in case the initial vorticity has small enough total variation. The paper of T. Kato and G. Ponce [12] deals with the free (NS) in $I\!\!R^n$, $n\ge 2$, with a belonging to the Sobolev spaces with negative indices and having a small norm. Finally we cite the recent article of M. Ben-Artzi [1] on the planar free Navier-Stokes and Euler equations with $\omega\in L^1(I\!\!R^2)$ and $\omega\in L^1(I\!\!R^2)\bigcap L^r(I\!\!R^2)$, $r>2$, respectively. In particular [1] contains continuous dependence results for the Cauchy problem for (NS) and (V).

Apart from the presence of forcing terms we allow the initial vorticity to be a fractional derivative of certain type of measures. The initial velocity becomes also singular and the spaces of initial vorticity and initial velocity which we introduce are strictly larger than those considered in [7], [9], [12], while comparing with [13] we point out that we have examples of initial data not covered in [13] and that in many cases we are able to remove the smallness restrictions imposed there. We require the forcing term to be L^q in $I\!R^2$ for certain $1 < q < \infty$ while in t we assume L^1_{loc} without any decay conditions as $t \to \infty$. The main result is an existence-uniqueness theorem for strong global (in time) solutions as well as L^q estimates on the solutions of (NS) and (V) down to $t = 0$. Concerning the uniqueness we have either no restrictions on the initial data or explicit bounds which, in case of zero forcing term $F \equiv 0$, $\nu = 1$ and ω finite Radon measure, coincide with the bounds on the atomic part of ω proved in [9]. We stress that the local existence and uniqueness result could be extended when the forcing term is more singular in x (e.g. finite Radon measure). We show also polynomial estimates of the L^q norms of the derivatives of the vorticity $\zeta(t, \cdot)$ and the velocity $u(t, \cdot)$ uniformly in t up to $t = 0$ when both the initial data and the forcing terms are $W^{\infty,q}$ in x, provided $4/3 \leq q < 2$. These polynomial estimates (a new result by itself as far as we know) or rather the method of the proof have at least two implications: first, we are able to generalize the continuous dependence results in [1] and secondly, they allow us to examine the approximate regular solutions ζ^ε (respectively u^ε), $\varepsilon > 0$ of (V) (respectively (NS)) when the initial vorticity ω (respectively velocity a) is strongly singular and $\lim_{\varepsilon \searrow 0} \zeta^\varepsilon(0, \cdot) = \omega$ (respectively $\lim_{\varepsilon \searrow 0} u^\varepsilon(0, \cdot) = a$). Actually we can extend the polynomial estimates in the framework of L^q spaces for all indices $1 \leq q \leq \infty$ and hence the corresponding continuous dependence results in [1] using a rather sophisticated version of the techniques for estimates in weighted spaces developed here. Moreover, we can get some results on uniqueness of the zero viscosity limits provided the viscosity goes to zero slowly enough as $\varepsilon \searrow 0$ with respect to the perturbation of the initial data. This will be worked out in another paper.

Finally we exhibit a family of strongly singular radially symmetric solutions to (V) when the initial data is $bD^k\delta(x)$, where $D = (-\triangle)^{1/2}$, $k \geq 0$, $b \in I\!\!R$ and $\delta(x)$ stands for the Dirac measure massed at 0 (a well known phenomenon, see e.g. the paper of C. Oseen [15] in 1911 for $k = 0$).

The paper is organized as follows. In section 2 we introduce some weighted spaces and relate them to other spaces used in the literature. In section 3 we deal with (V) when the initial data and the forcing term are regular and prove polynomial type estimates for the derivatives of ζ. The approach of this section is used essentially in section 4 which deals with (V) and (NS) with singular initial data and forcing terms. Section 5 gives examples of strongly singular solutions of (V) and (NS).

2. Weighted spaces and the heat semigroup

We define several types of subspaces of $\mathcal{S}'(I\!\!R^n)$. First we recall the weighted Hölder spaces used in [9], namely for $\alpha \in I\!\!R$, $0 < T \leq \infty$, $1 \leq p \leq \infty$ we set

$$C_\alpha(L^p(I\!\!R^n) : T) := \{u :]0, T[\to L^p(I\!\!R^n);\ \|u\|_{C_\alpha(L^p(I\!\!R^n):T)} < \infty\}, \tag{2.1}$$

where $\|u\|_{C_\alpha(L^p(I\!\!R^n):T)} := \sup\limits_{0<t<T} (t^\alpha \|u(t)\|_p)$ will be also denoted by $\|u\|_{\alpha,p,T}$ and $\|\cdot\|_p$ stands for the L^p norm. The set of all $u \in C_\alpha(L^p(I\!\!R^n) : T)$ such that $\limsup\limits_{t\searrow 0}(t^\alpha\|u(t)\|_p) = 0$ is denoted by $\dot{C}_\alpha(L^p(I\!\!R^n) : T)$. We point out that if $\alpha < 0$ then $C_\alpha(L^p(I\!\!R^n) : T)$ is embedded in $\dot{C}_0(L^p(I\!\!R^n) : T)$.

Following [12] we denote by $\dot{L}^{s,p}(I\!\!R^n)$, $s \leq 0$, $1 \leq p < \infty$ the spaces

$$\dot{L}^{s,p}(I\!\!R^n) := D^{-s}L^p(I\!\!R^n), \quad \|\phi\|_{\dot{L}^{s,p}} = \|D^s\phi\|_p,\ \phi \in \dot{L}^{s,p}(I\!\!R^n). \tag{2.2}$$

The Morrey space $\mathcal{M}_{p,\lambda}(I\!\!R^n)$, $0 \leq \lambda \leq n$, $1 \leq p < \infty$ is defined as the set of all $f \in L^p_{loc}(I\!\!R^n)$ such that $\|f\|_{p,\lambda} := \sup_{x\in I\!\!R^n, R>0}(\|f\|_{L^p(B(x,R))}R^{-\lambda/p}) < \infty$, with the convention that if $p = 1$ we allow f to be a measure with $\|\cdot\|_{L^1(B(x,R))}$ being the total variation of f on the ball $B(x, R)$. We have $\mathcal{M}_{p,0}(I\!\!R^n) = L^p(I\!\!R^n)$ for $p > 1$ and $\mathcal{M}_{1,0}(I\!\!R^n)$ coincides with the space of finite Radon measures $\mathcal{M}(I\!\!R^n)$.

