OSCILLATIONS INDUCED BY NUMERICAL VISCOSITIES

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1. Introduction

We investigate the numerical behavior of shock capturing methods for the computation of slowly moving shocks. Here slowly moving means that the ratio of the shock speed to the maximum wave speed in the domain is much less than one. Earlier literatures have reported the difficulty of computing slowly moving shocks [13,10,9], where first order Godunov or Roe type methods produce spurious long wave oscillations behind the shock and eventually ruin the downstream pattern. Several heuristic arguments, or improvements on the Riemann solver have been made in [13,10,5,1]. However, none of these improvements were robust enough to completely eliminate these downstream oscillations.

The goal of this article is to carefully study this peculiar numerical phenomenon, and to understand its formation and propagation. Our study shows that these downstream oscillations are generated by the perturbation of the discrete shock profile. They propagate along characteristics and decay in L^2 and L^∞. The perturbing nature of the viscous shock profile is the constant source for the generation of the downstream oscillations for all time. In our numerical experiments we also observed periodic structure of the perturbing viscous shock profile. The period is exactly the time for the shock to propagate one spatial grid.

The outline of the paper follows. In section 2 we present a numerical example, using a Roe type scheme, on a Riemann problem of the compressible

*Research was supported in part by NSF grant No. DMS-9404157.
†Research was supported in part by NSF grant No. DMS-9505275.

Euler equations that admits slow shocks. Among the numerical artifacts observed in this example are the momentum spikes and downstream oscillations. In section 3 a traveling wave analysis on a viscous isentropic Euler equations (Euler equations with linear viscosity terms in both the continuity and momentum equations) is presented to show the existence of the momentum spike, which differs from the momentum profile of the Navier-Stokes Equations. In section 4 we study the downstream oscillations, and establish its relation with the stability of the discrete shock.

2. Numerical Solutions of a Slowly Moving Shock

Consider the 1-D compressible Euler equations of gas dynamics,

$$\begin{aligned} &\partial_t \rho + \partial_x m = 0\,, \\ &\partial_t m + \partial_x(\rho u^2 + p) = 0\,, \\ &\partial_t E + \partial_x(u(E+p)) = 0\,. \end{aligned} \tag{2.1}$$

Here ρ, u, $m = \rho u$, p and E are respectively the density, velocity, momentum, pressure and total energy. For a polytropic gas, the equation of state is given by

$$p = (\gamma - 1)(E - \tfrac{1}{2}\rho u^2)\ . \tag{2.2}$$

Let A denote the Jacobian matrix $\partial F(U)/\partial U$. The Euler equations (2.1-2.2) is hyperbolic with eigenvalues

$$a^1 = u - c\,, \qquad a^2 = u\,, \qquad a^3 = u + c\,, \tag{2.3}$$

where $c = \sqrt{\gamma p/\rho}$ is the local speed of sound. The right eigenvectors of A form the matrix $R = (R^1, R^2, R^3)$ given by

$$R = \begin{pmatrix} 1 & 1 & 1 \\ u-c & u & u+c \\ H - uc & \frac{1}{2}u^2 & H + uc \end{pmatrix}\,, \tag{2.4}$$

with $H = \frac{c^2}{\gamma-1} + \frac{u^2}{2}$. The inverse of R defines the left eigenvectors $(L^1, L^2, L^3) = R^{-1}$ of A by

$$R^{-1} = \begin{pmatrix} \frac{1}{2}(b_1 + \frac{u}{c}) & \frac{1}{2}(-b_2 u - \frac{1}{c}) & \frac{1}{2}b_2 \\ 1 - b_1 & b_2 u & -b_2 \\ \frac{1}{2}(b_1 - \frac{u}{c}) & \frac{1}{2}(-b_2 u + \frac{1}{c}) & \frac{1}{2}b_2 \end{pmatrix}\,, \tag{2.5}$$

with

$$b_2 = \frac{\gamma - 1}{c^2}, \qquad b_1 = b_2 \frac{u^2}{2}. \tag{2.6}$$

Let $U = (\rho, m, E)^T$ be the vector of conserved quantity, $\hat{A}_{j+1/2}$ be the Roe matrix satisfying [11]

