THE SEMIGROUP APPROACH TO SYSTEMS OF CONSERVATION LAWS

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1. Introduction

This survey paper provides an account of recent developments concerning the uniqueness and continuous dependence of solutions to systems of conservation laws. Progress has been achieved by a new approach, based on the construction of a Lipschitz semigroup compatible with the standard solutions of Riemann problems. We shall try to give here an overwiew of the main ideas and techniques involved in this "semigroup approach", with the aid of several pictures. Proofs will often be given in a rather sketchy and informal way. Readers interested in technical details can find them in the references [3, 6, 7, 8, 11, 13].

Consider the Cauchy problem for a strictly hyperbolic $n \times n$ system of conservation laws in one space dimension:

$$u_t + [f(u)]_x = 0, \tag{1.1}$$

$$u(0,x) = \bar{u}(x). \tag{1.2}$$

We assume that each characteristic field is either linearly degenerate or genuinely nonlinear, in the sense of Lax [26].

In 1965, the global existence of weak solutions was proved in the fundamental paper of Glimm [22], within a class of functions with small total variation. Since then, the uniqueness of these solutions has been investigated by several authors.

A classical approach, followed in [18, 20, 24, 28, 31], establishes the uniqueness of those weak solutions which satisfy additional entropy-admissibility conditions, together with suitable regularity assumptions. Proofs are usually obtained by a Holmgren type argument.

The semigroup approach, developed at S.I.S.S.A. in the past two years, can be summarized in the following steps:

(1) Construct a uniformly Lipschitz semigroup S of weak solutions of (1.1), consistent with the standard solutions of the Riemann problems.

(2) Show that this semigroup is essentially unique, and that each trajectory $t \mapsto S_t \bar{u}$ provides the unique "good" solution of the corresponding Cauchy problem (1.1), (1.2).

(3) Provide an intrinsic characterization of the trajectories of the semigroup, in terms of local integral estimates.

Compared with previous results, this approach has two main advantages:

(i) It provides the uniqueness of solutions within the same class of **BV** functions where an existence theorem can be proved.

(ii) It yields the Lipschitz continuous dependence of solutions on the initial data, as well as error estimates concerning the $\mathbf{L}^1$ distance between an approximate solution and the exact one.

The plan of the paper is as follows. Section 2 contains a brief survey of the main results. Sections 3 to 5 present the basic ideas involved in the construction of the semigroup. We first examine how the distance between two "infinitesimally close" solutions changes at times of wave-front interactions. This leads to the introduction of a Riemann type metric, uniformly equivalent to the standard $\mathbf{L}^1$ distance, which is contractive w.r.t. the flow generated by the conservation equations. In section 5 we describe the key steps in the construction of approximate solutions which depend Lipschitz continuously on the initial data. In the limit, we obtain a "Standard Riemann Semigroup" of solutions of (1.1).

Section 6 reviews a construction of piecewise constant approximate solutions by wave-front tracking [4, 8]. These approximations play a key role in the proof of uniqueness of the semigroup, which is given in section 7. In section 8 we motivate the characterization of semigroup trajectories in terms of local integral estimates, sketching the main ideas of the proof.

Various extensions of the basic theory have recently appeared. For 2×2 systems, the Lipschitz continuous dependence of solutions is proved in [17] for non-homogeneous Cauchy problems, in [1] for initial-boundary value problems, and in [10] (locally) for the Cauchy problem with large **BV** data. For some $n \times n$ systems in the Temple class, a continuous semigroup exists whose domain contains functions with arbitrarily large total variation [2]. Rigorous error estimates on the $\mathbf{L}^1$ distance between a semigroup trajectory and an approximate solution constructed by the Glimm scheme can be found in [14].

2. Survey of the main results

The basic result is concerned with the existence of a continuous semigroup generated by (1.1). In the scalar case, the evolution equation is determined by a monotone operator, and the standard theory of nonlinear contractive semigroups can be applied [15]. When $n > 1$, however, we are no longer within the framework of this theory. Speaking of "the semigroup generated by a system of conservation laws", a different definition should thus be adopted.

Figure 1

Our framework is as follows. We consider a closed domain $\mathcal{D} \subset \mathbf{L}^1(I\!R;\ I\!R^n)$ consisting of functions with small total variation. Inside $\mathcal{D}$ there will be a dense subset $\mathcal{D}_{pc}$ of piecewise constant functions. For every initial data $\bar{u} \in \mathcal{D}_{pc}$, a local solution of (1.1), (1.2) can be obtained by piecing together the standard solutions of the Riemann problems determined by the jumps in $\bar{u}$ (fig.1). This construction yields a unique, piecewise Lipschitz, entropy-admissible weak solution, defined up to the first time $\tau = \tau(\bar{u})$ where waves originating from distinct Riemann problems interact together. (Of course, by Glimm's theorem a weak solution exists in **BV** also for times $t > \tau(\bar{u})$, but its structure may be very complicated as a result of successive wave interactions. Moreover, its uniqueness is not clear, at this stage.)

From an abstract point of view, we thus have a local flow, say

$$t \mapsto S_t \bar{u}, \qquad \bar{u} \in \mathcal{D}_{pc}, \quad t \in [0,\ \tau(\bar{u})]. \tag{2.1}$$

We wish to extend this local flow to a globally Lipschitz continuous semigroup $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$. If this can be done, we say that S is a *Standard Riemann Semigroup* generated by the system of conservation laws (1.1).

The following theorems show that this abstract extension problem admits a unique solution. It is a remarkable fact that all trajectories of the semigroup S thus constructed are indeed weak, entropy-admissible solutions of (1.1). Actually, the conservation equations and the entropy conditions are both encoded in the requirement that the semigroup be compatible with the standard solutions of the Riemann problems. Since the set of piecewise constant functions is dense, the entire semigroup is uniquely determined by the way in which these Riemann problems are solved.

Theorem 1. *Let $\Omega \subseteq I\!R^n$ be an open set containing the origin, and let $f : \Omega \mapsto I\!R^n$ be a smooth map. Assume that the system (1.1) is strictly hyperbolic and that each characteristic field is either linearly degenerate or genuinely nonlinear. Then there exists a closed domain $\mathcal{D} \subset \mathbf{L}^1(I\!R;\ I\!R^n)$ and a* **Standard Riemann Semigroup** *on $\mathcal{D}$, i.e. a continuous map $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$ and constants $L, \delta_0 > 0$ with the properties:*

(i) $S_0\bar{u} = \bar{u}, \quad S_s S_t \bar{u} = S_{s+t}\bar{u}$,

(ii) *Every function* $\bar{u} \in \mathbf{L}^1$ with Tot.Var.$(\bar{u}) \leq \delta_0$ lies in $\mathcal{D}$,

(iii) *For all* $\bar{u}, \bar{v} \in \mathcal{D}, \ \ t, s \geq 0$ *one has* $\|S_t\bar{u} - S_s\bar{v}\|_{\mathbf{L}^1} \leq L(|t-s| + \|\bar{u} - \bar{v}\|_{\mathbf{L}^1})$.

(iv) *If* $\bar{u} \in \mathcal{D}$ *is piecewise constant, then for* $t > 0$ *sufficiently small the function* $u(t,\cdot) = S_t\bar{u}$ *coincides with the solution of (1.1)-(1.2) obtained by piecing together the standard self-similar solutions of the corresponding Riemann problems.*

The positively invariant domain $\mathcal{D}$ has the form

$$\begin{aligned} \mathcal{D} = \quad & cl\,\{u \in \mathbf{L}^1(I\!R; I\!R^n); \quad u \text{ is piecewise constant}, \\ & V(u) + C \cdot Q(u) < \delta\}, \end{aligned} \tag{2.2}$$

for suitable constants $C, \delta > 0$. Here $V(u)$ and $Q(u)$ denote the total strength of waves and the wave interaction potential of u, as defined by Glimm [21], while cl denotes closure.

We remark that the standard constructive methods found in the literature [4, 19, 21, 22, 29, 33] are all based on a compactness argument. They yield the existence of weak solutions, but provide no information concerning uniqueness or continuous dependence. To construct a Lipschitz semigroup of solutions, somewhat different techniques must be introduced.

Theorem 1 was proved in [3, 5] for some classes of $n \times n$ systems with coinciding shock and rarefaction curves, and in [9] for general 2×2 systems. Other special cases were studied in the earlier papers [30, 32]. The construction of the semigroup in the general $n \times n$ case is worked out in the forthcoming paper [11].

Observe that the statement of Theorem 1 does not explicitly say that the trajectories of the semigroup are actually weak solutions of (1.1). This fact, however, can be deduced as a consequence of (i)–(iv), together with a number of additional properties which are collected below.

Theorem 2. *On a given domain $\mathcal{D}$ of the form (2.2), there exists at most one semigroup $S : \mathcal{D} \times [0,\infty[\mapsto \mathcal{D}$, satisfying the conditions (i)–(iv) listed in Theorem 1. For such a semigroup, the following properties also hold:*

(v) *Each trajectory $t \mapsto u(t,\cdot) = S_t\bar u$ is a weak, entropy-admissible solution of the corresponding Cauchy problem (1.1), (1.2).*

(vi) *Let $(u_\nu)_{\nu\geq 1}$ be a sequence of approximate solutions of (1.1), (1.2) generated by a wave-front tracking algorithm, or by the Glimm scheme with uniformly distributed sampling. Then, as $\nu \to \infty$, one has $\mathbf{L}^1$-$\lim u_\nu(t,\cdot) = S_t\bar u$, for every $t \geq 0$.*

(vii) *Let $u = u(t,x)$ be a piecewise Lipschitz, entropic solution of (1.1), (1.2), defined on some strip $[0,T]\times I\!R$. Then $u(t,\cdot) = S_t\bar u$ for all $t \in [0,T]$.*

A proof of Theorem 2 is contained in [7].

All of the above properties indicate that the semigroup trajectory $t \mapsto S_t\bar u$ should be regarded as the one and only "good" solution of the corresponding Cauchy problem. In connection with the specific Cauchy problem (1.1), (1.2), these results show that a solution u coincides with the unique semigroup trajectory if either u is a limit of wave-front tracking or Glimm approximations, or else if u can be embedded in a suitable flow of solutions. It is convenient to have an intrinsic characterization of semigroup trajectories, which does not refer to the method used to construct such solutions. The characterization given below is based on two types of integral estimates, involving local approximate parametrices for (1.1). One is derived from the self-similar solution of a Riemann problem, the other is obtained by "freezing" the coefficients of the corresponding quasilinear hyperbolic system in a neighborhood of a given point.

Figure 2

Consider a function $u : [0,T] \times I\!R \mapsto I\!R^n$, and fix any point (τ, ξ) in the domain of u. If $u(\tau, \cdot) \in \mathcal{D}$, the bound on the total variation implies the existence of the limits

$$u^- = \lim_{x \to \xi-} u(\tau, x), \qquad\qquad u^+ = \lim_{x \to \xi+} u(\tau, x).$$

Call $\omega = \omega(t,x)$ the self-similar solution of the Riemann problem

$$u_t + [f(u)]_x = 0, \qquad\qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0, \\ u^+ & \text{if} \quad x > 0. \end{cases} \tag{2.3}$$

Let $\hat\lambda$ be an upper bound for all characteristic speeds. For $t > \tau$, define (fig.2)

$$U^\sharp_{(u;\tau,\xi)}(t,x) \doteq \begin{cases} \omega(t-\tau,\ x-\xi) & \text{if} \quad |x-\xi| \le \hat\lambda(t-\tau), \\ u(\tau,x) & \text{if} \quad |x-\xi| > \hat\lambda(t-\tau). \end{cases} \tag{2.4}$$

The function $t \mapsto U^\sharp_{(u;\tau,\xi)}(t,\cdot)$ is then Lipschitz continuous with values in $\mathbf{L}^1$, and approaches $u(\tau,\cdot)$ as $t \to \tau+$. Notice that this last property would fail, had we defined simply $U^\sharp(t,x) = \omega(t-\tau, x-\xi)$ for all t, x.

Figure 3

Next, call $\widetilde{A} \doteq Df(u(\tau,\xi))$ the Jacobian matrix of f computed at the point $u(\tau,\xi)$. For $t > \tau$, define $U^\flat_{(u;\tau,\xi)}(t,x)$ as the solution of the linear hyperbolic Cauchy problem with constant coefficients (fig.3)

$$w_t + \widetilde{A} w_x = 0, \qquad\qquad w(\tau, x) = u(\tau, x). \tag{2.5}$$

In the following, by Tot.Var.$\{u(\tau);\ I\}$ we denote the total variation of the function $u(\tau,\cdot)$ over the set I.