Now, for every $0 \le d \le n$, $k \ge 0$, $1 \le p \le \infty$, $0 < T \le \infty$, we set

$$M^k_{d,p}(I\!\!R^n : T) = \{f \in \mathcal{S}'(I\!\!R^n);\ \|f\|_{M^k_{d,p}(T)} < \infty\} \tag{2.3}$$

where

$$\|f\|_{M^k_{d,p}(T)} := \sup_{0<t<T}(t^{k/2}\|e^{t\triangle}f\|_p) + \sup_{0<t<T}(t^{k/2+d/(2p)}\|e^{t\triangle}f\|_\infty).$$

Clearly $M^k_{d,p}(I\!\!R^n : T)$ is a Banach space. We put $M^k_{d,p}(I\!\!R^n) := M^k_{d,p}(I\!\!R^n : \infty)$, $M^k_d(I\!\!R^n : T) := M^k_{d,1}(I\!\!R^n : T)$, $M^k_d(I\!\!R^n) := M^k_d(I\!\!R^n : \infty)$. One notes that $L^p(I\!\!R^n) \hookrightarrow M^0_{n,p}(I\!\!R^n : T)$, $0 < T \le \infty$, $1 \le p \le \infty$. Finally we define for $r \in Z\!\!\!Z_+ \bigcup \{\infty\}$ the space $C^r(M^k_{d,p}(I\!\!R^n : T))$ consisting of all $f \in M^k_{d,p}(I\!\!R^n : T)$ such that $\partial^\alpha f \in M^{k+|\alpha|}_{d,p}(I\!\!R^n : T)$ for $\alpha \in Z\!\!\!Z^n_+$, $|\alpha| \le r$.

The following result relates the spaces $M^k_{d,p}(I\!\!R^n)$ with $C_\alpha(L^p(I\!\!R^n) : T)$, $\mathcal{M}_{p,\lambda}(I\!\!R^n)$ and $\dot{L}^{s,p}(I\!\!R^n)$ and the heat group.

Theorem 1. *We have*

i) if $0 < T' < T \le \infty$, $1 \le p \le p' \le \infty$, $k, k' \ge 0$, $0 \le d, d' \le n$ then

$$M^k_{d,p}(I\!\!R^n : T) \hookrightarrow M^{k'}_{d',p'}(I\!\!R^n : T'), \tag{2.4}$$

provided $\theta = \min\{k' - k + \frac{d'}{p'} - \frac{d}{p}, k' - k - d(\frac{1}{p} - \frac{1}{p'})\} \ge 0$, and

$$M^k_{d,p}(I\!\!R^n) \hookrightarrow M^{k'}_{d,p'}(I\!\!R^n), \qquad \textit{if}\ \ k' - k = d(\frac{1}{p} - \frac{1}{p'}). \tag{2.5}$$

ii) the heat semigroup acts continuously

$$M^k_{d,p}(I\!\!R^n : T) \ni \omega \longrightarrow e^{t\triangle}\omega \in \bigcap_{q=p}^{\infty} C_{k/2+d/2(1/p-1/q)}(L^q(I\!\!R^n) : T), \tag{2.6}$$

with

$$\|e^{t\triangle}\omega\|_{k/2+d/2(1/p-1/q),q,T} \le \|\omega\|_{M^k_{d,p}(I\!\!R^n:T)}, \quad \omega \in M^k_{d,p}(I\!\!R^n : T), \tag{2.7}$$

if $1 \le p \le q \le \infty$, $k \ge 0$, $0 \le d \le n$, $0 < T \le \infty$.

iii) for all $0 < d \leq n$, $k \geq 0$ we have

$$(-\triangle)^{k/2}(\mathcal{M}_{1,n-d}(I\!R^n)\bigcap \mathcal{M}(I\!R^n)) \hookrightarrow C^\infty(M_d^k(I\!R^n)). \tag{2.8}$$

iv) for every $1 < p < \infty$, $k \geq 0$ we have

$$\dot{L}^{-k,p}(I\!R^n) \hookrightarrow C^\infty(M_n^k(I\!R^n)). \tag{2.9}$$

Proof. In order to show i) we note that for $0 < t \leq T' < T$, $\omega \in M_{d,p}^k(I\!R^n : T)$ we can write, taking into account the convexity property $\|f\|_q \leq (\|f\|_p)^{p/q}(\|f\|_\infty)^{1-p/q}$, $1 \leq p \leq q \leq \infty$,

$$t^{k'/2}\|e^{t\triangle}\omega\|_{p'} \leq t^{(k'-k)/2-d/2(1/p-1/p')}\|\omega\|_{M_{d,p}^k(T)}, \tag{2.10}$$

$$t^{k'/2+d'/(2p')}\|e^{t\triangle}\omega\|_\infty \leq t^{(k'-k+d'/p'-d/p)/2}\|\omega\|_{M_{d,p}^k(T)} \tag{2.11}$$

and combining (2.10) and (2.11) with the definition of θ we deduce (2.4). The proof of (2.5) is analogous and (2.6) and (2.7) are shown by similar arguments.

Concerning (2.8) we note that (2.2), Lemma 2.1, [8], yields the following two estimates: $\sup_{t>0} t^{d/2}\|e^{t\triangle}\omega\|_\infty < \infty$ (respectively $\sup_{t>0}\|e^{t\triangle}\omega\|_1 < \infty$) for each $\omega \in \mathcal{M}_{1,n-d}(I\!R^n)$ (respectively $\omega \in \mathcal{M}(I\!R^n)$). The arguments in [8] imply that $\sup_{t>0} t^{(k+d)/2}\|e^{t\triangle}D^k\omega\|_\infty < \infty$ (respectively $\sup_{t>0} t^{k/2}\|e^{t\triangle}D^k\omega\|_1 < \infty$) for each $\omega \in \mathcal{M}_{1,n-d}(I\!R^n)$ (respectively $\omega \in \mathcal{M}(I\!R^n)$) and all $k \geq 0$ which proves (2.8). The rest of the theorem follows from the estimates on the heat kernel acting on the spaces $\mathcal{M}(I\!R^n)$, $\mathcal{M}_{p,\lambda}(I\!R^n)$ and $\dot{L}^{s,p}(I\!R^n)$ cf. [8], [9], [12], [16].