$$F(U_{j+1}) - F(U_j) = \hat{A}_{j+1/2}(U_{j+1} - U_j). \tag{2.7}$$

By projecting $U_{j+1} - U_j$ onto $\{R_{j+1/2}\}$ one obtains the characteristic decomposition

$$U_{j+1} - U_j = \sum_{p=1}^{3} \alpha^p_{j+1/2} R^p_{j+1/2}. \tag{2.8}$$

In this decomposition the local characteristic variables $\alpha^p_{j+1/2}$ can be obtained using Roe's average which perfectly resolves stationary discontinuities.

We let $x_{j+1/2}$ be the grid points, $U_{j+1/2}$ be the pointwise value of U at $x = x_{j+1/2}$, and U_j be the cell center value of U at $x_j = (x_{j+1/2} + x_{j-1/2})$. We use a first order upwind, Roe type scheme, called ENO1-Roe, by Shu and Osher [12], that has the numerical flux defined by

$$F_{j+1/2} = \frac{1}{2}(F(U_j) + F(U_{j+1})) - \frac{1}{2}\mathrm{sgn}(\lambda^p_{j+1/2})(\gamma^p_{j+1} - \gamma^p_j)R^p_{j+1/2}, \tag{2.9a}$$

where γ^p_j is the component of $F(U_j)$ in the p-th characteristic family,

$$F(U_j) = \sum_{p=1}^{3} \gamma^p_j R^p_j. \tag{2.9b}$$

We carry out the following 1-D test on a Riemann problem of the Euler equations (2.1)-(2.2).

Example 2.1. We take the following initial data [9] that gives a Mach-3 shock moving to the right with a speed $s = 0.1096$:

$$\begin{aligned} U_L &= \begin{pmatrix} 3.86 \\ -3.1266 \\ 27.0913 \end{pmatrix} \quad \text{if} \quad 0 \le x < 0.5\,; \\ U_R &= \begin{pmatrix} 1 \\ -3.44 \\ 8.4168 \end{pmatrix} \quad \text{if} \quad 0.5 \le x \le 1\,. \end{aligned} \tag{2.12}$$

Figure 1: A slowly moving shock at $t = 0.95$ computed by ENO1-Roe using $\Delta x = 0.01$ and $\Delta t = 0.001$.

We take $\gamma = 1.4$ and output the results at $t = 0.95$ in Fig.1. The computation is carried out in the domain $[0, 1]$ with $\Delta x = 0.01, \Delta t = 0.001$. One can see that there is a momentum spike and the post-shock solution is oscillatory.

3. The Momentum Spikes

Here we present a traveling wave analysis on a viscous Euler equations, which shows precisely the formation of the momentum spike. After comparing it with the traveling wave solution of the Navier-Stokes equation we find out that this spike is a numerical artifact.

Consider the following viscous isentropic Euler equations for density ρ and momentum m:

$$\begin{aligned} &\partial_t \rho + \partial_x m = \epsilon\, \partial_{xx} \rho\,, \\ &\partial_t m + \partial_x \left(\tfrac{m^2}{\rho} + p(\rho) \right) = \epsilon\, \partial_{xx} m\,. \end{aligned} \tag{3.1}$$

Here the pressure $p(\rho) = k\rho^\gamma$ for some constants k and γ. This hyperbolic system has two distinct eigenvalues $u \pm c$ where $u = m/\rho$ is the velocity, $c = \sqrt{\gamma k \rho^{\gamma-1}}$ is the sound speed. Although the true numerical viscosity is far more complicated than those appeared on the right hand side of Eq.(3.1), a study on (3.1) is sufficient for a full understanding of the numerical momentum spike.