Definition 1. Let $u : [0,T] \mapsto \mathbf{BV}$ be continuous w.r.t. the $\mathbf{L}^1$-norm. We say that u is a *viscosity solution* of the system of conservation laws (1.1) if there exists a constant $C > 0$ such that, at each point $(\tau,\xi) \in [0,T[\times I\!R$, for every $\rho, \varepsilon > 0$ sufficiently small one has

$$\begin{aligned} \frac{1}{\varepsilon} & \int_{\xi-\rho+\varepsilon\hat\lambda}^{\xi+\rho-\varepsilon\hat\lambda} \Big| u(\tau+\varepsilon, x) - U^\sharp_{(u;\tau,\xi)}(\tau+\varepsilon, x) \Big|\, dx \\ & \leq C \cdot \text{Tot.Var.}\{u(\tau);\ ]\xi-\rho,\ \xi[\ \cup\ ]\xi,\ \xi+\rho[\,\}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \frac{1}{\varepsilon} & \int_{\xi-\rho+\varepsilon\hat\lambda}^{\xi+\rho-\varepsilon\hat\lambda} \Big| u(\tau+\varepsilon, x) - U^\flat_{(u;\tau,\xi)}(\tau+\varepsilon, x) \Big|\, dx \\ & \leq C \cdot \Big(\text{Tot.Var.}\{u(\tau);\ ]\xi-\rho,\ \xi+\rho[\,\} \Big)^2. \end{aligned} \tag{2.7}$$

If also $u(0,\cdot) = \bar u$, then we say that u is a viscosity solution of the Cauchy problem (1.1), (1.2).

Remark 1. In the above definition we require that, at each point (τ, ξ), both estimates hold for $\rho, \varepsilon > 0$ sufficiently small. However, it is not difficult to prove the implications

(2.7) $\Longrightarrow$ (2.6) if $u(\tau, \cdot)$ is continuous at ξ,

(2.6) $\Longrightarrow$ (2.7) if $u(\tau, \cdot)$ has a jump at ξ.

Therefore, it would be just the same to require only that (2.6) holds at points of jump and that (2.7) holds at points of continuity.

Remark 2. Even at a superficial glance, it is clear that the above definition is entirely different from the one introduced by Crandall and Lions [16] in connection with Hamilton-Jacobi equations. Indeed, in [16] solutions are sought within a class of scalar, continous functions. It is thus natural to characterize them by inequalities involving pointwise upper and lower gradients. In the case of systems of conservation laws, on the other hand, solutions are found in a class of vector-valued, discontinuous functions; they can be appropriately characterized by means of local integral estimates. The two definitions, however, have the same goal: to single out a class of physically admissible solutions, nicely depending on the data of the problems.

The next result, proved in [7], provides a complete characterization of semigroup trajectories.

Theorem 3. *Let $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$ be a Standard Riemann Semigroup generated by the system of conservation laws (1.1). Then a continuous map $u : [0, T] \mapsto \mathcal{D}$ is a viscosity solution of (1.1), (1.2) if and only if it coincides with the semigroup trajectory $t \mapsto S_t \bar{u}$ for $t \in [0, T]$.*

Remark 3. It remains an open problem to prove that the limits as $\varepsilon \to 0$ of the viscous approximations

$$u_t + [f(u)]_x = \varepsilon u_{xx}$$

actually coincide with semigroup trajectories. In this connection we observe that, if the initial condition $u(0, \cdot)$ is piecewise constant, then for $t > 0$ small the semigroup solution $u(t, \cdot) = S_t \bar{u}$ is piecewise smooth. By a result of Goodman and Xin [23], it can thus be obtained as a limit of viscous approximations. If the vanishing viscosity method produces solutions with bounded variation which depend Lipschitz continuously on the initial data, then, by the uniqueness of the flow stated in Theorem 2, these solutions must coincide with semigroup trajectories. In particular, they must be "viscosity solutions" in the sense of Definition 1.

3. Basic interaction estimates

For a scalar conservation law, even in several space dimensions, it is well known that the entropic solutions constitute a contractive semigroup [15, 25]. This is definitely not the case for systems [36]. To understand how the distance between two solutions of (1.1) changes in time, we first study two basic examples.

Figure 4

Example 1. Let $u = u(t, x)$ be a piecewise constant solution, consisting of two shocks of different families (say, $i_1 > i_2$) which interact at some time τ (fig.4). Denote by $x_1(t)$, $x_2(t)$ the locations of these two fronts and call σ_1^-, σ_2^- their strengths before interaction. After time τ, as a result of the interaction, their strengths will be different, say σ_1^+, σ_2^+ respectively. In addition, the interaction will produce new wave-fronts in the other characteristic families, say with strengths σ_j, $(j \neq i_1, i_2)$.

Next, consider a slightly perturbed solution $\tilde{u}$, obtained from u by shifting the initial positions of the shocks (along the x-axis) in the amounts ξ_1^-, ξ_2^- respectively. After the interaction has taken place (both in u and in $\tilde{u}$), the perturbed solution $\tilde{u}$ will still contain wave-fronts of exactly the same strengths as u. The locations of these wave-fronts, however, will be shifted by certain amounts, say ξ_1^+, ξ_2^+, ξ_j. The quantities

$$E^- \doteq |\sigma_1^-\xi_1^-| + |\sigma_2^-\xi_2^-|, \qquad E^+ \doteq |\sigma_1^+\xi_1^+| + |\sigma_2^+\xi_2^+| + \sum_j |\sigma_j\xi_j| \tag{3.1}$$

provide a measure of the distance (in the $\mathbf{L}^1$ sense) between the two solutions $u, \tilde{u}$, before and after the time of interaction. In general, we may well have $E^+ > E^-$. The increase in the distance, however, can be precisely controlled in terms of the amount of interaction.

To see this, recall the standard interaction estimates [22, 35]

$$|\sigma_1^+ - \sigma_1^-| + |\sigma_2^+ - \sigma_2^-| + \sum_j |\sigma_j| \leq C_0|\sigma_1^-\sigma_2^-|, \tag{3.2}$$

$$|\dot{x}_1(\tau+) - \dot{x}_1(\tau-)| \leq C_0|\sigma_2^-|, \qquad |\dot{x}_2(\tau+) - \dot{x}_2(\tau-)| \leq C_0|\sigma_1^-|, \tag{3.3}$$

for some constant C_0. In the t-x plane, denote by $P, \widetilde{P}$ be the interaction points in the solutions $u, \tilde{u}$, respectively. An elementary calculation shows that the vector $\widetilde{P} - P$ has components

$$\widetilde{P} - P = \left(\frac{\xi_2 - \xi_1}{\dot{x}_1 - \dot{x}_2}, \ \frac{\xi_2\dot{x}_1 - \xi_1\dot{x}_2}{\dot{x}_1 - \dot{x}_2} \right).$$

An outgoing front with speed $\dot{x}_\alpha^+$ is thus shifted in the amount

$$\xi_\alpha \doteq \frac{\xi_2^-(\dot{x}_1^- - \dot{x}_\alpha^+) - \xi_1^-(\dot{x}_2^- - \dot{x}_\alpha^+)}{\dot{x}_1^- - \dot{x}_2^-} \tag{3.4}$$

From (3.4) and the estimates (3.3) we thus obtain

$$\begin{aligned} |\xi_1^+ - \xi_1^-| &= O(1) \cdot |\sigma_2^-|(|\xi_1^-| + |\xi_2^-|), \\ |\xi_2^+ - \xi_2^-| &= O(1) \cdot |\sigma_1^-|(|\xi_1^-| + |\xi_2^-|), \\ |\xi_j| &= O(1) \cdot (|\xi_1^-| + |\xi_2^-|), \end{aligned} \tag{3.5}$$

In turn, this implies the basic estimate

$$\begin{aligned} E^+ - E^- &\leq |\sigma_1^+ \xi_1^+ - \sigma_1^- \xi_1^-| + |\sigma_2^+ \xi_2^+ - \sigma_2^- \xi_2^-| + \sum_j |\sigma_j \xi_j| \\ &= O(1) \cdot |\sigma_1^- \sigma_2^-| (|\xi_1^-| + |\xi_2^-|). \end{aligned} \tag{3.6}$$

Observe that the increase in the distance is estimated in (3.6) by an expression which is linear w. r. t. the shifts $|\xi_\alpha|$ and contains the amount of interaction $|\sigma_1^- \sigma_2^-|$ as a coefficient.

Figure 5

Figure 6

Example 2. Consider now a solution u consisting of two shocks, both of the i-th (genuinely nonlinear) family, which interact at some time τ (fig.5). Let $x'(t), x''(t)$ be the locations of the shocks for $t < \tau$, and let $x(t)$ be the location of the i-shock emerging from the interaction after time τ. Call σ', σ'' the sizes of the two incoming waves, and σ the size of the outgoing i-shock. Of course, $\sigma', \sigma'', \sigma < 0$. In general, the interaction will also produce new wave-fronts of different characteristic families, say with sizes $\sigma_j \quad (j \neq i)$.

As before, consider a slightly perturbed solution $\tilde{u}$, obtained from u by shifting the positions of the initial shocks (along the x-axis) in the amounts ξ', ξ'' respectively. After the interaction has taken place, the perturbed solution $\tilde{u}$ will still contain exactly the same waves as u. The outgoing i-shock will be shifted, say by an amount ξ, while the other outgoing fronts will be shifted by certain amounts ξ_j.

In this case, the distance between u and $\tilde{u}$ can be measured as

$$E^- \doteq |\sigma'\xi'| + |\sigma''\xi''|, \qquad\qquad E^+ \doteq |\sigma\xi| + \sum_j |\sigma_j\xi_j|, \tag{3.7}$$

respectively before and after the interaction. To estimate the increase in this distance, we now use the a-priori bounds

$$|\sigma - \sigma' - \sigma''| + \sum_{j\neq i} |\sigma_j| \leq C_0|\sigma'\sigma''|(|\sigma'| + |\sigma''|), \tag{3.8}$$

$$\left| \dot{x} - \frac{|\sigma'|\dot{x}' + |\sigma''|\dot{x}''}{|\sigma'| + |\sigma''|} \right| \leq C_0|\sigma'\sigma''|, \tag{3.9}$$

$$|\dot{x}' - \dot{x}''| \geq c \cdot (|\sigma'| + |\sigma''|) \qquad \text{for some } c > 0. \tag{3.10}$$

A proof of the interaction estimate (3.8) is given in [35, p.372]. The estimate (3.10) is a straightforward consequence of the genuine nonlinearity of the i-th characteristic field. For a proof of (3.9), see Lemma 3 in [2].

The shift in the outgoing i-shock is now computed by

$$\xi = \frac{\xi''(\dot{x}' - \dot{x}) - \xi'(\dot{x}'' - \dot{x})}{\dot{x}' - \dot{x}''} \tag{3.11}$$

while the shifts in the new j-waves generated by the interaction are

$$\xi_j = \frac{\xi''(\dot{x}' - \dot{x}_j) - \xi'(\dot{x}'' - \dot{x}_j)}{\dot{x}' - \dot{x}''}. \tag{3.12}$$

Observe that the shifts ξ_j in (3.12) may be very large compared with ξ', ξ'', because of the small denominator. Thanks to (3.8), however, these large shifts are more than compensated by the very small sizes σ_j of the new wave-fronts. From (3.8)–(3.12) we obtain an estimate entirely similar to (3.6), namely

$$E^+ - E^- \leq |\xi\sigma - \xi'\sigma' - \xi''\sigma''| + \sum_{j\neq i} |\sigma_j \xi_j| = O(1) \cdot |\sigma'\sigma''|(|\xi'| + |\xi''|). \quad (3.13)$$

We now consider the more general situation of a piecewise constant solution u, with several wave-fronts, say located at $x_\alpha(t)$ $(\alpha = 1, \ldots, N)$, interacting at various points (fig.6). Let $\tilde{u}$ be a slightly perturbed solution, with the same wave-front configuration, obtained from u by shifting the initial positions of the fronts at time $t = 0$. If σ_α, ξ_α denote respectively the size and the amount of shift of the front at x_α, the quantity

$$E \doteq \sum_\alpha |\sigma_\alpha \xi_\alpha| \quad (3.14)$$

provides a measure of the distance between u and $\tilde{u}$. Of course, this quantity will change (and possibly increase) at times where interactions take place. We seek an a-priori bound on E, uniformly valid for all times $t \geq 0$. To obtain such a bound, it is convenient to work with a weighted distance of the form

$$E_\star \doteq \sum_\alpha |\sigma_\alpha \xi_\alpha| W_\alpha. \quad (3.15)$$

The key idea is to choose the weights W_α in such a way that the quantity $E_\star$ is non-increasing at each interaction. Such weights will have the form

$$\begin{aligned} W_\alpha \ = \ 1 \ & + C_1 \cdot [\text{amount of waves in } u \text{ which approach} \\ & \qquad \text{the front at } x_\alpha] \\ & + C_2 \cdot [\text{ global interaction potential of waves in } u]. \end{aligned} \quad (3.16)$$