Remark 1. We note that:

i) $\delta(x) \in \mathcal{M}(I\!R^n)$ and $\delta(x) \notin \dot{L}^{n/p-2,p}(I\!R^n)$ for all $p \geq n$.

ii) Let $\mu = \delta(x_1)g(x_2)$, with $g(x_2) \in L^1(I\!R)\bigcap L^\infty(I\!R)$. Then for each $k \geq 0$ $D^k\mu$ belongs both to $M_1^k(I\!R^2)$ (in view of (2.8)) and to the functional space $\mathcal{N}_{1,2,\infty}^{-k}(I\!R^2)$ defined by H. Kozono and T. Yamazaki in [13] which contains $D^k\mathcal{M}_{1,1}(I\!R^2)$ as a subspace (see Theorem 2.5, [13]). Straightforward calculations show that if $D_1 = |\partial_{x_1}|$ then the distribution $D_1\mu$ belongs to $M_1^k(I\!R^2)$

while one can not apply Theorem 2.5, [13] for $k \notin \mathbb{Z}_+$ since the symbol $|\xi_1|^k$ is not smooth as a function of $(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus 0$. We point out that we are able to give a kind of microlocal versions of the spaces $M^k_{d,p}(\mathbb{R}^n)$.

3. Polynomial estimates for (V)

The main result of this section is the following theorem.

Theorem 1. *Let $\omega \in W^{\infty,r}(\mathbb{R}^2)$ and let $F \in W^{\infty,1}_{loc}([0,\infty[: W^{\infty,r}(\mathbb{R}^2))$, where $1 \le r < 2$. Then there is a global solution $\zeta(t) = \zeta(t,\cdot)$ on $t > 0$ of (V) which has the following properties:*

i) $\zeta \in W^{\infty,1}_{loc}([0,+\infty[: W^{\infty,q}(\mathbb{R}^2))$ for every $q \in [r,\infty]$.

ii) (uniqueness) for each $q \in [q^*, 2[$, $q^* = \max\{4/3, r\}$ there is a unique $\zeta \in C([0,\infty[: L^q(\mathbb{R}^2))$ solving (V) with $\zeta(0) = \omega$.

iii) for each $k, s \in \mathbb{Z}_+$, $q \in [q^*, 2[$, there exists a positive constant $C = C_{q,s,k}$ such that

$$\|\partial_t^k \zeta(t)\|_{s,q} \le C\Theta^k_{s,q}(\omega, F; T)\left(\nu^{-1}(1 + \Theta^k_{s,q}(\omega, F; T))\right)^{sq/(2(q-1))}, \qquad (POL)_q$$

for all $\omega \in W^{\infty,q}(\mathbb{R}^2)$, $F \in W^{\infty,1}_{loc}([0,\infty[: W^{\infty,q}(\mathbb{R}^2))$, $0 < \nu \le 1$ and where

$$\Theta^k_{s,q}(\omega, F; T) = \|\omega\|_{s+2k,q} + \sum_{\mu=0}^{k} \sup_{0<t\le T} \|\partial_t^{\mu-1} F(t)\|_{s+2(k-\mu),q}, \qquad (3.1)$$

with the convention

$$\sup_{0<t\le T} \|\partial_t^{-1} F(t)\|_{s,q} := \int_0^T \|F(\tau)\|_{s,q} d\tau, \quad \|f\|_{s,q} = \sum_{|\alpha|\le s} \|\partial^\alpha f\|_q.$$

Moreover changing, if necessary, the constant C we claim that

$$\|\partial_t^k(\zeta_1(t) - \zeta_2(t))\|_{s,q} \le C\Theta^k_{s,q}(\omega_1 - \omega_2, F_1 - F_2; T)\times$$

$$\left(\nu^{-1}(1 + \Theta^k_{s,q}(\omega_1, F_1; T) + \Theta^k_{s,q}(\omega_2, F_2; T))\right)^{sq/(2(q-1))}, \qquad (CD)_q$$

provided $\omega_j \in W^{\infty,q}(\mathbb{R}^2)$, $F_j \in W^{\infty,1}_{loc}([0,\infty[: W^{\infty,q}(\mathbb{R}^2))$, $j = 1, 2$.

The proof of this theorem will be divided into several steps. For the sake of simplicity we deal with $(POL)_q$ when $k=0$. If $k \geq 1$ we observe that the L^q norm of $\partial_t^k \zeta(t)$ is equivalent to the L^q norm of $(\nu\triangle)^k\zeta(t) + \partial_t^{k-1}F(t)$. We set $\Theta_{s,q}(\omega, F; T) = \Theta^0_{s,q}(\omega, F; T)$. Further on, in order to simplify the proof, we shall assume $\nu = 1$, $r = 1$, so that $q^* = 4/3$.

First we will establish a local existence-uniqueness result.

Proposition 2. *Let $F \in L^1_{\mathrm{loc}}([0, +\infty) : L^q(I\!R^2))$ and let $\omega \in L^q(I\!R^2)$ for some $\frac{4}{3} \leq q < 2$. Then the inhomogeneous vorticity equation (V) (we consider $\nu = 1$) with initial data $\zeta|_{t=0} = \omega$ has a unique local solution in $C_{1-\frac{1}{q}}(L^q(I\!R^2) : T_o)$, for some $T_o > 0$ chosen below.*

Proof. Let $B^q(R) = \{\zeta \in C_{1-\frac{1}{q}}(L^q(I\!R^2) : T_o) : \|\zeta\|_{1-\frac{1}{q},q,T_o} \leq R\}$. We choose $R > 0$ and T_o such that J, defined on $B^q(R)$ by

$$J(\zeta)(t) = e^{t\triangle}\omega - \int_0^t e^{(t-\tau)\triangle}[\partial \cdot (\zeta S * \zeta)(\tau) - F(\tau)]d\tau, \tag{3.2}$$

$0 \leq t \leq T_o$, maps $B^q(R)$ into itself and it is a contraction. We have, for $\zeta \in B^q(R)$, using (1.1) and Lemma 1.1 in [9] with $r = \frac{2q}{4-q}$, $\chi_q = \chi(1/r - 1/q)$

$$\begin{aligned} t^{1-\frac{1}{q}}\|J(\zeta)(t)\|_q &\leq t^{1-\frac{1}{q}}\Theta_{0,q}(\omega, F, t) + \chi_q t^{1-\frac{1}{q}} \int_0^t \|\zeta S * \zeta(\tau)\|_{\frac{2q}{4-q}} (t-\tau)^{-\frac{1}{q}} d\tau \\ &\leq A(t) + c(\sup_{0\leq\tau\leq t} \tau^{1-\frac{1}{q}}\|\zeta(\tau)\|_q)^2, \end{aligned} \tag{3.3}$$