We look at the traveling wave solution to (3.1). Let $\xi = \frac{x-st}{\epsilon}$ where s is the shock speed. Then the traveling wave solution takes the form

$$\rho(x,t) = \phi(\xi)\,, \qquad m(x,t) = \psi(\xi) \tag{3.2}$$

with asymptotic states

$$\phi(\pm\infty) = \phi_{\pm}\,, \qquad \psi(\pm\infty) = \psi_{\pm}\,. \tag{3.3}$$

Applying the traveling wave solution (3.2) in (3.1a), after some manipulations one gets the following ODE:

$$\partial_\xi \phi = -s(\phi - \phi_-) + \psi - \psi_-\,,$$

or

$$\psi = s\phi + \partial_\xi \phi + (\psi_- - s\phi_-)\,. \tag{3.4}$$

If the density is smeared, then ϕ is monotone, and $\partial_\xi \phi$ becomes a spike. (3.4) shows that *ψ is a superposition of a monotone profile $s\phi$ with a spike corresponding to $\partial_\xi \phi$. When s is small (for stationary or slowly moving shock), the monotone profile $s\phi$ becomes small and the spike term $\partial_\xi \phi$ dominates. Thus the shock profile of ψ is a non-monotone spike.* Therefore the spike is usually generated in a stationary or slowly moving shock, as shown in our earlier example. For a strong shock the monotone profile $s\phi$ dominates so the shock profile of the momentum is monotone.

Although our traveling wave analysis applies to linear viscosities, one can similar argue about the existence of the momentum spike for nonlinear scheme (such as the ENO1-Roe).

Note that the momentum spike is solely numerical artifacts. By solving the Riemann problem exactly one obtains a monotone momentum.

In [4] we also studied the traveling wave solution of the Navier-Stokes equations (where there is no viscosity term in tha continuity equation). The result shows that the momentum profile given by the Navier-Stokes equations is again monotone, thus does not have the spike. It is conjectured that the solution of the Euler equations is the zero viscosity limit of the solution of the Navier-Stokes equations. Since the Navier-Stokes equations are more physical, the momentum spike in unphysical.

By examining the interrelation between the viscous Euler and the Navier-Stokes equation one can come up with a change of variable that recovers the Navier-Stokes equations in an asymptotic sense from the Euler equations. This also motivates a numerical change of variable, i.e., from the cell-center or cell average momentum to the mass flux, which eliminates the momentum spike exactly. For details see [4]. However, what is more catastrophic is the downstream oscillations, which can not be easily eliminated, as will be studied next.

4. The Downstream Oscillations

We bear in mind that almost all shock capturing methods are in conservative form. Due to the conservation of momentum, the total mass of momentum carried by the spike profile should be compensated by an equal amount of momentum mass elsewhere. This explains the formation of the downstream waves.

In Fig.2 we output the result of ENO1-Roe for example 2.1 after 5 time steps to illustrate the formation of the momentum spike and downstream wave. As the density is smeared, the momentum forms a spike and a downstream wave. The spike and the downstream wave carry the same mass so the total momentum is conserved.

In order to demonstrate that the downstream oscillations propagate along characteristics and are diffusive, we use the Roe decomposition (2.8), where α^p represents the component of $U_{j+1} - U_j$ in the p-th characteristic family. We define the numerical "characteristic" variable as

$$\beta_j^p = \sum_{i \le j} \alpha_{i+1/2}^p \Delta x \,. \tag{4.1}$$

A distinction between the dispersive oscillations associated with a center difference schemes and the downstream oscillations studied here is that the latter lie only in its own characteristic family. For example a wave appears in β^p does not appear in β^q for $p \ne q$. These can be seen in Fig.3. We also see that each wave

Figure 2: The formation of the momentum spike and the downstream wave in the ENO1-Roe calculation of example 2.1 after 5 time steps. As the density is smeared, the momentum forms a spike, and a downstream wave to balance the mass of the spike for momentum conservation.

moves away with the corresponding characteristic speed, and behaves diffusively (spread out and decay).

In Fig.4 we display the time evolution of the momentum profile of the ENO1-Roe for example 2.1. One can see that the spike (viscous) profile keeps fluctuating in an $O(1)$ manner, causing the downstream oscillations for all time. The diffusive nature of the downstream oscillations is evident in the picture.