More precisely, at any given time t, denote by $\mathcal{A}$ the set of all couples of approaching wave-fronts in $u(t, \cdot)$, and define

$$W_\alpha \ = \ 1 + C_1 \cdot \left(\sum_{(\alpha,\beta)\in\mathcal{A}} |\sigma_\beta| \right) + C_2 \cdot \left(\sum_{(\beta,\beta')\in\mathcal{A}} |\sigma_\beta \sigma_{\beta'}| \right). \quad (3.17)$$

We will show that, for a suitable choice of the constants C_1, C_2, if the total variation of $u(t, \cdot)$ remains suitably small, then the quantity $E_\star$ decreases at every interaction. Indeed, let τ be an interaction time, and denote by $E_\star^-, E_\star^+$ the values of $E_\star$ before and after the interaction. The difference $E_\star^+ - E_\star^-$ will be estimated by considering separately the fronts which take part in the interaction and those which do not. We claim that

$$E_\star^+ - E_\star^- = (E_{int}^+ - E_{int}^-) + (E_{nonint}^+ - E_{nonint}^-) \leq 0. \tag{3.18}$$

Assume that the incoming wave-fronts belong to distinct families, say $i_1 > i_2$ and have sizes σ_1^-, σ_2^-, as in Example 1. Concerning the interacting fronts, with obvious meaning of the notation we then have

$$\begin{aligned} E_{int}^+ - E_{int}^- = \;& |\sigma_1^+ \xi_1^+| W_1^+ + |\sigma_2^+ \xi_2^+| W_2^+ \\ & + \sum_{j \neq i_1, i_2} |\sigma_j \xi_j| W_j - |\sigma_1^- \xi_1^-| W_1^- - |\sigma_2^- \xi_2^-| W_2^-. \end{aligned} \tag{3.19}$$

Call

$$Q \doteq \sum_{(\beta,\beta') \in \mathcal{A}} |\sigma_\beta \sigma_{\beta'}|$$

the global interaction potential of waves in u. If the total variation remains sufficiently small, standard interaction estimates imply that the decrease of Q is comparable to the amount of interaction:

$$Q^+ - Q^- < -\frac{1}{2} |\sigma_1^- \sigma_2^-|. \tag{3.20}$$

Since $Q^+ < Q^-$, by (3.17) it follows

$$W_1^+ \leq W_1^- - C_1 \sigma_2^-, \qquad\qquad W_2^+ \leq W_2^- - C_1 \sigma_1^-.$$

Therefore, recalling (3.5) and (3.6) we have

$$\begin{aligned} E_{int}^+ - E_{int}^- \leq \;& -C_1 \big(|\sigma_1^- \xi_1^-| |\sigma_2^-| + |\sigma_2^- \xi_2^-| |\sigma_1^-| \big) + O(1) \cdot \\ & |\sigma_1^- \sigma_2^-| \big(|\xi_1^-| + |\xi_2^-| \big) \leq 0, \end{aligned} \tag{3.21}$$

if C_1 is large and Tot.Var.(u) is sufficiently small. Next, consider the fronts not taking part of the interaction. It is clear that the sizes σ_α and the shifts ξ_α of these fronts remain the same. However, their weights W_α do change, because of the change in the interaction potential. We observe that

$$E_{nonint}^+ - E_{nonint}^- = \sum_{nonint} |\sigma_\alpha \xi_\alpha| (W_\alpha^+ - W_\alpha^-) \leq 0. \tag{3.22}$$

Indeed, for each non-interacting front, by (3.2) and (3.20) it follows

$$\begin{aligned} W_\alpha^+ - W_\alpha^- &\leq C_1\Big(|\sigma_1^+ - \sigma_1^-| + |\sigma_2^+ - \sigma_2^-| + \sum_j |\sigma_j|\Big) + C_2(Q^+ - Q^-) \\ &\leq C_1 C_0 |\sigma_1^- \sigma_2^-| - \tfrac{C_2}{2} |\sigma_1^- \sigma_2^-| \\ &\leq 0, \end{aligned}$$

provided that C_2 is sufficiently large. Together, (3.21) and (3.22) yield (3.18).

An entirely similar computation shows that the weighted distance (3.17) does not increase also in the case of interactions between fronts of the same characteristic family, considered in Example 2.

We remark that, working with functions u having small total variation, in (3.17) we can assume that $W_\alpha \approx 1$. The weighted distances (3.14) and (3.15) will therefore be uniformly equivalent.

At this stage, we are only able to compare "infinitesimally close" piecewise constant solutions $u, \tilde{u}$, having the same wave-front configuration. For such solutions, the weighted distance (3.15) does not increase in time. Of course, one would like to compare two arbitrary solutions, not necessarily close to each other. For this purpose, the idea is to define a Riemann-type metric on a suitable space of tangent vectors, with weights of the form (3.16). Because of (3.18), we expect that the corresponding distance will be contractive w.r.t. the flow determined by the system (1.1). The construction of this metric is described in the next section.

4. A locally contractive Riemann metric

The basic strategy for obtaining a Lipschitz semigroup of solutions to (1.1) is to construct suitable approximate solutions, carefully controlling how their distance varies in time. Throughout the following, we shall work with piecewise Lipschitz functions, having finitely many points of jump. At the present date, this seems to be the largest class in which a local existence and uniqueness theorem is known [27, 34], together with a linearized evolution equation for first order tangent vectors [13]. For various applications to variational problems in connection with conservation laws, it would be interesting to extend this differential calculus to a wider class of **BV** functions. A tentative step in this direction was recently taken in [12].

Although the distance between solutions will be eventually measured in the standard norm of $\mathbf{L}^1(I\!\!R;\ I\!\!R^n)$, it is convenient to adopt the point of view of differential geometry, and regard the space **BV** as a Riemann manifold. In this case, the usual way to measure the distance between two functions u, u' is to consider the set of all suitably regular paths $\gamma : [0,1] \mapsto \mathbf{BV}$ connecting u with u', and define the distance $d(u, u')$ as the infimum among the lengths of all these paths.

Figure 7

To estimate how the distance between two solutions of (1.1) varies in time, the following approach now seems quite natural (fig.7). Given two initial conditions $\bar u, \bar u'$, consider any sufficiently regular path $\gamma_0 : [0,1] \mapsto \mathbf{BV}$ with $\gamma_0(0) = \bar u$, $\gamma_0(1) = \bar u'$. For each $\theta \in [0,1]$, let $t \mapsto u^\theta(t)$ be the solution of (1.1) with initial data $u^\theta(0) = \gamma_0(\theta)$. Call $\gamma_t : \theta \mapsto u^\theta(t)$ the curve connecting the various solutions at time t. Clearly, the distance between $u(t)$ and $u'(t)$ is bounded by the length of γ_t. Therefore, if we can show that

$$\|\gamma_t\| \leq L \cdot \|\gamma_0\| \tag{4.1}$$

for all such curves γ_0, we will have proved that

$$\|u(t) - u'(t)\| \leq L \cdot \|u(0) - u'(0)\| \qquad \forall t > 0. \tag{4.2}$$

By continuity we can then extend this flow to much wider class of **BV** functions.

If the path γ is sufficiently regular, its length $\|\gamma\|$ can be measured by integrating the norm of a tangent vector, say

$$\|\gamma\| = \int_0^1 \left\| \frac{d\gamma(\theta)}{d\theta} \right\| \, d\theta. \tag{4.3}$$

An estimate of the form (4.1) can thus be obtained by showing that the norm of any tangent vector increases at most by a factor L.

In the following, we shall implement the above strategy using paths of piecewise Lipschitz functions. Let $\theta \mapsto u^\theta$ be a one-parameter family of piecewise Lipschitz functions, each u^θ having the same number of jumps (fig.8), say at the points $x_1^\theta < \ldots < x_N^\theta$. Assume that there exist the functions

$$v^\theta(x) \doteq \lim_{h\to 0} \frac{u^{\theta+h}(x) - u^\theta(x)}{h} \qquad \text{for a.e. } x \tag{4.4}$$

and the numbers

$$\xi_\alpha^\theta \doteq \lim_{h\to 0} \frac{x_\alpha^{\theta+h} - x_\alpha^\theta}{h} \qquad \alpha = 1, \ldots, N. \tag{4.5}$$

Figure 8

Then, under suitable regularity conditions, the $\mathbf{L}^1$ length of the path $\gamma : \theta \mapsto u^\theta$ is computed by

$$\|\gamma\|_{\mathbf{L}^1} = \int_0^1 \|v^\theta\|_{\mathbf{L}^1} \, d\theta + \sum_{\alpha=1}^N \int_0^1 |u^\theta(x_\alpha+) - u^\theta(x_\alpha-)||\xi_\alpha^\theta| \, d\theta. \tag{4.6}$$

We stress the fact that, in general, the path $\theta \mapsto u^\theta$ is not differentiable w.r.t. the usual differential structure of $\mathbf{L}^1$. Indeed, if the shift rates ξ_α^θ are not equal to zero, as $h \to 0$ the ratio $[u^{\theta+h} - u^\theta]/h$ does not converge to any limit in $\mathbf{L}^1$. In order to correctly measure the length of a path γ, it is essential to work with a class of "generalized tangent vectors" of the form $(v, \xi) \in T_u \doteq \mathbf{L}^1(\mathbb{R};\ \mathbb{R}^n) \times \mathbb{R}^N$. Observe that the tangent space T_u actually depends on the function u, through the number of points of discontinuity.

Example 3. Consider the one-parameter family of functions

$$u^\theta(x) = \begin{cases} 2\theta x & \text{if} \quad x \in [0,\ 1+\theta] \\ 0 & \text{otherwise} \end{cases} \qquad \theta \in [0,1].$$

The corresponding generalized tangent vector $(v^\theta, \xi^\theta) \in T_{u^\theta} \doteq \mathbf{L}^1 \times \mathbb{R}$ is given by

$$\xi_\theta \equiv 1, \qquad v^\theta(x) = \begin{cases} 2x & \text{if} \quad x \in [0,\ 1+\theta], \\ 0 & \text{otherwise.} \end{cases}$$

The length of the path $\gamma : \theta \mapsto u^\theta$, computed by (4.6), is

$$\|\gamma\|_{\mathbf{L}^1} = \int_0^1 \left(\int_0^{1+\theta} 2x\ dx \right) d\theta + \int_0^1 2\theta(1+\theta) \cdot 1\ d\theta = 4.$$

Next, assume that each function $u^\theta(t, \cdot)$ is a solution of the system of conservation laws. We wish to derive a linearized evolution equation for the corresponding tangent vectors (v^θ, ξ^θ).

For this purpose, some preliminary notations must be introduced. Let $A(u)$ be the $n \times n$ Jacobian matrix of f at u, and call $\lambda_1(u) < \cdots < \lambda_n(u)$ its eigenvalues. Choose dual bases of right and left eigenvectors $r_i(u)$, $l_i(u)$ of $A(u)$, so that

$$(D\lambda_i) r_i \geq 0, \qquad \langle l_i,\ r_j \rangle = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{4.7}$$

Here $(D\lambda_i) r_i$ denotes the directional derivative of the eigenvalue λ_i in the direction of the eigenvector r_i, while $\langle \cdot\,, \cdot \rangle$ denotes an inner product. For $u, u' \in \mathbb{R}^n$, define the averaged matrix

$$A(u, u') \doteq \int_0^1 A(\theta u + (1-\theta) u') d\theta \tag{4.8}$$

with eigenvalues $\lambda_i(u,u')$, and choose right and left eigenvectors $r_i(u,u')$, $l_i(u,u')$ of $A(u,u')$ as in (4.7). The differential of λ_i at (u,u') is written $D\lambda_i(u,u')$. We thus have

$$D\lambda_i(u,u')\cdot(v,v') = \lim_{\varepsilon\to 0}\varepsilon^{-1}[\lambda_i(u+\varepsilon v, u'+\varepsilon v') - \lambda_i(u,u')].$$

The same notation will be used for the differentials of the eigenvectors r_i and l_i.