where $A(t) = t^{1-1/q}\Theta_{0,q}(\omega, F, t)$ and $c = c_q = \chi_q\sigma_q B(\frac{2-q}{q}, \frac{q-1}{q})$, $B(\cdot,\cdot)$ being the Beta function. T_o may be chosen as the unique solution of

$$t^{1-\frac{1}{q}} = \frac{3}{16c\Theta_{0,q}(\omega, F, t)}. \tag{3.4}$$

If $R = \frac{1}{4c}$ we have, from (3.3), that $t^{1-\frac{1}{q}}\|J(\zeta)(t)\|_q \leq R$ for $0 < t \leq T_o$. J is also a contraction: in fact let $\zeta_1, \zeta_2 \in B^q(R)$. Analogously to (3.3) we get

$$t^{1-\frac{1}{q}}\|[J(\zeta_1) - J(\zeta_2)](t)\|_q \leq 2cR\|\zeta_1 - \zeta_2\|_{1-\frac{1}{q},q,t} \leq \frac{1}{2}\|\zeta_1 - \zeta_2\|_{1-\frac{1}{q},q,T_o},$$

for $0 < t \leq T_o$.

Now, in the following proposition, we prove linear estimates for $\|\partial^\alpha \zeta(t)\|_q$ uniformly in an interval $[0, T_k]$, $k = |\alpha|$, $T_k \leq T_o$. First we recall the monotonicity property of $\zeta(t)$ which follows from Lemma 4.1 in [11]

$$\|\zeta(t)\|_p \leq \|\zeta(t')\|_p + \int_{t'}^{t} \|F(\tau)\|_p d\tau, \quad 1 \leq p \leq \infty,\ 0 \leq t' \leq t, \tag{IM}$$

provided the norms exist.

Proposition 3. *Let now $F \in L^1_{loc}([0, +\infty[: W^{\infty,q}(I\!R^2))$ and $\omega \in W^{\infty,q}(I\!R^2)$. If $\Theta_{k,q}(\omega, F, T_o)$ and T_k denote, respectively,*

$$\Theta_{k,q}(\omega, F, T_o) = \|\omega\|_{q,k} + \int_0^{T_o} \|F(\tau)\|_{q,k} d\tau \quad \textit{and} \quad T_k = \Big(\frac{3}{16c\Theta_{k,q}(\omega, F, T_o)}\Big)^{\frac{q}{q-1}}$$

then there is $c_k > 0$ such that for all $|\alpha| \leq k$ the following estimate holds

$$\sup_{0 \leq t \leq T_k} \|\partial^\alpha \zeta(t)\|_q \leq c_k \Theta_{k,q}(\omega, F, T_o). \tag{3.5}$$

Proof. The result holds for $k = 0$ with $c_o = 1$ in view of (IM). Let us assume that (3.5) holds for every $|\beta| < k$. Let $|\alpha| = k$ and $0 \leq t \leq T_k$. Then

$$\partial^\alpha \zeta(t) = \partial^\alpha \zeta_0(t) - \partial \cdot \int_0^t e^{(t-\tau)\triangle} \sum_{0 \leq \beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \partial^\beta \zeta(\tau) (S * \partial^{\alpha-\beta}\zeta)(\tau) d\tau \tag{3.6}$$

where $\zeta_0(t) = e^{t\triangle}\omega + \int_0^t e^{(t-\tau)\triangle} F(\tau) d\tau$. Taking the L^q-norm of (3.6) gives, using the estimates above and induction

$$\begin{aligned} \|\partial^\alpha \zeta(t)\|_q \quad \leq \quad & \Theta_{k,q}(\omega, F, T_o) + \frac{t^{1-\frac{1}{q}}}{1-\frac{1}{q}} \chi_q \sigma_q 2\Theta_{0,q}(\omega, F, T_o) \sup_{0 \leq \tau \leq t} \|\partial^\alpha \zeta(\tau)\|_q \\ & + \sum_{0 < \beta < \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} c_{|\beta|} c_{k-|\beta|} \Theta_{|\beta|,q}(\omega, F, T_o) \Theta_{k-|\beta|,q}(\omega, F, T_o) \frac{T_k^{1-\frac{1}{q}}}{1-\frac{1}{q}}. \end{aligned}$$

This implies, since $B(\frac{2-q}{q}, \frac{q-1}{q}) \geq 2\sqrt{2}$ for $\frac{4}{3} \leq q < 2$, that

$$\sup_{0 \leq t \leq T_k} \|\partial^\alpha \zeta(t)\|_q \leq \frac{5}{2} \Theta_{k,q}(\omega, F, T_o) \Big(1 + \frac{3}{16c(1-\frac{1}{q})} \sum_{0 < \beta < \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} c_{|\beta|} c_{k-|\beta|}\Big) \tag{3.7}$$

which shows (3.5).

Proposition 4. *Given $k \in \mathbb{Z}_+$, there is $C_{k,q} > 0$ such that, for every $t' \geq 0$ there is $h_k(t') > 0$ satisfying:*

i) if ζ solves (V) in $[0, t'] \times \mathbb{R}^2$ then ζ is extended uniquely as a solution of (V) to $[0, t' + h_0(t')] \times \mathbb{R}^2$ and for all $0 \leq h \leq h_0(t')$ we have

$$\sup_{0 \leq t - t' \leq h} (t - t')^{1-\frac{1}{q}} \|\zeta(t)\|_q \leq C_{0,q} \tag{3.8}$$

ii) if $k \in \mathbb{N}$ then for all $0 \leq h \leq h_k(t')$, $\alpha \in \mathbb{Z}^2$, $|\alpha| = k$ we have

$$\sup_{0 \leq t - t' \leq h} (t - t')^{\frac{k}{2}} \|\partial^\alpha \zeta(t)\|_q \leq C_{k,q}(\|\omega\|_q + \int_0^{t'+h} \|F(\tau)\|_{k,q} d\tau) \tag{3.9}$$

provided the solution exists in $[0, t] \times \mathbb{R}^2$, $t' \leq t \leq t' + h$.

Proof. Let us consider the Cauchy problem (V) with initial data at $t = t'$. Analogously to the local existence proof in $[0, T_o]$, one proves using (IM) that if $h(t') > 0$ is chosen as the unique solution of the equation

$$h^{1-\frac{1}{q}} = \frac{3}{16c\Theta_{0,q}(\omega, F, t' + h)} \leq \frac{3}{16c(\|\zeta(t')\|_q + \int_{t'}^{t'+h} \|F(\tau)\|_q d\tau)}$$

then J is a contraction in

$$B_{t'}(R) = \{\zeta : \sup_{t' \leq t \leq t' + h(t')} (t - t')^{1-\frac{1}{q}} \|\zeta(t)\|_q \leq R\}$$

and (3.8) is true. Here $R = (4c)^{-1}$ as in Proposition 2.