In Fig.5 we plot the peak of the momentum spike as a function of time. The fluctuating nature of the spike is evident. *The more the mass of the spike profile varies the more strongly the diffusion waves emerge for momentum conservation.* Interesting is that *the peaks are periodic, with the duration of each period agrees with the time for the shock to move one grid point.* Thus the discrete shock profile is stable only modulu this period. However, within each period it is fluctuating, which becomes the source of the new downstream waves.

Recall that the definition of a discrete traveling wave solution Φ_j^n, an approximation of $U(x_j, t_n), t_n = n\Delta t$, requires

$$\Phi_j^{nq} = \Phi_{j-np}^0 \,, \tag{4.2}$$

Figure 3: The downstream waves in "characteristic" variables. Note that each wave belongs only to one characteristic family and diffuses.

where $s\Delta t/\Delta x = p/q$ for some relative prime integers p and q. The stability of such discrete shock for the Lax-Friedrichs scheme was established by Jennings [3] for scalar equations and by Majda and Ralston [8] and J.-G. Liu and Xin [6] for nonlinear systems. The periodicity of the momentum peaks in Fig.5 shows the stability of the discrete traveling wave solution Φ_j^n for these schemes modulus the time for the shock to travel one grid point. This is because, when $s\Delta t << \Delta x$, there exists a sufficiently large q such that $|q(s\Delta t) - \Delta x| < s\Delta t$, or $|s\Delta t/\Delta x - 1/q| \leq 2/q^2$. However within each period the numerical shock layer is unsteady and corresponds to different travelling wave profiles, which becoming the source of the new downstream waves in all time for these schemes.

Figure 4: Time evolution of the momentum spike and the downstream waves by ENO1-Roe for $t \in [0.5, 0.8]$. For better visualization these graphs are displayed upside down. The diffusive nature of the downstream pattern is apparent .

A scheme that completely eliminates the downstream oscillations in later time should have a steady viscous profile beyond the initial formation of the spike, i.e., the momentum spike peak should remain a constant in later time. However this is impossible as long as the shock is moving and it takes many times for the shock to move to the next cell.

The discrete shock profile perturbs even when the shock does not move slowly. Thus the downstream oscillations exist even for fast shocks. However, in the fast shock case the momentum profile is monotone, thus does not leave much room for the shock profile to perturb. In other words, each perturbation does not change the mass of the viscous profile much, and the downstream errors become negligible. For slow shock the momentum profile has a spike, which increases the mass of the viscous profile and the relative mass change in each perturbation, so the downstream errors become more significant. This also illustrates why the downstream errors in the density is far less significant. Since the density is monotone, thus the relative change in the mass of the viscous profile is smaller than that of the momentum.

In summary, although each family of the downstream waves decay time-asymptotically, the perturbing spike or viscous profile is a constant source for

Figure 5: Time evolution of the peak of the momentum spike by ENO1-Roe.

the generation of new downstream waves, causing the downstream noise for *all* time. Higher order methods use higher order interpolations, which amplify the noises and exhibit rich but spurious post-shock structures.

5. Discussions and Conclusions

As studied in [4], similar behavior occurs in schemes that are of monotone, TVD or ENO type. Note that all these monotonicity theories are established only for scalar equations, or linear systems via the characteristic variables. For nonlinear systems there are no global characteristic variables, thus these methods are usually extended to nonlinear systems using the idea for linear systems, i.e., via the so-called local characteristic decomposition (using the Roe matrix for example). Since there is no theory for the monotonicity of these methods for nonlinear systems, it is not surprising to see the non-monotone behavior represented by the spike and the downstream oscillations reported here. It seems to us that, to fully solve this problem, instead of applying scalarly monotone, TVD or ENO scheme to nonlinear systems, one needs a method that is *systematically* "monotone, TVD or ENO". One also needs to choose numerical viscosity properly so it mimics the physical viscosity of the Navier-Stokes equations. the ultimate goal is to have a scheme that not only provides a high resolution but, more importantly, has a more stable viscosity profile. These require good theories for both inviscid and viscous nonlinear systems (such as those in [7]), and remain open and challenging research subjects for the future.

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Received November 14, 1995
Revised January 19, 1996