Let $u = u(t,x)$ be a piecewise Lipschitz continuous weak solution of (1.1). At almost every point (t,x), the function u thus satisfies the quasilinear system

$$u_t + A(u)u_x = 0, \tag{4.9}$$

while, along the shock lines $x = x_\alpha(t)$, the Rankine-Hugoniot equations hold:

$$\dot x_\alpha[u^+ - u^-] = [f(u^+) - f(u^-)], \tag{4.10}$$

with $u^+ = u(t, x_\alpha+)$, $u^- = u(t, x_\alpha-)$. If the jump at x_α occurs in the k_α-th characteristic family, this implies that $u^+ - u^-$ is a right eigenvector of the matrix $A(u^+, u^-)$, with corresponding eigenvalue

$$\dot x_\alpha = \lambda_{k_\alpha}(u^+, u^-). \tag{4.11}$$

The eigenvector condition can also be written as

$$\langle l_i(u^+,u^-),\ u^+ - u^-\rangle = 0 \qquad \forall i \neq k_\alpha. \tag{4.12}$$

A system of linearized evolution equations for the generalized tangent vector (v,ξ) can now be derived from (4.9)-(4.12). Namely

$$v_t + A(u)v_x + [DA(u)\cdot v]u_x = 0 \tag{4.13}$$

outside the lines of discontinuity, together with the conditions

$$\begin{aligned}\Big\langle Dl_i \quad (u^+,u^-)\cdot(\xi_\alpha u_x^+ + v^+,\ \xi_\alpha u_x^- + v^-),\quad u^+ - u^-\Big\rangle \\ +\Big\langle l_i(u^+,u^-),\quad \xi_\alpha u_x^+ + v^+ - \xi_\alpha u_x^- - v^-\Big\rangle = 0 \qquad \forall i\neq k_\alpha,\end{aligned} \tag{4.14}$$

$$\dot{\xi}_\alpha = D\lambda_{k_\alpha}(u^+, u^-) \cdot (\xi_\alpha u_x^+ + v^+, \quad \xi_\alpha u_x^- + v^-), \tag{4.15}$$

on each line $x = x_\alpha(t)$ where u has a shock (or contact discontinuity) in the k_α characteristic field. Observe that the equations (4.13) are formally derived from (4.9), replacing u by $u + \varepsilon v$ and differentiating w.r.t. ε. Similarly, the equations (4.14), (4.15) are obtained from (4.12), (4.11), replacing the right and left limits u^+, u^- by $u^+ + \varepsilon v^+ + \varepsilon u_x^+ \xi_\alpha$ and $u^- + \varepsilon v^- + \varepsilon u_x^- \xi_\alpha$ respectively, and differentiating w.r.t. ε. For a rigorous derivation of (4.13)–(4.15) see [13].

By (4.6), the standard $\mathbf{L}^1$ length of a path γ can be computed by integrating the norm of the corresponding tangent vector, defined as

$$\|(v,\xi)\|_{\mathbf{L}^1} \doteq \|v\|_{\mathbf{L}^1} + \sum_\alpha |u(x_\alpha+) - u(x_\alpha-)||\xi_\alpha|. \tag{4.16}$$

This norm may well increase along solutions of the linearized system (4.13)-(4.15). In [6], however, the following was proved.

Theorem 4. *There exists a set $\mathcal{U}$, containing all piecewise Lipschitz functions with sufficiently small total variation, and a family of norms $\|\cdot\|_u$, defined for $u \in \mathcal{U}$, with the following properties.*

(i) *If $u = u(t,x)$ is a piecewise Lipschitz continuous solution of (1.1) with $u(t) \in \mathcal{U}$, and if $(v(t), \xi(t))$ is any solution of the corresponding linearized system (4.13)–(4.15), then the norm $\|(v(t), \xi(t))\|_{u(t)}$ is a non-increasing function of time, even at times where two shocks interact.*

(ii) *There exists a constant L such that, for all $u \in \mathcal{U}$ and $(v,\xi) \in T_u$, one has*

$$\|(v,\xi)\|_{\mathbf{L}^1} \le \|(v,\xi)\|_u \le L \cdot \|(v,\xi)\|_{\mathbf{L}^1}. \tag{4.17}$$

This "infinitesimally contractive" Riemann metric is constructed as follows. Let u be a piecewise Lipschitz function having jumps at the points $x_1 < x_2 < \cdots < x_N$. Assume that the jump at x_α occurs in the k_α-th characteristic family, and has strength measured by $J_\alpha > 0$. For any $(v,\xi) \in T_u \doteq \mathbf{L}^1 \times I\!R^N$, define the components

$$v_i(x) \doteq \langle l_i(u(x)),\ v(x)\rangle, \quad u_x^i(x) \doteq \langle l_i(u(x)),\ u_x(x)\rangle \quad i = 1, \ldots, n. \tag{4.18}$$

The weighted norm of a generalized tangent vector $(v,\xi)\in T_u=\mathbf{L}^1\times I\!\!R^N$ is then defined as

$$\|(v,\xi)\|_u \doteq \sum_{i=1}^{n}\int_{-\infty}^{\infty}|v_i(x)|W_i^u(x)\,dx+\sum_{\alpha=1}^{N}J_\alpha|\xi_\alpha|W^u_{k_\alpha}(x_\alpha). \tag{4.19}$$

Here $W_i^u(x)$ is the weight assigned to a wave of the i-th family located at x. It has the form

$$W_i^u(x)\doteq 1+C_1R_i^u(x)+C_2Q(u), \tag{4.20}$$

where

$$\begin{aligned} R_i^u(x) &\doteq \left[\sum_{j\le i}\int_x^\infty+\sum_{j\ge i}\int_{-\infty}^x\right]|u_x^j(y)|dy+\left[\sum_{\substack{k_\alpha\le i\\ x_\alpha>x}}+\sum_{\substack{k_\alpha\ge i\\ x_\alpha<x}}\right]J_\alpha,\\ Q(u) &\doteq \sum_{i\le j}\iint_{x<y}|u_x^j(x)||u_x^i(y)|\,dxdy+\sum_{k_\alpha\le k_\beta,\,x_\alpha>x_\beta}J_\alpha J_\beta\\ &\quad+\sum_\alpha\left[\sum_{i\le k_\alpha}J_\alpha\int_{x_\alpha}^\infty|u_x^i(x)|dx+\sum_{i\ge k_\alpha}J_\alpha\int_{-\infty}^{x_\alpha}|u_x^i(x)|dx\right], \end{aligned} \tag{4.21}$$

while C_1, C_2 are suitable constants. We regard $|u_x^i(x)|dx$ as the strength of an infinitesimal i-wave in u located at x. Of course, at each point x_α, u also contains a jump of strength J_α in the k_α-th family. In analogy with (3.16), the weight (4.20) can thus be interpreted as

$$\begin{aligned} W_i^u(x)= {}& 1+C_1\cdot[\text{amount of waves in } u \text{ which}\\ &\qquad\text{approach an } i\text{-wave located at } x]\\ &+C_2\cdot[\text{global interaction potential of waves in } u]. \end{aligned} \tag{4.22}$$

In turn, the Riemann metric $\|\cdot\|_u$ determines a weighted distance $d^\star$ on $\mathcal{U}$, constructed as follows. Given any two piecewise Lipschitz continuous functions $u,u'\in\mathcal{U}$, call $\Sigma_{u,u'}$ the family of all continuous paths $\gamma:[0,1]\mapsto\mathcal{U}$ with $\gamma(0)=u$, $\gamma(1)=u'$, such that, for almost all $0<\theta<1$, the differential $D\gamma(\theta)=(v^\theta,\xi^\theta)$ is a well defined element in $T_{\gamma(\theta)}$. The weighted length of a path $\gamma\in\Sigma_{u,u'}$ is defined as

$$\|\gamma\|_\star\doteq\int_0^1\|D\gamma(\theta)\|_{\gamma(\theta)}d\theta, \tag{4.23}$$

while the Riemannian distance between u and u' is given by

$$d^\star(u,u') \doteq \inf \{\|\gamma\|_\star; \quad \gamma \in \Sigma_{u,u'}\}. \tag{4.24}$$

Because of (ii) in Theorem 4, the distance $d^\star$ is uniformly equivalent to the standard $\mathbf{L}^1$ distance. Hence, it can be extended to the $\mathbf{L}^1$ closure of $\mathcal{U}$, by continuity. Moreover, by (i), the length of every regular path does not increase in time along the flow of (1.1). This suggests that the flow generated by (1.1) should be globally contractive w.r.t. the weighted distance $d^\star$, and hence uniformly Lipschitz continuous w.r.t. the usual $\mathbf{L}^1$ distance.

Unfortunately, a rigorous proof of this fact runs into a major difficulty. Namely, the estimates on the norm of a tangent vector, and on the length of a path $\theta \mapsto \gamma_t(\theta)$, are valid assuming that all solutions remain piecewise Lipschitz throughout a given interval $[0,T]$. This is not the case in general. Indeed, a piecewise Lipschitz solution may lose its regularity for two main reasons:

(i) The number of shock fronts may become infinite in finite time, due to repeated shock interactions.

(ii) The Lipschitz constant outside the shocks may become infinite, due to the genuine nonlinearity of some characteristic fields.

Figure 9

The general situation is depicted in fig.9. Each initial condition $\bar{u}^\theta$ determines a unique local solution u^θ which will remain piecewise Lipschitz up to a time τ^θ where breakdown of regularity occurs. For every time $t < \inf_\theta \tau^\theta$, the length of the path $\gamma_t : \theta \mapsto u^\theta(t)$ can be computed by integrating the norm of the corresponding tangent vectors, as in (4.3). For $t > \inf \tau^\theta$ some of these tangent vectors no longer exist, and the whole construction breaks down.

An attempt to get around this difficulty goes as follows. Consider a one-parameter family of solutions. If some of these solutions lose their regularity before the terminal time T, define $t_1 > 0$ as the supremum of the times where all solutions remain piecewise Lipschitz. At time t_1, we choose $\varepsilon_1 > 0$ and construct a new path $\gamma_{t_1+} : \theta \mapsto u^\theta(t_1+, \cdot)$ such that **(i)** the new path is close to the old one, and **(ii)** the weighted length of the new path is almost the same as the length of the old one. More precisely,

Figure 10

$$\|u^\theta(t_1+,\cdot) - u^\theta(t_1,\cdot)\|_{\mathbf{L}^1} \leq \varepsilon_1 \qquad \forall\theta, \tag{4.25}$$

$$\|\gamma_{t_1+}\|_\star \leq \|\gamma_{t_1}\|_\star + \varepsilon_1. \tag{4.26}$$

We then construct solutions of (1.1) with these new initial conditions, up to the first time $t_2 > t_1$ where one of the u^θ loses its regularity. At time t_2 we apply the same restarting procedure (fig.10), etc...

The problem with this approach is that, as $\varepsilon_1 \to 0$, the length of the time interval over which the solutions will remain regular may also tend to zero.

Therefore, it is not clear whether these paths of approximate solutions can actually be constructed on the whole interval $[0, T]$. A crucial property needed here is

(P) Given a path $\gamma_\tau : \theta \mapsto u^\theta(\tau, \cdot)$, the following holds. For every $\epsilon_1 > 0$ one can construct a new path $\gamma_{\tau+} : \theta \mapsto u^\theta(\tau+, \cdot)$ satisfying (4.25), (4.26), so that all the corresponding solutions u^θ of (1.1) remain regular over a time interval $[\tau,\ \tau']$ independent of ϵ_1.

Working with exact solutions of (1.1), it does not seem possible to achieve the property **(P)**. Neverthless, as we shall see in the next section, our approach can be successfully implemented for suitable classes of approximate solutions.

5. Construction of the semigroup

We shall apply the techniques introduced in the previous section in connection with piecewise Lipschitz approximate solutions. More precisely, we partition a given interval $[0, T]$ inserting the times $0 = t_0 < t_1 < \cdots < t_\nu = T$. On each subinterval $[t_{i-1}, t_i]$ our approximate solution is piecewise Lipschitz, with jumps along finitely many lines, say $x = x_\alpha(t)$. Outside these jumps, one has

$$u_t + A(u)u_x = 0 \qquad \text{a. e.} \tag{5.1}$$

On the other hand, along the jumps, a suitably modified version of the Rankine-Hugoniot equations holds. This will yield more favourable qualitative properties of the functions u. At each time t_i, in order to prevent a loss of regularity, a "restarting procedure" is applied. The function $u(t_i)$ is thus replaced by a nicer function $u(t_i+)$, so that the solution can be prolonged to a new interval $[t_i, t_{i+1}]$ (still in the class of piecewise Lipschitz functions), etc.

A modified set of jump conditions is now described. For any state $\omega \in I\!R^n$, let $\sigma \mapsto S_i(\sigma)(\omega)$ and $\sigma \mapsto R_i(\sigma)(\omega)$ be the parametrized shock and rarefaction curves through ω, respectively.

Definition 2. We say that the jump (u^-, u^+), travelling with speed $\dot{x}$, satisfies the *Modified Rankine-Hugoniot conditions* (MRH) if

$$u^+ = R_i(\sigma)(u^-) \qquad \text{for some } \ \sigma < 0, \quad i \in \{1, \ldots, n\}, \tag{5.2}$$

$$\dot{x} = \lambda_i^r \doteq \frac{1}{|\sigma|} \int_\sigma^0 \lambda_i(R_i(s)(u^-))\, ds. \tag{5.3}$$

Figure 11

In other words, we artificially impose that the left and right states of the jump be connected by a rarefaction curve, instead of a true shock curve. Piecewise Lipschitz functions, which satisfy (5.2), (5.3) at points of jump, clearly will not satisfy the conservation equations (1.1). An additional set of approximate Rankine-Hugoniot conditions, more closely related with the true ones, is introduced below. In the following, we fix a smooth, non-increasing map $\varphi : I\!\!R \mapsto [0,1]$ such that

$$\begin{cases} \varphi(\sigma) = 1 & \text{if} \quad \sigma < -2, \\ \varphi(\sigma) = 0 & \text{if} \quad \sigma > -1. \end{cases} \tag{5.4}$$

Definition 3. We say that the jump (u^-, u^+), travelling with speed $\dot{x}$, satisfies the *ε-approximated Rankine-Hugoniot conditions* (ε-RH) if, for some $\sigma < 0$ and $i \in \{1, \ldots, n\}$, we have

$$u^+ = \varphi(\sigma/\varepsilon) \cdot S_i(\sigma)(u^-) + (1 - \varphi(\sigma/\varepsilon)) \cdot R_i(\sigma)(u^-), \tag{5.5}$$

$$\dot{x} = \hat{\lambda}_i \doteq \varphi(\sigma/\varepsilon)\lambda_i^s + (1 - \varphi(\sigma/\varepsilon))\lambda_i^r. \tag{5.6}$$

Here λ_i^r is defined by (5.3), while λ_i^s is the speed of a true shock connecting u^- with $S_i(\sigma)(u^-)$.