Let, for each $t' > 0$ and $k = 1, 2, \cdots,$ $\tilde{h}_k(t')$ be the unique solution of the equation

$$h^{1-\frac{1}{q}} = \frac{3}{16c2^{\frac{k}{2}}[\|\omega\|_q + \int_0^{t'+h} \|F(\tau)\|_{q,k} d\tau]} \tag{3.10}$$

and let $h_k(t') = \min\{1, \tilde{h}_k(t')\}$. Clearly $h_k(t')$ is nonincreasing in k and t'. We will prove (3.9) by induction. Let us denote $K = \chi_q \sigma_q$.

Now assume that (3.9) holds for $|\beta| < k$. If $0 \leq t - t' \leq h \leq h_k(t')$,

$$(t - t')^{\frac{k}{2}} \|\partial^\alpha \zeta(t)\|_q \leq \|\zeta(t')\|_q + (t - t')^{\frac{k}{2}} \int_{t'}^{t'+h} \|\partial^\alpha F(\tau)\|_q d\tau$$

$$+ K(t - t')^{\frac{k}{2}} \int_{t'}^{\frac{t+t'}{2}} (t - \tau)^{-\frac{k}{2} - \frac{1}{q}} d\tau \sup_{t' \leq \tau \leq t' + h} \|\zeta(\tau)\|_q^2$$

$$+K(t-t')^{\frac{k}{2}}\int_{\frac{t+t'}{2}}^{t}\Big[(t-\tau)^{-\frac{1}{q}}[2\|\zeta(\tau)\|_q\|\partial^\alpha\zeta(\tau)\|_q$$
$$+\sum_{0<\beta<\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\|\partial^\beta\zeta(\tau)\|_q\|\partial^{\alpha-\beta}\zeta(\tau)\|_q\Big]d\tau\le\|\omega\|_q+\int_0^{t'+h}\|\partial^\alpha F(\tau)\|_q d\tau$$
$$+K(t-t')^{1-\frac{1}{q}}\frac{1}{1-\frac{k}{2}-\frac{1}{q}}\Big(1-\frac{1}{2^{1-\frac{k}{2}-\frac{1}{q}}}\Big)\sup\|\zeta(\tau)\|_q^2$$
$$+K(t-t')^{1-\frac{1}{q}}\Big\{\Big[\|\omega\|_q+\int_0^{t'+h}\|F(\tau)\|_q d\tau]2\sup[(\tau-t')^{\frac{k}{2}}\|\partial^\alpha\zeta(\tau)\|_q\Big]$$
$$+\sum_{0<\beta<\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\sup\Big[(\tau-t')^{\frac{|\beta|}{2}}\|\partial^\beta\zeta(\tau)\|_q(\tau-t')^{\frac{k-|\beta|}{2}}\|\partial^{\alpha-\beta}\zeta(\tau)\|_q\Big]\Big\}\frac{2^{\frac{1}{q}+\frac{k}{2}-1}}{1-\frac{1}{q}}.$$

Notice that, since the right hand side of (3.10) is a decreasing function of h, we have that, for $t-t'\le h\le h_k(t')$ $(<h_{|\beta|}(t'))$,

$$(t-t')^{1-\frac{1}{q}}\le\frac{3}{16c2^{\frac{k}{2}}[\|\omega\|_q+\int_0^{t'+h}\|F(\tau)\|_{q,k}d\tau]}.$$

Also, by the choice of c and K, $\frac{3Kq2^{1/q}}{16c(q-1)}<\frac{1}{2}$. Then, using the induction hypothesis, we get (3.9).

Set now $t_0=h_0(0)$, $t_j=t_{j-1}+h_0(t_{j-1})$, $j\in I\!N$. Note that $\{t_j\}_{j=0}^\infty$ is a nondecreasing sequence. Clearly the globality of $\zeta(t)$ in $t>0$ will follow from

Proposition 5. *Under the hypotheses above* $\lim_{j\to\infty}t_j=\infty$.

Proof. Put $\tilde t=\sup_{j\ge0}t_j=\lim_{j\to\infty}t_j$. Assume that $\tilde t<\infty$. Then necessarily $h_j(t_{j-1})=t_j-t_{j-1}$ tends to zero as $j\to\infty$. On the other hand for j large enough $\tilde h_j(t_{j-1})=h_j(t_{j-1})$ and since $\tilde t\ge t_j$ we have

$$h_j(t_{j-1})=\frac{3}{16c(\|\omega\|_q+\int_0^{t_j}\|F(\tau)\|_q d\tau)}\ge\frac{3}{16c(\|\omega\|_q+\int_0^{\tilde t}\|F(\tau)\|_q d\tau)}$$

which clearly contradicts $\lim_{j\to\infty}h_j(t_{j-1})=0$. The proof is complete.

Concerning the estimates $(POL)_q$ we note that their proof relies on the estimates (3.5) applied for $0\le t\le h_k(0)$ and the estimates (3.9), the definition of $h_k(t)$ and its monotonicity property for $t\ge h_k(0)$. Similar but technically more complicated arguments lead to $(CD)_q$.

Clearly now Theorem 1 follows from the assertions proved above.

Remark 1. We point out that the polynomial estimates yield the possibility to study more precisely the behaviour for $\varepsilon \to 0$ of the family of solutions ζ^ε of (V) with $\omega^\varepsilon \in W^{\infty,1}(I\!R^2)$, $F^\varepsilon \in W^{\infty,1}_{loc}([0,\infty[: W^{\infty,1}(I\!R^2))$ for $0 < \varepsilon \leq 1$ and ω^ε (respectively F^ε) approximating very singular initial velocity ω (respectively forcing term F). The estimates $(POL)_q$, $4/3 \leq q < 2$ combined with the property of the heat kernel and the operator S imply that $\|\zeta^\varepsilon(t,\cdot)\|_{s,q}$ will have polynomial growth in ε^{-1} provided $\|\omega^\varepsilon(\cdot)\|_{s,q}$ and $\|F^\varepsilon(t,\cdot)\|_{s,q}$ have polynomial growth in ε^{-1}. In particular we can obtain generalized solutions of Colombeau type without any logarithmic growth conditions required in the papers of H.A. Biagioni and M. Oberguggenberger [4], [5] and H.A. Biagioni and R.J. Iorio Jr. [3], where other equations of Mathematical Physics are considered.

4. Solving (V) and (NS) with singular initial data

First we state a rather general theorem.

Theorem 1. *Let $q \in [4/3, 2[$. Assume that $F \in L^1_{loc}([0,\infty[: L^q(I\!R^2))$ and $\omega \in \mathcal{S}'(I\!R^2)$ satisfying $\sup_{0<t\leq T} t^{1-1/q}\|e^{t\triangle}\omega\|_q < \infty$ for each $T > 0$. Then there is a unique solution $\zeta \in C([0,\infty[: \mathcal{S}'(I\!R^2)) \bigcap C_{1-1/q}(L^q(I\!R^2) : T)$, $T > 0$, of (V) with initial value $\zeta(0) = \omega$ provided*

$$\eta_q(\omega) := \limsup_{t \searrow 0}(t^{(1-1/q)}\|e^{t\triangle}\omega\|_q) < \nu\varepsilon_q, \quad \varepsilon_q = (4\tilde{B}_q)^{-1}, \tag{4.1}$$

where $\tilde{B}_q = \pi^{-1}\sigma_q B(\frac{q-1}{q}, \frac{2-q}{q})\sqrt{\frac{3q-2}{2q}}(2\pi\Gamma(\frac{q}{3q-2}))^{3/2-1/q}$.

Next we consider (V) under the following hypotheses on the forcing term

$$F \in L^1_{loc}([0,\infty) : L^1(I\!R^2) \bigcap L^\infty(I\!R^2)), \tag{4.2}$$

and on the initial vorticity

$$\omega \in M^k_d(I\!R^2 : T), \quad T > 0,\ 0 \leq d \leq 2,\ k \geq 0. \tag{4.3}$$

The crucial restriction on the admissible singularity of ω is

$$d = 2,\ k = 0 \ \text{ or } \ k + \frac{d}{2} < 1. \tag{4.4}$$

Assuming (4.4) to be true, we set

$$q^* = \begin{cases} \frac{2-d}{2-d-k} & \text{if } 4k + d > 2 \text{ and } d < 2 \\ \frac{4}{3} & \text{otherwise.} \end{cases} \tag{4.5}$$

One notes that (4.4) and (4.5) imply $\frac{4}{3} \le q^* < 2$. Let

$$\theta(q) = (1 - \frac{d}{2})(1 - \frac{1}{q}) - \frac{k}{2}, \quad q \in [q^*, 2). \tag{4.6}$$

We note that $\theta(q)$ is strictly increasing if $d < 2$, $\theta(q) \ge 0$ for $q \in [q^*, 2[$ and the definition of q^* shows that $\theta(q^*) = 0$ iff $d < 2$ and $2 < 4k + d$.

Remark 1. We point out that if $k = 0$ and $d = 2$ we include finite Radon measures as initial vorticity. Moreover then $q^* = 4/3$ and the quantity $\eta_q(\omega)$ in (4.1) coincides (in the case $\omega \in \mathcal{M}(I\!R^2)$) with ℓ^q norm of the sequence $\{b_j\}_{j=1}^{\infty}$, where $\sum_{j=1}^{\infty} b_j \delta(x - \xi^j)$, $\xi^j \in I\!R^2$, $j = 1, 2, \ldots$ stands for the atomic part of ω.

Now we state the main result on (V) with singular initial vorticity.

Theorem 2. *Let ω and F satisfy (4.2)-(4.4). Then there is a global solution $\zeta(t)$ on $t > 0$ which has the following properties:*

i) $\zeta \in C_{k/2+d/2(1-1/q)}(L^q(I\!R^2) : T)$, for every $T > 0$ when $1 \le q \le \infty$. Moreover if $F \in W^{\infty,1}_{loc}([0,\infty) : W^{\infty,1}(I\!R^2))$ then $\zeta \in C^\infty(I\!R^+ \times I\!R^2)$ and for all $\alpha \in Z\!\!Z^2_+$, $r \in Z\!\!Z_+$ and $s \ge 0$

$$\partial_t^r D^s \zeta \in C_{(k+s)/2+r+d/2(1-1/q)}(L^q(I\!R^2) : T), \quad T > 0,\ 1 < q \le \infty, \tag{4.7}$$

$$\partial_t^r \partial^\alpha \zeta \in C_{(k+|\alpha|)/2+r+d/2(1-1/q)}(L^q(I\!R^2) : T), \quad T > 0,\ 1 \le q \le \infty. \tag{4.8}$$

ii) $\zeta \in C([0,\infty[: \mathcal{S}'(I\!R^2))$ and $\zeta(0) = \omega$.

iii) (uniqueness) If $\theta(q) > 0$, $q \in [q^*, 2[$ or, if $\theta(q) = 0$ and (4.1) holds, then there exists a unique solution $\zeta \in C_{1-1/q}(L^q(I\!R^2) : T) \bigcap C([0,\infty[: \mathcal{S}'(I\!R^2))$ for each $T > 0$.

iv) $v(t)=\zeta(t)-\zeta_0(t)\in C_{k/2+d/2(1-1/q)-\theta(q)}(L^q(\mathbb{R}^2:T)$ *for all* $1\le p\le\infty$, *where* $\zeta_0(t)=e^{t\nu\Delta}\omega+\int_0^t e^{(t-\tau)\nu\Delta}F(\tau)d\tau$.

Proof. We write the integral equation:

$$\zeta(t)=K(\zeta)(t)=\zeta_0(t)+K_0(\zeta)(t), \tag{4.9}$$

where

$$K_0(\zeta)(t)=-\partial\cdot\int_0^t e^{(t-\tau)\nu\Delta}(\zeta S*\zeta)(\tau,\cdot)d\tau. \tag{4.10}$$

Set $\|f\|_{q,T}:=\|f\|_{k/2+d/2(1-1/q),q,T}$. For every $q\in[4/3,2[$ we have, taking into account (1.1) and the estimates for the heat kernel,

$$t^{k/2+d/2(1-1/q)}\|K_0(\zeta)(t)\|_q\le B_q\nu^{-1/q}t^{\theta(q)}(\sup_{0<\tau<t}\|\zeta(\tau)\|_q)^2, \tag{4.11}$$

where $\theta(q)$ is given by (4.6).