In other words, a jump (u^-, u^+) satisfies (ε-RH) if either its size is $\sigma < -2\varepsilon$ and the true Rankine-Hugoniot conditions hold, or if its size is $\sigma \in [-\varepsilon, 0]$ and the (MRH) equations hold, or else if its size is $\sigma \in [-2\varepsilon, \ -\varepsilon]$ and a particular interpolation between the exact and the modified Rankine-Hugoniot equations holds (fig.11). Clearly, as $\varepsilon \to 0$, the ε-approximate Rankine-Hugoniot conditions approach the true ones.

The changes in the Rankine-Hugoniot conditions, introduced above, in turn determine new ways of approximately solving a Riemann problem. More precisely, let

$$u(0, x) = \begin{cases} u^- & \text{if} \quad x < 0, \\ u^+ & \text{if} \quad x > 0, \end{cases} \tag{5.7}$$

be the initial data. The standard solution of the Riemann problem with data (5.7) is then obtained [26, 34, 35] by setting

$$\psi_i(\sigma)(\omega) \doteq \begin{cases} S_i(\sigma)(\omega) & \text{if} \quad \sigma < 0, \\ R_i(\sigma)(\omega) & \text{if} \quad \sigma \geq 0, \end{cases}$$

and determining wave sizes $\sigma_1, \ldots \sigma_n$ such that

$$u^+ = \psi_n(\sigma_n) \circ \cdots \circ \psi_1(\sigma_1)(u^-).$$

Instead, adopting (MRH) one is lead to

(MRS) (Modified Riemann Solver: Shock and rarefaction curves coinciding) Given u^-, u^+, we first determine the wave sizes σ_i so that

$$u^+ = R_n(\sigma_n) \circ \cdots \circ R_1(\sigma_1)(u^-). \tag{5.8}$$

Consider the intermediate states

$$\omega_0 = u^-, \ \ \omega_1 = R_1(\sigma_1)(\omega_0), \ \ldots \ , \ \omega_n = R_n(\sigma_n)(\omega_{n-1}) = u^+.$$

If $\sigma_i \geq 0$, connect the states ω_{i-1} and ω_i by a centered rarefaction wave as usual. If $\sigma_i < 0$, connect these two states by a single jump, travelling with the speed λ_i^r defined at (5.3).

(ε-RS) (ε-approximate Riemann Solver: Small shocks coinciding with rarefactions) We start by defining the parametrized curves

$$\sigma \mapsto \psi_i^\varepsilon(\sigma)(\omega) \doteq \varphi(\sigma/\varepsilon) \cdot S_i(\sigma)(\omega) + (1 - \varphi(\sigma/\varepsilon)) \cdot R_i(\sigma)(\omega), \tag{5.9}$$

obtained as an interpolation between the i-shock and the i-rarefaction curves through ω. We then find wave sizes $\sigma_1, \ldots, \sigma_n$ such that

$$u^+ = \psi_n^\varepsilon(\sigma_n) \circ \cdots \circ \psi_1^\varepsilon(\sigma_1)(u^-). \tag{5.10}$$

Let $\omega_0 = u^-$, $\omega_1, \ldots, \omega_n = u^+$ be the corresponding intermediate states. If $\sigma_i \geq 0$, the states ω_{i-1} and ω_i are connected by a centered rarefaction wave, as usual. If $\sigma_i < 0$, these two states are connected by a single jump, travelling with speed $\hat{\lambda}_i$, defined at (5.6).

The motivation for introducing the approximate Riemann Solvers (MRS) or (ε-RS) is that, in these cases, the techniques described in the previous section actually work, allowing us to construct a globally Lipschitz semigroup $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$. This semigroup will satisfy the conditions (i)–(iii) in Theorem 1, with (iv) replaced by

(iv′) *If $\bar{u} \in \mathcal{D}$ is piecewise constant, then for $t > 0$ sufficiently small the function $u(t, \cdot) = S_t\bar{u}$ coincides with the approximate solution of (1.1), (1.2) obtained by piecing together the solutions of the corresponding Riemann problems constructed by (MRS) [or (ε-RS), respectively].*

As $\varepsilon \to 0$, the Lipschitz constant of the semigroup S^ε generated by (ε-RS) remains uniformly bounded. In the limit, we will thus obtain a "Standard Riemann Semigroup" generated by (1.1).

We first describe the construction of the semigroup generated by (MRS). The additional technicalities involved in the construction of a semigroup generated by (ε-RS) will be addressed at a later stage.

Let us begin with an easy case, assuming that all characteristic fields are linearly degenerate. In this case, solutions which are initially smooth remain smooth for all times $t \geq 0$. Therefore, no restarting procedure is ever needed and the strategy outlined in the previous section can be easily implemented [3]. To construct the semigroup, one first consider the flow defined by (5.1) on a set of $\mathcal{C}^2$ functions with small total variation. By continuity, this flow can then be extended to a larger domain of **BV** functions.

The fact that linearly degenerate systems preserve regularity will play an important role in the sequel. We thus give here a sketch of the proof. For details, see [3].

Figure 12

Let $u = u(t,x)$ be a smooth solution of (5.1), defined on some initial interval $[0, T[$. Recalling (4.18), define

$$E(t) \doteq \sup_{j,x} |u_x^j(t,x)|$$

as the largest value among all components of the gradient of u at time t. We claim that, if the total variation is sufficiently small, then

$$E(t) < 2E(0) \qquad \text{for all} \quad t. \tag{5.11}$$

In particular, a gradient catastrophe can never occur. To prove (5.11), assume, on the contrary, that

$$E(\tau) = |u_x^i(\tau, \bar{x})| = 2E(0) \tag{5.12}$$

at some first time τ, for some $i, \bar{x}$. Let $x = x_i(t)$ denote the i-th characteristic through the point $(\tau, \bar{x})$ as in fig.12. From (5.1) it follows that, along this line, the component u_x^i satisfies the equation

$$\frac{d}{dt} u_x^i(t, x_i(t)) = \sum_j \sum_{k \neq j} G_{ijk}(u) u_x^j u_x^k. \tag{5.13}$$

The continuous functions G_{ijk} depend on certain directional derivatives of the eigenvalues and eigenvectors of $A(u)$. Their exact expression can be found in [3], but is not needed here. For some $\tau' \in [0, \tau[$, we can assume that

$$\begin{aligned} &|u_x^i(\tau', x_i(\tau'))| = E(0), \\ &E(0) \leq |u_x^i(t, x_i(t))| \leq 2E(0), \\ &|u_x^k(t, x_i(t))| \leq 2E(0), \end{aligned} \tag{5.14}$$

for all $t \in [\tau', \tau]$ and $k \in \{1, \ldots, n\}$. Because of strict hyperbolicity, for $j \neq i$ the differences $|\lambda_j(u) - \lambda_i(u)|$ are bounded away from zero. We can thus find a constant C such that

$$\frac{d}{dt} |u_x^i(t, x_i(t))| \leq C |u_x^i| \cdot \left[\sum_{j \neq i} |\lambda_j(u) - \lambda_i(u)| |u_x^j| \right]. \tag{5.15}$$

Indeed, in (5.13) the coefficients $G_{ijk}(u)$ are uniformly bounded, while (5.14) implies

$$|u_x^k(t, x_i(t))| \leq 2E(0) \leq 2|u_x^i(t, x_i(t))| \qquad \forall t \in [\tau', \tau], \; k \in \{1, \ldots, n\}.$$

We now recognize the expression within square brackets in (5.15) as the instantaneous amount of waves which cross the i-th characteristic $x = x_i(t)$. Integrating (5.15) with initial condition $(5.14)_1$, we obtain

$$\begin{aligned} |u_x^i(\tau, \bar{x})| \; &\leq \exp \Big\{ C \cdot \text{ total amount of waves which cross the} \\ &\qquad \text{characteristic } x_i(\cdot) \text{during the time interval} \\ &\qquad [\tau', \tau] \Big\} \cdot |u(\tau', x_i(\tau'))| \\ &< 2E(0), \end{aligned} \tag{5.16}$$

provided that the total variation (and hence the total amount of waves that can cross the characteristic x_i) is sufficiently small. This contradiction with (5.12) proves (5.11).

Next, consider the somewhat more general case where $n-1$ characteristic families are linearly degenerate and only one, say the k-th family, is genuinely nonlinear. In this case, we construct approximate trajectories of (MRS) in the class of functions which are Lipschitz continuous except for a finite number of jumps in the k-th characteristic field. Assume that the initial data $u = u(0,x)$ satisfies the following assumptions:

$$u_x^k \geq -1, \qquad |u_x^j| \leq M \qquad (j \neq k), \qquad \text{for a.e. } x \in I\!R. \tag{5.17}$$

Moreover, let u have finitely many points of discontinuity, say $x_1 < \ldots, x_N$. At each point x_α, assume

$$u(x_\alpha+) = R_k(\sigma_\alpha)(u(x_\alpha-)) \qquad \text{for some } \ \sigma_\alpha < 0. \tag{5.18}$$

Then, on some interval $[0, t_1]$, a piecewise Lipschitz function $u = u(t,x)$ exists, which satisfies (5.1) a.e., and the (MRH) conditions (5.2), (5.3) along every line of jump.

A key observation, at this point, is that the time interval $[0, t_1]$ depends only on the bound M in (5.17), and not on the number of jumps. Indeed, whenever two jumps in u collide, since they both belong to the k-th family, they simply stick together as a single k-jump. No new waves in the other characteristic families are produced by the interaction. (Here lies the great advantage of using the approximate Riemann solver (MRS): jumps occur precisely along the rarefaction curves!) As a result, the number of jumps can only decrease.

Concerning the Lipschitz constant outside the jumps, for $i \neq k$ the gradient components u_x^i defined at (4.18) satisfy an estimate of the form (5.16), and thus remain uniformly bounded. Along each k-characteristic $x = x_k(t)$, the k-th component u_x^k satisfies

$$\frac{d}{dt} u_x^k(t, x_k(t)) = -[(D\lambda_k) r_k(u)] \cdot (u_x^k)^2 + \sum_i \sum_{j \neq i} G_{kij}(u) u_x^i u_x^j. \tag{5.19}$$

Due to the squared term on the right hand side of (5.19), the component u_x^k may actually approach $-\infty$ within finite time. However, this component can be bounded from below by the solution of the O. D. E.

$$\dot{Z} = -aZ^2 + bZ - c, \qquad\qquad Z(0) = -1, \tag{5.20}$$

for suitable constants $a, b, c > 0$, depending only on M and on the coefficients in (5.19). If $\tau > 0$ is the blow-up time for the solution of the Cauchy problem (5.20), we have

$$u_x^k(t, x_k(t)) \geq Z(t) > -\infty \qquad\qquad \forall t \in [0, \tau[\,.$$

Therefore, a piecewise Lipschitz solution $u = u(t,x)$ is well defined on some interval $[0, t_1]$, with $t_1 \geq \tau$.

Figure 13

We now describe a method for constructing approximate trajectories of the semigroup generated by (MRS), on a given interval $[0, T]$. Fix $\varepsilon > 0$ and $N >> 1$. Start with a piecewise Lipschitz function $u(0, \cdot)$ satisfying (5.17), (5.18). Construct an exact solution of (5.1)–(5.3) up to the first time t_1 where

$$\inf_x u_x^k(t, x) \leq -N. \tag{5.21}$$

At time t_1 we replace $u(t_1, \cdot)$ with a new function $u(t_1+, \cdot)$ satisfying (5.17) together with

$$\left\|u(t_1+) - u(t_1)\right\|_{\mathbf{L}^1} \leq \varepsilon.$$

This can be done by changing the graph of $u(t_1)$ in the regions where $u_x^k < -1$, replacing a large negative gradient with several small downward jumps in the k-th field, as shown in fig.13. We then extend u as a piecewise Lipschitz solution of (5.1)–(5.3) on a second interval $[t_1, t_2]$, up to the next time where (5.21) occurs, perform our "restarting procedure" again, etc. Observe that the lengths of the intervals $[t_i, t_{i+1}]$ are bounded away from zero. Indeed, they depend only on upper bound M in (5.17), which (with some care in the construction) we can assume uniformly valid on $[0, T]$. By applying our restarting procedure at a finite number of times $t_1 < t_1 < \cdots < t_\nu$, we thus define an approximate solution u on the whole interval $[0, T]$.

If now u, u' are approximate solutions constructed as above, let $\gamma_0 : \theta \mapsto u^\theta(0)$ be a regular curve of initial conditions, connecting $u(0)$ with $u'(0)$. For every θ, we can now apply the previous algorithm and obtain an approximate solution u^θ of (5.1)–(5.3), continuously depending on θ. A careful construction yields an estimate of the form

$$d^\star(u(t),\ u'(t)) \leq \|\gamma_t\|_\star \leq \|\gamma_0\|_\star + O(\varepsilon) \qquad \forall t \in [0, T],$$

where $\|\gamma_t\|_\star$ denotes the weighted length of the path $\theta \mapsto u^\theta(t)$. As $\varepsilon \to 0$, this implies that the approximate solutions converge to the trajectories of a continuous semigroup generated by (MRS), contractive w.r.t. the weighted distance $d^\star$.

Figure 14

At this stage, we are ready to treat the general case of a hyperbolic system with an arbitrary number of genuinely nonlinear characteristic families. Fix a state $u^* \in I\!R^n$ and let $\lambda_1^* < \ldots < \lambda_n^*$ be the eigenvalues of the matrix $A(u^*)$. For $k = 1, \ldots, n$, let $A_k(u)$ be the matrix with the same eigenvectors $r_i(u)$ as $A(u)$, but whose eigenvalues are

$$\lambda_1^*, \ \ldots, \ \lambda_{k-1}^*, \quad \lambda_k^* + n(\lambda_k(u) - \lambda_k^*), \ \ldots, \ \lambda_n^*. \tag{5.22}$$

Clearly, for each k the system

$$u_t + A_k(u)u_x = 0 \tag{5.23}$$

has $n-1$ linearly degenerate fields. By the previous analysis, the corresponding (MRS) generates a semigroup S^k, contractive w.r.t. a suitable weighted metric $d^\star$ which we can take to be the same for all k. To construct the semigroup S for the original system (5.1), we fix a time step $\Delta t > 0$ and concatenate the flows of the semigroups $S^1, \ldots, S^n$ cyclically, on subintervals of length $\Delta t/n$. More precisely, we define by induction an approximate semigroup $S^{(\Delta t)}$ as follows. If

$$t \in \left[\Big(q + \frac{k-1}{n}\Big)\Delta t, \quad \Big(q + \frac{k}{n}\Big)\Delta t \right]$$

for some integers $q \geq 0$, $k \in \{1, \ldots, n\}$, we define

$$S_t^{(\Delta t)} \bar u \doteq S^k_{t-(q+(k-1)/n)\Delta t} \Big(S^{(\Delta t)}_{(q+(k-1)/n)\Delta t} \bar u \Big).$$

As $\Delta t \to 0$, it is clear that $S^{(\Delta t)}$ tends to a contractive semigroup S, because all semigroups S^k are contractive w.r.t. the same weighted distance $d^\star$. To prove that S is precisely the semigroup generated by (MRS), it suffices to check that it behaves correctly on the Riemann problems. In turn, it suffices to check that it behaves correctly in connection with elementary waves. Let (u^-, u^+) be a jump in the i-th family, satisfying (5.2), whose correct speed, according to (MRS), is given by λ_i^r in (5.3). Observe that, in the flow of $S^{(\Delta t)}$, this jump travels with speed λ_i^* during the time intervals of the form

$$\left[\Big(q + \frac{j}{n}\Big)\Delta t, \quad \Big(q + \frac{j+1}{n}\Big)\Delta t \right], \qquad j \neq i, \quad q \geq 0,$$

and with speed $\lambda_i^* + n(\lambda_i^r - \lambda_i^*)$ during the intervals

$$\Big[\big(q+\frac{i}{n}\big)\Delta t,\ \ \big(q+\frac{i+1}{n}\big)\Delta t\Big], \qquad q \geq 0.$$

Over any interval $[q\Delta t,\ (q+1)\Delta t]$, its average speed thus coincides with the correct speed λ_i^r. Hence, as $\Delta t \to 0$, the convergence is clear (fig.14).

The case of a centered rarefaction wave of the i-th family is entirely similar (fig.15).

Figure 15

Figure 16

Having studied the semigroup generated by the Modified Riemann Solver (MRS), where all jumps occur along rarefaction curves, we now tackle the more difficult problem of constructing a semigroup generated by (ε-RS). Because of the bound on the total variation, any solution $u = u(t, \cdot)$ will contain finitely many shock fronts $x = x_\alpha(t)$ with strength $> \varepsilon$. Outside these fronts, the evolution determined by (ε-RS) is the same as for (MRS). Trajectories of (ε-RS) are thus the same as in the previous case, except for the presence of a finite number of "free boundaries" corresponding to shock fronts of strength $> \varepsilon$.

These solutions will be constructed first by a local procedure. Let an initial condition $\bar{u}$ be given. In a neighborhood of any point where the Riemann problem does not involve any jump of size $\geq \varepsilon$, a local solution is constructed by the same method used for (MRS). If, at some point $\bar{x}$, the jump in $\bar{u}$ determines shocks of size $\geq \varepsilon$, the construction of approximate solutions proceeds as follows. We choose small constants $\Delta t, \delta > 0$. We divide the time axis into steps of length $\Delta t/n$. We insert strips

$$\Gamma_\alpha \doteq \{(t, x); \quad |x - x_\alpha(t)| < \delta\}$$

of width δ around each large shock front (fig.16). Outside the strips, on each interval of the form

$$I_k \doteq [(q + (k-1)/n)\Delta t, \quad (q + k/n)\Delta t],$$

the function u is a piecewise Lipschitz solution of a quasilinear system of the form (5.23), with only the k-th field genuinely nonlinear. Inside each strip Γ_α, u satisfies a quasilinear system where all characteristic speeds are constant. Approximate solutions are sought within the class of functions which are Lipschitz continuous inside each strip, have jumps in the k-th field outside the strips (on the time intervals I_k, $k = 1, \ldots, n$), and satisfy the ε-approximate Rankine-Hugoniot equations (5.5), (5.6) along the big shock fronts. In this case, during each time interval I_k, we have two types of restarting operations to perform:

(1) replace a function $u(t, \cdot)$, that has large negative gradient component u_x^k, with a function $u(t+, \cdot)$ such that $u_x^k \geq -1$, meanwhile causing minimal changes in the total variation, in the interaction potential, and in the sizes of the other components of the gradient.

(2) replace a function that has jumps inside the regions $\{|x - x_\alpha(t)| < \delta\}$ with one which is Lipschitz in these regions.

Moreover, at times $t = (q + (k-1)/n)\Delta t$, we must replace $u(t, \cdot)$ with a new function $u(t+, \cdot)$ having small jumps only in the k-th family. The key point here is that the time between a restarting and the next one can be predicted in advance, and does not depend on the accuracy of the restarting procedure.

Letting $\Delta t, \delta \to 0$, by compactness we obtain a local solution. Patching together all these local solutions, and showing that they coincide on overlapping regions, for every initial data $\bar{u}$ we construct a unique weak solution $u(t, \cdot) = S_t \bar{u}$.

The next step is to prove that the semigroup S is uniformly Lipschitz continuous. This fact is not obvious, because the approximation scheme (with the insertion of "boundary layers" around each large shock) is different for different solutions. At this preliminary stage, we are able to estimate the distance of two solutions u, u' only if their wave-front configuration is the same. By this we mean that u and u' have the same number of shock fronts of strength $> \varepsilon$, which interact at the same number of points, in the same order.

Figure 17

Continuity of the flow is proved by a local argument, as in [9]. To establish the uniform Lipschitz continuity, we first define a solution $u = u(t, x)$ as *Struc-*

turally Stable if, roughly speaking, it does not contain three shocks of strength $> \varepsilon$ interacting at a single point, nor a centered compression wave of strength $> \varepsilon$ (fig.17,18). This property guarantees that nearby solutions $u' \approx u$ will have the same wave-front structure as u.

Figure 18

Given a continuous, one-parameter family of solutions $\theta \mapsto u^\theta$, we show that, if each u^θ is structurally stable (and hence if all u^θ have the same wave-front configuration), then the weighted length of the path $\gamma_t : \theta \mapsto u^\theta(t)$ does not increase in time. By the continuity of the flow w.r.t. the initial condition, it follows that the weighted length $\|\gamma_t\|_\star$ is non-increasing also in the case where finitely many of the solutions u^θ are allowed to be structurally unstable.

The final step consists in showing that Structural Stability is a "generic" property. More precisely: an arbitrary continuous path γ can be uniformly approximated by a path containing finitely many structurally unstable solutions. For this purpose, we consider a perturbed system of the form

$$u_t + A^\varphi(\theta, t, x; u)u_t = 0 \tag{5.23}$$

where $\varphi = (\varphi_1, \ldots, \varphi_n)(\theta, t, x)$ is a function with arbitrarily small $\mathcal{C}^2$ norm, and

A^φ is a matrix with the same eigenvectors as $A(u)$, and eigenvalues

$$\lambda_i^\varphi = \lambda_i(u) + \varphi_i(\theta, t, x). \tag{5.24}$$

We show that suitable functions φ can be chosen, with arbitrarily small $\mathcal{C}^2$ norm, such that the lengths of the paths of corresponding solutions change by arbitrarily little, and these paths have finitely many structurally unstable points. Intuitively, the perturbations φ_i in (5.24) slightly change the speeds of the shock fronts. As a result, we can avoid the unstable configurations shown in fig. 17 for all but finitely many values of the parameter θ.

This establishes the uniform Lipschitz continuity of each approximate semigroup S^ε, with a constant independent of ε. Letting $\varepsilon \to 0$, the semigroups S^ε converge uniformly to the Standard Riemann Semigroup generated by (1.1). Details of this construction will appear in [11].

6. Wave-front Tracking

In this section we review a wave-front tracking algorithm developed in [4, 8]. Its purpose is to construct a sequence of approximate solutions u_ν of the Cauchy problem, within a class of piecewise constant functions with small total variation. By a compactness argument, an exact solution can then be obtained, in the limit. Other constructions, sharing similar ideas, can be found in [19, 33].

The basic idea of wave-front tracking is quite simple (fig.22). Start with a piecewise constant initial data $\bar{u}$. At each point of jump, solve the Riemann problem within a class of piecewise constant functions. If all waves present in the exact solutions are shocks or constant discontinuities, the exact solution is already piecewise constant and no modification is needed. If the exact solution contains centered rarefaction waves, these are approximated by piecewise constant rarefaction fans, inserting additional intermediate states. The solution is thus prolonged up to a time t_1 where the first couple of wave-fronts interact. This new Riemann problem is again solved within a class of piecewise constant functions. The solution is then prolonged up to a time t_2 where the second interaction takes place, etc.

The main technical difficulty in this approach is the fact that the number of wave-fronts may become infinite in finite time (fig.19). When this happens, the algorithm breaks down. To avoid this situation, in [8] two distinct procedures are used for approximately solving the Riemann problem: an accurate method, which possibly introduces several new wave-fronts, and a simplified method, which avoids the introduction of new wave-fronts. These are now described.

Figure 19

Accurate Riemann Solver: Let $\omega_0 = u^-, \ \omega_1, \ldots, \omega_n = u^+$ be the states present in the standard solution of a Riemann problem, say generated by the interaction of an i-wave with a j-wave. The approximate solution (fig.20) is then obtained from the exact solution by replacing each centered rarefaction wave of a characteristic family $k \neq i, j$ with a rarefaction fan. This is done by dividing the jump $(\omega_{k-1}, \ \omega_k)$ into ν equal jumps, inserting the intermediate states

$$\omega_{k,0} = \omega_{k-1}, \ \omega_{k,1}, \ \ldots \ ,\omega_{k,\nu-1}, \ \omega_{k,\nu} = \omega_k.$$

Each small jump $(\omega_{k,\ell-1}, \ \omega_{k,\ell})$ will travel with speed $\lambda_k(\omega_{k,\ell})$, i.e. with the k-th characteristic speed of its right state. When $k = i$ or $k = j$, a rarefaction wave of the same family of an incoming front is not further partitioned: it is propagated as a single front, travelling with the characteristic speed $\lambda_k(\omega_k)$ of its right state.

Figure 20

Figure 21

Simplified Riemann Solver: Assume that the Riemann problem is determined by the interaction of two waves of distinct families, say $i < j$, with sizes σ_i, σ_j. Call u^l, u^m, u^r the left, middle and right states before the interaction (fig.21). We then solve the Riemann problem by two outgoing wave-fronts of the same families, still with sizes σ_i, σ_j. The solution will thus involve a new middle state $\tilde{u}^m$, connected on the right of u^l by an i-wave of size σ_i, and a new right state $\tilde{u}^r$, connected on the right of $\tilde{u}^m$ by a j-wave of size σ_j. In general, $\tilde{u}^r \neq u^r$. In our approximate solution, this jump is propagated along a "non-physical" wave-front with constant speed $\hat{\lambda}$, larger than all characteristic speeds.