In a similar way we deduce the estimate

$$\|K(\zeta_1)-K(\zeta_2)\|_{q,T}\le 2B_q\nu^{-1/q}T^{\theta(q)}\|\zeta_1-\zeta_2\|_{q,T}\max\{\|\zeta_1\|_{q,T},\|\zeta_2\|_{q,T}\}, \tag{4.12}$$

where $B_q=\pi^{-1}\sigma_q B(\frac{q-1}{q},\frac{(1-k-d)q+d}{q})\sqrt{\frac{3q-2}{2q}}(2\pi\Gamma(\frac{q}{3q-2}))^{3/2-1/q}$.

Hence (4.10), (4.11) and (4.12) show that in order to apply "Fixed Point Theorem" (FPT) we have to examine the equation $a_{q,\nu}(T)-z+B_q\nu^{-1/q}T^{\theta(q)}z^2=0$ where $a_{q,\nu}(T)=a_{q,\nu}(\omega,F;T)=\|\zeta_0\|_{q,T}$. If $D(T)=1-4\nu^{-1/q}T^{\theta(q)}a_{q,\nu}(T)B_q>0$ for some $T>0$ we can apply the FPT in the ball $B^q(r)=\{f:\|f\|_{q,T}\le r\}$ with r being the smallest root of the quadratic equation, namely

$$r=\frac{1-\sqrt{D(T)}}{2B_q\nu^{-1/q}T^{\theta(q)}}=\frac{2a_{q,\nu}(T)}{1+\sqrt{D(T)}}. \tag{4.13}$$

Indeed, the estimate (4.11) implies that K acts in the ball $B^q(r)$ while for the contraction (4.12) implies $\|K(\zeta_1)-K(\zeta_2)\|_{q,T}\le(1-\sqrt{D(T)})\|\zeta_1-\zeta_2\|_{q,T}$ for all $\zeta_1,\zeta_2\in B^q(r)$.

We deal first with the case $\theta(q)>0,\ q\in[q^*,2[$. We recall the estimate

$$a_{q,\nu}(t)\le A_q\nu^{-k/2-d/2(1-1/q)},\quad 0<t\le1,\ 0<\nu\le1, \tag{4.14}$$

where A_q depends linearly on $\|\zeta_0\|_{q,1}$. We have $D(T) \geq 1/4$ for all $T \in]0, T_{\nu,q}[$ provided $T_{\nu,q} = \left(\frac{3\nu^{(k+d)/2+(1-d/2)1/q}}{16A_qB_q}\right)^{1/\theta(q)}$.

Next we consider the case $\theta(q) = 0$. Taking into account the definition of $\eta_q(\omega)$ we can write $a_{q,\nu}(T) = \nu^{-k/2-d/2(1-1/q)}(\eta_q(\omega) + o(1))$ as $T \searrow 0$ uniformly in $\nu \in]0,1]$. Hence the smallness condition $\eta_q(\omega) < \frac{\nu^{(k+d)/2+(1-d/2)/q}}{4B_q}$ guarantees the application of FPT in $B^q(r)$ for $T_{\nu,q} > 0$ small enough. However we do not have control of the dependence of $T_{\nu,q}$ on ν and ζ_0.

Now we show the estimates up to $t = 0$ in $L^r(I\!\!R^2)$, $1 \leq r < q^*$. Take first $r \in [r^*, q^*[$, where $r^* = \frac{2q^*}{4-q^*}$ and set $q = 4r/(2+r) \in [4/3, 2[$. Putting $z_q(t) = \|\zeta\|_{q,t}$ we can write

$$(\nu t)^{k/2+d/2(1-1/r)}\|\zeta(t)\|_r \leq C_r + \nu^{1/2-k/2-d/4}K_r T_{\nu,q}^{1/2-k/2-d/4}(z_q(T_{\nu,q}))^2 \quad (4.15)$$

for $0 < \nu \leq 1$, $0 < t \leq T_{\nu,q}$ and $C_r > 0$, $K_r > 0$. If $r^* > 1$ we proceed further on by setting $q_1 = \max\{4/3, r^*\}$ and then $r_1 = 2q_1/(4-q_1)$. We point out that $r_1 = 1$ if $q_1 = 4/3$. Since we have already proved that

$$\zeta \in C_{k/2+d/2(1-1/q)}(L^q(I\!\!R^2) : T), \quad (4.16)$$

for $q \in [q_1, 2[$ we repeat the arguments used above and show that (4.16) is true for $q \in [r_1, 2[$. Then if $r_1 > 1$ we get $q_2 = \max\{4/3, r_1\}$ and $r_2 = 2q_2/(4-q_2)$ and so on until we arrive for some s at $q_s = 4/3$ and hence $r_s = 1$.

The validity of (4.16) for $q \geq 2$ follows from the inequality

$$\|K_0(\zeta)(t)\|_q \leq C_{q,\tilde{q}}\nu^{-2/\tilde{q}}(z_{\tilde{q}}(t))^2, \quad C_{q,\tilde{q}} > 0 \quad (4.17)$$

where $\tilde{q}$ satisfies $2/\tilde{q} - 1/q < 1$, $\tilde{q} \in [4/3, 2[$ if $q \geq 2$.

In order to extend (4.7) for $s > 0$ we modify for the inhomogeneous case the approach of [9]. Once done this we must deduce (4.8) globally. For $1 < q < \infty$ we can use the identity $\partial^\alpha = (\partial^\alpha(-\triangle)^{-|\alpha|/2})(-\triangle)^{|\alpha|/2}$ and Mikhlin's theorem on multipliers [14] ($\xi^\alpha|\xi|^{-|\alpha|}$ is (positively) homogeneous of zero order) in order to verify (4.7) for all $T > 0$. The case $q = \infty$ follows from Gagliardo-Nirenberg inequalities while for $q = 1$ we use that the L^1 norm of $\partial^\alpha\zeta(t)$ is estimated via the $L^{4/3}$ norms of $\partial^\beta\zeta(t)$, $\beta < \alpha$, taking into account (3.6), the proof of (3.9) and (4.15) with $r = 1$, $\tilde{q} = 4/3$.

Now for the initial condition in ii) we proceed as in [9]. The claim that $\zeta(t)$ is global in $t > 0$ follows from the arguments used in the previous section since $\zeta(t) \in L^1(I\!R^2) \bigcap L^\infty(I\!R^2)$ for $t > 0$ small enough. The uniqueness was also proved in view of the exact estimates on the length of the time interval in order to apply FPT and the monotonicity. The proof is complete.