In the case where both incoming wavefronts belong to the same i-th family and have sizes σ', σ'', the Riemann problem is solved by a single outgoing i-wave of size $\sigma = \sigma' + \sigma''$, together with a non-physical wave-front with speed $\hat{\lambda}$.

Figure 22

To complete the description of the front tracking algorithm, one should specify which Riemann solver is used at any given interaction. For this purpose, to each front we associate a "generation number", keeping track of how many interactions were needed to generate that particular front (fig.22). Waves originating from the initial Riemann problems at $t = 0$ are the primary ancestors, and have order 1. The newly born waves, originating from the interaction of fronts of orders k_1, k_2 have generation order $\max\{k_1, k_2\} + 1$.

In the construction of the ν-th approximation u_ν, the Riemann problems generated by the interaction of two wave-fronts of orders $k_1, k_2 < \nu$ are solved accurately, dividing each newly born centered rarefaction wave into a fan of ν small jumps. On the other hand, if one of the interacting fronts has order $\geq \nu$, then the simplified Riemann solver is used.

This method guarantees that the total number of fronts remains finite. Indeed, if N_1 fronts of order 1 are present at time $t = 0$, there can be $< N_1^2$ points of interaction among them. At each such interaction, at most $(n-1)\nu$ new wave-fronts of order 2 can be produced. Therefore, the total number of fronts of order 2 is $N_2 \leq N_1^2(n-1)\nu$. Similarly, waves of order 3 are produced when a front of order 2 hits a front of order 1 or 2. This implies that the number of third order waves is $N_3 \leq N_2(N_1 + N_2)(n-1)\nu$. By induction, the number of fronts of order $k+1$ is bounded by

$$N_{k+1} \leq N_k(N_1 + \ldots + N_k)(n-1)\nu.$$

Since there are no waves of order $> \nu + 1$, the total number of wave-fronts is a priori bounded.

To prove the global existence of weak solutions of (1.1), one first checks that, as $\nu \to \infty$

(i) the total variation of $u_\nu(t, \cdot)$ remains uniformly small.

(ii) the maximum size of each rarefaction front in u_ν approaches zero.

(iii) The total strength of all non-physical waves approaches zero.

The proof of (i) is based on the standard interaction estimates of Glimm. By (i), Helly's theorem guarantees the existence of a subsequence strongly convergent in $\mathbf{L}^1_{loc}$, while (ii) and (iii) imply that any strong limit $u = \lim_{\nu\to\infty} u_\nu$ is a weak, entropic solution to the system of conservation laws.

As we shall see in the next section, the above algorithm plays a key role in the proof of uniqueness of a Standard Riemann Semigroup. Indeed, it provides the existence of piecewise constant approximate solutions which satisfy the evolution equation (1.1) with an arbitrary degree of accuracy. More precisely, the following holds.

Lemma 1. *Let $\mathcal{D}$ be a domain of the form (2.2). Then, for each piecewise constant initial data $\bar{u} \in \mathcal{D}$ and $\varepsilon > 0$ there exists an approximate solution*

$u^\varepsilon = u^\varepsilon(t,x)$ of (1.1) with $u^\varepsilon(0,x) = \bar u(x)$, $u^\varepsilon(t,\cdot) \in \mathcal{D}$ for all $t \geq 0$. Moreover, the discontinuities of u^ε lie on a finite number of polygonal lines in the t-x plane. These jumps can be of three types, denoted by $\mathcal{S}, \mathcal{R}, \mathcal{N}$, respectively.

(i) *Contact discontinuities or entropy-admissible shocks, satisfying the Rankine-Hugoniot equations exactly.*

(ii) *Rarefaction fronts, with size $\sigma_\alpha \in [0,\ \varepsilon]$, travelling with speed equal to the characteristic speed of the state to the right.*

(iii) *Non-physical fronts, all travelling with a fixed speed $\hat\lambda$ larger than all characrteristic speeds. Their total strength satisfies $\sum_\ell |\sigma_\ell| < \varepsilon$.*

Details of the proof can be found in [4] or in Chapter 7 of [8].

7. Uniqueness of the semigroup

Let $S : \mathcal{D} \times [0,\infty[\mapsto \mathcal{D}$ be a Standard Riemann Semigroup with Lipschitz constant L, and let $u : [0,T] \mapsto \mathcal{D}$ be any continuous map. We regard u as an approximate solution to the Cauchy problem (1.1), (1.2), and seek a formula expressing the distance between $u(T)$ and the value of the exact solution $S_T \bar u$. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be any finite partition of the interval $[0,T]$. Calling L the Lipschitz constant of the semigroup, we have (fig.23)

Figure 23

$$
\begin{aligned}
\|u(T) - S_T \bar u\| \quad &\le \|S_T u(0) - S_T \bar u\| \\
&\quad + \sum_{i=0}^{N-1} \left\| S_{T-t_{i+1}} u(t_{i+1}) - S_{T-t_i} u(t_i) \right\| \\
&\le L \cdot \|u(0) - \bar u\| + L \cdot \sum_{i=0}^{N-1} \left\| \frac{u(t_i + h_i) - S_{h_i} u(t_i)}{h_i} \right\| \cdot \\
&\quad (t_{i+1} - t_i),
\end{aligned}
\tag{7.1}
$$

where $h_i = t_{i+1} - t_i$. Considering finer and finer partitions, in the limit we formally obtain the error estimate

$$
\begin{aligned}
\|u(T) - S_T \bar u\|_{\mathbf{L}^1} \quad &\le L \cdot \|u(0) - \bar u\|_{\mathbf{L}^1} \\
&\quad + L \cdot \int_0^T \left(\limsup_{h \to 0+} \left\| \frac{u(t+h) - S_h u(t)}{h} \right\|_{\mathbf{L}^1} \right) dt.
\end{aligned}
\tag{7.2}
$$

For a rigorous proof, see [7]. Observe that the integrand in (7.2) can be interpreted as an instantaneous rate of error, i.e. the amount by which the function u fails to satisfy the evolution equation at a given time t.

Figure 24

We shall now apply (7.2) to estimate the distance between a semigroup trajectory and any ε-approximate solution constructed by wave-front tracking. For each fixed t, the exact solution $h \mapsto S_h u^\varepsilon(t)$ is obtained by solving the Riemann problems generated by the jumps of $u^\varepsilon(t, \cdot)$. According to Lemma 1, these jumps fall into the three classes $\mathcal{S}, \mathcal{R}, \mathcal{N}$: shocks (or contact discontinuities), rarefactions and non-physical waves (fig.24).

By (i), if $u^\varepsilon(t,\cdot)$ has a shock or a contact discontinuity at $x_\alpha(t)$, then in a neighborhood of this point the exact solution $S_h u^\varepsilon(t)$ coincides with $u^\varepsilon(t+h,\cdot)$. By (ii), if u^ε has a rarefaction front of size $\sigma_\alpha > 0$ through the point $x_\alpha(t)$, then in a neighborhood of this point the exact solution $S_h u^\varepsilon(t)$ consists of a centered rarefaction wave of the same size σ_α. The difference in speed between this wave and the wave-front in u^ε is $O(\sigma_\alpha)$. For $\rho, h > 0$ sufficiently small, we thus have

$$\frac{1}{h}\int_{x_\alpha(t)-\rho}^{x_\alpha(t)+\rho} \Big| u^\varepsilon(t+h,x) - (S_h u^\varepsilon(t))(x)\Big|\, dx \le C_1|\sigma_\alpha|^2. \tag{7.3}$$

By (iii), if $u^\varepsilon(t)$ has a non-physical wave-front of strength $|\sigma_\alpha|$ at the point $x_\alpha(t)$, in a neighborhood of this point the difference between $u^\varepsilon(t+h)$ and the exact solution $S_h u^\varepsilon(t)$ can be estimated by

$$\frac{1}{h}\int_{x_\alpha(t)-\rho}^{x_\alpha(t)+\rho} \Big| u^\varepsilon(t+h,x) - (S_h u^\varepsilon(t))(x)\Big|\, dx \le C_2|\sigma_\alpha|. \tag{7.4}$$

Combining together the above estimates, and recalling Lemma 1, we obtain

$$\begin{aligned} &\limsup_{h\to 0+}\left\| \frac{u^\varepsilon(t+h) - S_h u^\varepsilon(t)}{h}\right\|_{\mathbf{L}^1} = \\ &= \sum_{\alpha\in\mathcal{S}\cup\mathcal{R}\cup\mathcal{N}} \left(\lim_{h\to 0+}\frac{1}{h}\int_{x_\alpha(t)-\rho}^{x_\alpha(t)+\rho}\Big|u^\varepsilon(t+h,x) - (S_h u^\varepsilon(t))(x)\Big| dx\right) \\ &\le 0 + \sum_{\alpha\in\mathcal{R}} C_1|\sigma_\alpha|^2 + \sum_{\alpha\in\mathcal{N}} C_2|\sigma_\alpha| \\ &\le \Big(\sum_{\alpha\in\mathcal{R}} C_1|\sigma_\alpha|\Big)\varepsilon + C_2\varepsilon \end{aligned} \tag{7.5}$$

for some constants C_1, C_2. By (7.5), as $\varepsilon \to 0$, the integrand in (7.2) approaches zero uniformly on $[0,T]$. As a result, we have the equality

$$S_T \bar u = u(T,\cdot) \qquad \bar u \in \mathcal{D},\ T \ge 0, \tag{7.6}$$

valid for every Standard Riemann Semigroup with domain $\mathcal{D}$, and every solution u of the Cauchy problem (1.1), (1.2), obtained as limit of wave-front tracking approximations.

This clearly implies that the semigroup is unique and that the wave-front tracking scheme converges to a unique limit, for every initial data $\bar{u} \in \mathcal{D}$. Moreover, each semigroup trajectory is a weak, entropy-admissible solution of (1.1), because this is true for every limit of front tracking approximations.

We conclude this section observing that, due to the finite propagation speed, a localized version of (7.2) can be obtained. Namely, let L be the Lipschitz constant of the semigroup and let $\hat{\lambda}$ be an upper bound for all characteristic speeds. Fix an interval $J_0 = [a, b]$, and define $J_t \doteq [a + \hat{\lambda}t,\ b - \hat{\lambda}t]$ a domain of dependency for J_0 at time $t < (b-a)/2$. Then, for any initial data $\bar{u}, \bar{v} \in \mathcal{D}$ one has

$$\|S_t\bar{u} - S_t\bar{v}\|_{\mathbf{L}^1(J_t)} \leq L \cdot \|\bar{u} - \bar{v}\|_{\mathbf{L}^1(J_0)}. \tag{7.7}$$

Moreover,

$$\Big\|u(T,x) - S_T u(0)\Big\|_{\mathbf{L}^1(J_T)} \leq L \cdot \int_0^T \limsup_{h\to 0+} \left\| \frac{u(t+h) - S_h u(t)}{h} \right\|_{\mathbf{L}^1(J_{t+h})} dt. \tag{7.8}$$

8. Characterization of semigroup trajectories

In this section we motivate the characterization of trajectories in terms of local integral estimates, and sketch the proof of Theorem 3.

Observe that, if $v = v(t,x)$ is any approximate solution of (1.1), one can rewrite (7.8) in the slightly more general form

$$\begin{aligned} &\Big\|v(\tau+\varepsilon) - S_\varepsilon v(\tau)\Big\|_{\mathbf{L}^1(J_\varepsilon)} \\ &\leq L \cdot \int_\tau^{\tau+\varepsilon} \limsup_{h\to 0+} \left\| \frac{v(t+h) - S_h v(t)}{h} \right\|_{\mathbf{L}^1(J_{t+h-\tau})} dt, \end{aligned} \tag{8.1}$$

with $J_\varepsilon = [a + \varepsilon\hat{\lambda},\ b - \varepsilon\hat{\lambda}]$.