We note that the arguments used in showing Theorem 2 yield the proof of Theorem 1. We also show in an analogous way

Theorem 3. *Let ω and F satisfy (4.2)-(4.4). Let ζ be a solution of (V) given by the previous theorem. Then $u = S * \zeta$ is a solution of (NS) with the following properties:*

i) $u \in C_{(k-1)/2+d/2(1-1/q)}(L^q(I\!R^2) : T)$, for every $T > 0$ when $2 < q \leq \infty$. Moreover if $F \in W^{\infty,1}_{loc}([0, \infty[: W^{\infty,1}(I\!R^2))$ then for all $\alpha \in Z\!\!Z^2_+$, $r \in Z\!\!Z_+$ such that $r + |\alpha| \geq 1$

$$\partial_t^r \partial^\alpha u \in C_{(k-1+|\alpha|)/2+r+d(1-1/q)}(L^q(I\!R^2) : T), \quad T > 0, \, 1 < q \leq \infty. \tag{4.18}$$

ii) $u \in C([0, \infty[: \mathcal{S}'(I\!R^2))$, $2 < p < \infty$ and $u(0) = a = S * \omega$.

iii) (uniqueness) If $\theta(q) > 0$, $q \in [q^*, 2[$ or, if $\theta(q) = 0$ and $\eta_p(\omega)$ is small enough, then there exists a unique solution $u \in C_{1-1/p}(L^p(I\!R^2) : T) \bigcap C([0, \infty[: \mathcal{S}'(I\!R^2))$, $p = \frac{2q}{2-q}$, for each $T > 0$.

iv) $w(t) = u(t) - u_0(t) \in C_{(k-1)/2+d/2(1-1/q)-\theta(2p/(p+2))}(L^p(I\!R^2) : T)$, $T > 0$, for all $2 < p \leq \infty$, where $u_0(t) = e^{t\nu\triangle}a - \int_0^t e^{(t-\tau)\triangle}G(\tau)d\tau$.

Remark 2. We stress that the methods used in the proofs of the results on the Cauchy problem for (NS) and (V) and the estimates $(POL)_q$ and $(CD)_q$ will serve us in further extensions. More precisely:

i) We can show continuous dependence on the initial data and the right-hand side generalizing in particular Theorem B in [1].

ii) We can obtain local uniqueness, existence and continuous dependence results in weighted spaces when the forcing term F in (V) belongs to $C([0,\infty[: \mathcal{M}(I\!R^2))$.

iii) We can treat also the Euler equation with singular forcing term.

5. Strongly singular solutions

The main result in this section is the following one

Theorem 1. *Let $k \geq 0$. Then $\zeta(t) = (-\triangle)^{k/2}\delta(x)$ is a solution of (V) with initial vorticity $\omega = (-\triangle)^{k/2}\delta(x)$ which clearly is radially symmetric in $x \in I\!R^2$. Moreover $u(t) = S * \zeta(t)$ satisfies (NS) with initial velocity $(-\triangle)^{k/2}S$.*

Proof. Concerning (V) it is enough to show that

$$\partial \cdot (\zeta S * \zeta)(t) \equiv 0 \quad \text{for} \quad t > 0. \tag{5.1}$$

We sketch the calculations on a formal level. We note that $\zeta(t) \in L^q(I\!R^2)$, $1 \leq q \leq \infty$ and $u(t) \in L^p(I\!R^2)$, $2 < p < \infty$ for every $t > 0$. Hence in view of the Hölder inequality $\zeta(t)u(t) \in L^1(I\!R^2)$. Therefore applying the Fourier transform $\mathcal{F}_{x\to\xi}$ to (5.1) we see that (5.1) is equivalent to

$$\hat{u}_1 * \widehat{\partial_1\zeta}(t,\xi) + \hat{u}_2 * \widehat{\partial_2\zeta}(t,\xi) \equiv 0 \quad \text{for} \quad \xi \in I\!R^2,\, t > 0. \tag{5.2}$$

We have $\zeta(t) = e^{t\triangle}D^k\delta$, $u(t) = S * \zeta(t)$ and taking into account the properties of the Fourier transform and the convolution we deduce that the left-hand side $H(t,\xi)$ of (5.2) could be written as follows:

$$H(t,\xi) = i(2\pi)^{-1}\int_{I\!R^2}(\xi_1\eta_2 - \xi_2\eta_1)e^{-t(|\xi-\eta|^2+|\eta|^2)}\frac{|\eta|^k}{|\xi-\eta|^2}d\eta. \tag{5.3}$$

We note that the integral above is absolutely convergent and $H(t,\xi)$ is continuous in $t > 0$ and $\xi \in I\!R^2$. We introduce the polar coordinates $\eta_1 = \rho\cos\theta$, $\eta_2 = \rho\sin\theta$, $\rho > 0$, $0 \leq \theta \leq 2\pi$. Set $a_\xi(\theta) = \xi_1\cos\theta + \xi_2\sin\theta$. Then we have

$|\xi-\eta|^2 = \xi^2 - 2\rho a_\xi(\theta) + \rho^2$, $\xi_1\eta_2 - \xi_2\eta_1 = \rho a'_\xi(\theta)$. Now we can rewrite $H(t,\xi)$ as

$$H(t,\xi) = \int_0^\infty \left(\int_0^{2\pi} \frac{d}{d\theta}(Q(\xi,\rho,\theta))d\theta \right) d\rho, \tag{5.4}$$

with

$$Q(\xi,\rho,\theta) = \frac{\rho^{k+1}}{\rho^2 - 2\rho a_\xi(\theta) + \xi^2} e^{-t(2\rho^2 - 2\rho a_\xi(\theta) + \xi^2)}.$$

Evidently the integral in θ is zero if we integrate formally by parts since the function is 2π-periodic in θ. One can justify that this is rigorous by considering the integration with respect to ρ in $|\rho - |\xi|| > \varepsilon$ for $\varepsilon > 0$ and then letting $\varepsilon \searrow 0$.

One deals with (NS) in a similar way.

Remark 1. We can prove that the above assertion remains true if $\omega = (-\triangle)^{k/2}\delta_{S_r}$, $k \geq 0$, $r > 0$, with δ_{S_r} being the delta function on the circle $S_r = \{x \in I\!R^2 : \|x\| = r\}$. In particular, taking into account Theorem 2 in the previous section, we can show that if $k < 1/2$ the radially symmetric solution $e^{t\nu\triangle}\omega$ is unique in $C_{1-1/q}(L^q(I\!R^2) : \infty) \bigcap C([0,\infty[: \mathcal{S}'(I\!R^2))$ for $\max\{4/3, 1/(1-k)\} \leq q < 2$.

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