Let now $u(t,\cdot) = S_t\bar{u}$ be any semigroup trajectory. To derive (2.6), we apply (8.1) with

$$[a,\ b] \doteq [\xi - \rho,\ \xi + \rho], \qquad J_\varepsilon = [\xi - \rho + \varepsilon\hat{\lambda},\ \xi + \rho - \varepsilon\hat{\lambda}],$$

regarding $U^\sharp_{(u;\tau\xi)}$ as the approximate solution and u as the exact solution. The estimate (2.6) will thus be proved if we can show that

$$\begin{aligned}&\limsup_{h\to 0+}\left\|\frac{U^\sharp(t+h)-S_hU^\sharp(t)}{h}\right\|_{\mathbf{L}^1(J_{t-\tau})}\\ &\leq C_0\cdot \text{Tot.Var.}\{u(\tau);\ \ ]\xi-\rho,\ \xi[\ \cup\ ]\xi,\ \xi+\rho[\,\}\end{aligned}\tag{8.2}$$

for some constant C_0 and all $t\in[\tau,\ \tau+\varepsilon]$. Roughly speaking, the left hand side of (8.2) measures the "instantaneous error rate" of the approximation $U^\sharp$ at time t. Since $U^\sharp$ is an exact solution of (1.1) inside the triangular region where $|x-\xi|<(t-\tau)\hat\lambda$, the only contributions to the left hand side of (8.2) are given by the waves contained in the two intervals where

$$(t-\tau)\hat\lambda<|x-\xi|<\rho-(t-\tau)\hat\lambda,$$

travelling with speed 0, and by the two jumps at $x=\xi\pm(t-\tau)\hat\lambda$, travelling with speed $\pm\hat\lambda$. Observing that the total strength of all these waves is $O(1)\cdot$ Tot.Var.$\{u(\tau);\ ]\xi-\rho,\ \xi[\cup]\xi,\ \xi+\rho[\ \}$, we deduce (8.2) and hence (2.6).

The idea behind the estimate (2.7) is similar. We now apply (8.1), with $U^\flat_{(u;\tau,\xi)}$ playing the role of the approximate solution, while u is the exact solution. In this case, instead of (8.2), one has to show that

$$\begin{aligned}&\limsup_{h\to 0+}\left\|\frac{U^\flat(t+h)-S_hU^\flat(t)}{h}\right\|_{\mathbf{L}^1(J_{t+h-\tau})}\\ &\leq C_0\cdot\Big(\text{Tot.Var.}\{u(\tau);\ \ ]\xi-\rho,\ \xi+\rho[\,\}\Big)^2\end{aligned}\tag{8.3}$$

for some constant C_0 and all $t\in[\tau,\tau+\varepsilon]$. Observe that, for every point (t,x) in the domain of dependency, the waves in $U^\flat$ are propagated with constant speeds $\tilde\lambda_1,\ldots,\tilde\lambda_n$, corresponding to the eigenvalues of $\widetilde A$. In an exact solution, these waves should travel with speeds equal to the eigenvalues of $A(U^\flat(t,x))$. The difference between these eigenvalues can be estimated in terms of the difference between the corresponding matrices. In turn, this is given by

$$\begin{aligned}\Big\|A\ \ (U^\flat(t,x))-\widetilde A\Big\|&=\Big\|A(U^\flat(t,x))-A(u(\tau,\xi))\Big\|\\ &=O(1)\cdot|U^\flat(t,x)-u(\tau,\xi)|=O(1)\cdot\text{Tot.Var.}\{u(\tau);\ ]\xi-\rho,\ \xi+\rho[\ \}.\end{aligned}$$

At each time t, we thus expect that the left hand side of (8.3) be roughly measured by

$$\begin{aligned}&[\text{total strength of waves in } U^\flat \text{ at time } t] \times [\text{error in the wave speeds}]\\ &\qquad = O(1)\cdot\Big(\text{Tot.Var.}\{u(\tau);\ ]\xi-\rho,\ \xi+\rho[\ \}\Big)^2.\end{aligned}$$

This motivates the necessity of the condition (2.7). A rigorous proof can be found in [7].

In order to prove the sufficiency of the integral conditions (2.6), (2.7), let u be a viscosity solution of (1.1), (1.2), according to Definition 1. For any T, our plan is to prove that $\|u(T,\cdot)-S_T\bar u\|=0$ relying on the estimate (7.2), showing that the integrand on the right hand side of (7.2) actually vanishes for all t.

Let $\tau\in[0,T]$ be given, and call μ the positive Radon measure corresponding to the total variation of $u(\tau,\cdot)$. Fix $\varepsilon>0$ and a compact set $K\subset I\!R$. Since μ is finite, there will be finitely many atoms, say $x_1,\ldots,x_N$ with $\mu(\{x_j\})\geq\varepsilon$. For each one of these atoms, choose an open interval $I_j=]x_j-\rho_j,\ x_j+\rho_j[$ such that

$$\mu(I_j\setminus\{x_j\})<\frac{\varepsilon}{N}. \tag{8.4}$$

Observe that the two functions $u(t,\cdot)$ and $S_{t-\tau}u(\tau)$ are both viscosity solutions of (1.1). (The first by assumption, the second by the "necessary" part of Theorem 3.) We can thus assume that for both of these functions the estimates (2.6) hold on the interval I_j.

Next, choose finitely many open intervals $I'_\ell=]\xi'_\ell-\rho'_\ell,\ \xi'_\ell+\rho'_\ell[$ such that

(i) $\mu(I'_\ell)\leq\varepsilon$ for each ℓ,

(ii) The family $\big(\bigcup_j I_j\big)\cup\big(\bigcup_\ell I'_\ell\big)$ covers the compact set K,

(iii) No point $x\in I\!R$ lies inside more that two distinct intervals I'_ℓ.

(iv) The two viscosity solutions $u(t,\cdot)$ and $S_{t-\tau}u(\tau)$ both satisfy the estimates

(2.7) on each interval I'_ℓ.

Figure 25

For $h>0$, call $I_j(h)=]x_j-\rho_j+h\hat\lambda,\ x_j+\rho_j-h\hat\lambda[$ the domain of dependency of I_j at time $\tau+h$. Define $I'_\ell(h)$ similarly (fig.25). Observe that, for h small, the intervals $I_j(h), I'_\ell(h)$ still cover K. The difference $|u(\tau+h)-S_h u(\tau)|$ is now estimated using the bounds (2.6) over the intervals I_j and the bounds (2.7) over the intervals I'_ℓ. Using (8.4) and the properties of the intervals I_j, I'_ℓ, one obtains

$$
\begin{aligned}
&\tfrac{1}{h}\big\|u(\tau+h,\cdot)-S_h u(\tau)\big\|_{\mathbf{L}^1(K)}\\
&\le \sum_{j=1}^N \frac{1}{h}\int_{I_j(h)} \Big(\big|u(\tau+h,x)-U^\sharp_{(u;\tau,\xi_j)}(t+h,x)\big|\\
&\qquad +\big|U^\sharp_{(u;\tau,\xi_j)}(t+h,x)-(S_h u(\tau))(x)\big|\Big)\,dx\\
&\quad+\sum_\ell \frac{1}{h}\int_{I'_\ell(h)} \Big(\big|u(\tau+h,x)-U^\flat_{(u;\tau,\xi'_\ell)}(t+h,x)\big|\\
&\quad +\big|U^\flat_{(u;\tau,\xi'_\ell)}(t+h,x)-(S_h u(\tau))(x)\big|\Big)\,dx\\
&\le \sum_{j=1}^N 2C\cdot\mu(I_j\setminus\{x_j\})+\sum_\ell 2C\cdot\big(\mu(I'_\ell)\big)^2\\
&\le N\cdot 2C\cdot\tfrac{\varepsilon}{N}+2C\cdot\sum_\ell \varepsilon\mu(I'_\ell)\\
&\le 2C\varepsilon+2C\varepsilon\cdot 2\mu(\mathbb{R}).
\end{aligned}
$$

Since $\varepsilon>0$ and $K\subset\mathbb{R}$ were arbitrary, we conclude that the right hand side of (7.2) vanishes for all T. This yields the sufficiency part of Theorem 3.

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References

[1] Amadori, D. and Colombo, R.M., *Continuous dependence for* 2×2 *conservation laws with boundary*, Preprint S.I.S.S.A., (1995).

[2] Baiti, P. and Bressan, A., *The semigroup generated by Temple class systems with large data*, Preprint S.I.S.S.A., (1995).

[3] Bressan, A., *Contractive metrics for nonlinear hyperbolic systems*, Indiana Univ. Math. J., 37 (1988), 409-421.

[4] Bressan, A., *Global solutions of systems of conservation laws by wave-front tracking*, J. Math. Anal. Appl., 170 (1992), 414-432.

[5] Bressan, A., *A contractive metric for systems of conservation laws with coinciding shock and rarefaction waves*, J. Differential Equations, 106 (1993), 332-366.

[6] Bressan, A., *A locally contractive metric for systems of conservation laws*, Ann. Scuola Norm. Sup. Pisa, IV-22 (1995), 109-135.

[7] Bressan, A., *The unique limit of the Glimm scheme*, Arch. Rational Mech. Anal., 130 (1995), 205-230.

[8] Bressan, A., *Lecture Notes on Systems of Conservation Laws*, S.I.S.S.A., Trieste, (1995).

[9] Bressan, A. and Colombo, R. M., *The semigroup generated by* 2×2 *systems of conservation laws*, Arch. Rational Mech. Anal., 133 (1995), 1-75.

[10] Bressan, A. and Colombo, R.M., *Unique solutions of* 2×2 *conservation laws with large data*, Indiana Univ. Math. J., 44 (1995), 677-725.

[11] Bressan, A.; Crasta, G. and Piccoli, B., *Well-posedness of the Cauchy problem for $n \times n$ systems of conservation laws*, to appear.

[12] Bressan, A. and Guerra, G., *Shift-differentiability of the flow generated by a conservation law*, Preprint S.I.S.S.A., (1995).

[13] Bressan, A. and Marson, A., *A variational calculus for discontinuous solutions of systems of conservation laws*, Comm. Part. Diff. Equat., 20 (1995), 1491-1552.

[14] Bressan, A. and Marson, A., *Error bounds for a deterministic version of the Glimm scheme*, Preprint S.I.S.S.A., (1995).

[15] Crandall, M., *The semigroup approach to first-order quasilinear equations in several space variables*, Israel J. Math., 12 (1972), 108-132.

[16] Crandall. M. and Lions, P. L., *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 277 (1983), 1-42.

[17] Crasta, G. and Piccoli, B., *Viscosity solutions and uniqueness for systems of inhomogeneous balance laws*, Preprint S.I.S.S.A., (1995).

[18] Dafermos, C. and Geng, X., *Generalized characteristics, uniqueness and regularity of solutions in a hyperbolic system of conservation laws*, Ann. Inst. Henri Poincaré - Nonlinear Analysis, 8 (1991), 231-269.

[19] DiPerna, R., *Global existence of solutions to nonlinear hyperbolic systems of conservation laws*, J. Differential Equations, 20 (1976), 187-212.

[20] DiPerna, R., *Uniqueness of solutions to hyperbolic conservation laws*, Indiana Univ. Math. J., 28 (1979), 137-188.

[21] DiPerna, R., *Convergence of approximate solutions to conservation laws*, Arch. Rational Mech. Anal., 82 (1983), 27-70.

[22] Glimm, J., *Solutions in the large for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math., 18 (1965), 697-715.

[23] Goodman, J. and Xin, Z., *Viscous limits for piecewise smooth solutions to systems of conservation laws*, Arch. Rational Mech. Anal., 121 (1992), 235-265.

[24] Heibig, A., *Existence and uniqueness of solutions for some hyperbolic systems of conservation laws*, Arch. Rational Mech. Anal., 126 (1994), 79-101.

[25] Kruzkov, S., *First order quasilinear equations with several space variables*, Math. U.S.S.R. Sbornik, 10 (1970), 217-243.

[26] Lax, P., *Hyperbolic systems of conservation laws II*, Comm. Pure Appl. Math., 10 (1957), 537-566.

[27] Ta-tsien, L.and Wen-ci, Y. *"Boundary value problems for quasilinear hyperbolic systems"*, Duke University Mathemathics Series V, (1985).

[28] Liu, T. P., *Uniqueness of weak solutions of the Cauchy problem for general* 2×2 *conservation laws*, J. Differential Equations, 20 (1976), 369-388.

[29] Liu, T. P., *The deterministic version of the Glimm scheme*, Comm. Math. Phys., 57 (1977), 135-148.

[30] Ljapidevskii, V. J., *On correctness classes for nonlinear hyperbolic systems*, Soviet Math. Dokl., 16 (1975), 1505-1509.

[31] Oleinik, O. A., *Discontinuous solutions of nonlinear differential equations*, Usp. Mat. Nauk, 12 (1957), 3-73. English Transl. Amer. Math. Soc. Transl. Ser. 2, Vol. 26, 95-172.

[32] Pimbley, G., *A semigroup associated with a quasilinear system in which the coupling terms are linear - II*, Nonlinear Analysis, 12 (1988), 321-340.

[33] Risebro, N. H., *A front-tracking alternative to the random choice method*, Proc. Amer. Math. Soc., 117 (1993), 1125-1139.

[34] Rozdestvenskii, B. and Yanenko, N., *"Systems of Quasi-Linear Equations"*, A. M. S. Translations of Mathematical Monographs, Vol. 55, 1983.

[35] Smoller, J., *"Shock waves and reaction-diffusion equations"*, Second Edition, Springer-Verlag, New York, 1994.

[36] Temple, B., *No* $\mathbf{L}^1$*-contractive metric for systems of conservation laws*, Trans. Amer. Math. Soc., 288 (1985), 471-480.

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