MATHEMATICAL MODELING OF PLASTICITY IN METALS

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Abstract

We present a short introduction to continuum models for the plastic flow of metals. Our emphasis is on the physical principles underlying these models, the nature and validity of approximations involved, and the mathematical structure of the flow equations. Using the framework developed, we derive a simple, but realistic, model describing one-dimensional plastic flow.

1. Introduction

When a piece of metal is subjected to stress, it responds by deforming. If only small stresses are applied, then the metal returns to its original shape when stress is relieved. In this regime, the metal is elastic. If, however, the stress exceeds a threshold, the yield stress, then the metal suffers permanent deformation. This is plastic behavior.

Models of plasticity have been investigated by many scientists for many years (some of the fundamental work is referenced below). Early work assumed the deformations to be small; this approximation is often quite good, and it leads to some simplification. A primary focus was static solutions and linearized dynamics. More recent work has addressed the problems posed by large deformations and nonlinear dynamics, but much remains to be understood.

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*This work was supported in part by: the U. S. Army Research Office under Grant DAAL03-92-G-0185; the National Science Foundation under Grant INT-9512873; and the Conselho Nacional de Desenvolvimento Tecnológico e Cientíifico under Grant 301411/95-6.
1991 *Mathematics Subject Classification.*73E05, 73E70, 73D05.
Key words and phrases: rate-dependent plasticity.

Physically realistic models of dynamic plasticity are necessarily complicated. The kinematics of a three-dimensional continuum, the thermodynamics of materials, and the physics of microscopic defects all enter the description of plastic phenomena. Simplified versions of such models are, of course, valuable for understanding the mathematical features of plastic flow. It is essential, however, for these simplified models to be faithful to the physics. The purpose of this paper is to derive such a simplified model, paying close attention to the nature and validity of the approximations made.

Section 2 contains an overview of continuum models of plasticity. We discuss the governing conservation laws, measures of elastic and plastic strain, the form of constitutive relations dictated by thermodynamics, and the phenomenon of yielding. More specific constitutive models are considered in Sec. 3. The elastic response of a metal is isotropic, and the shear strain that it can support is tiny; these properties allow us to formulate a constitutive model with a manageable number of empirical parameters. In Sec. 4, we specialize the general three-dimensional flow equations to one-dimensional flow with uniaxial strain. This flow configuration is used in the experimental determination of material properties under conditions of high pressure and high strain rate. The governing equations resemble those for reactive gas dynamics, but there is an important difference, as we discuss. We conclude by briefly describing some general features of solutions of this simple model.

2. Large-Strain Plasticity

In this section, we develop a general framework for describing three-dimensional plastic flow, following Ref. [11].

2.1. Kinematics and conservation laws. To describe the motion of a continuous body, two distinct frames of reference are required: the material (or Lagrangian) frame, in which a coordinate X labels a point of the body in its reference configuration; and the spatial (or Eulerian) frame, in which a coordinate x represents a position in physical space. The placement of a body in space is an association of a position x to each material point X. Therefore the motion of a body is represented mathematically by a time-dependent map ϕ defining the spatial positions of material points through the relation

$$x = \phi(X, t). \tag{2.1}$$

A material point moves with the particle velocity

$$V = \frac{\partial \phi}{\partial t}, \tag{2.2}$$

and the deformation of a small neighborhood of the point is characterized by the deformation gradient

$$F = \nabla \phi. \tag{2.3}$$

In the absence of external forces, the principles of conservation of momentum and energy lead to the two equations

$$\rho_0 \dot{V} - \nabla \cdot P = 0, \tag{2.4}$$

$$\rho_0 \left(\frac{1}{2} |V|^2 + \mathcal{E} \right)^{\cdot} - \nabla \cdot \left(V^\top P \right) = 0. \tag{2.5}$$

Here the dot represents differentiation with respect to time, ρ_0 is the mass density in the reference configuration (which can depend on X), P is the (first) Piola-Kirchhoff stress tensor, and $\mathcal{E}$ is the specific internal energy. The principle of conservation of angular momentum is equivalent to requiring $F^{-1}P$ to be symmetric; therefore we write $P = FS$, where S is called the (second, or symmetric) Piola-Kirchhoff stress tensor. These Piola-Kirchhoff stress tensors are related to the familiar Cauchy stress tensor σ through

$$\sigma = J^{-1} P F^\top = J^{-1} F S F^\top. \tag{2.6}$$

Here

$$J = \det F \tag{2.7}$$

denotes the Jacobian determinant of ϕ. Whereas σ represents the force per unit spatial area, P represents the force per unit material area.

This system of conservation laws involves second-order derivatives of ϕ, but it can be reduced to a first-order system in the standard way. Indeed, the deformation gradient and velocity are related by the identity

$$\dot{F} - \nabla V = 0. \tag{2.8}$$

Conversely, if F and V solve this equation, and if $F|_{t=0} = \nabla\phi_0$ for some ϕ_0, then there exists a motion ϕ, with $\phi|_{t=0} = \phi_0$, such that $F = \nabla\phi$ and $V = \partial\phi/\partial t$ for $t > 0$. Therefore, by including Eq. (2.8) in the system of conservation laws, we can regard F and V as fundamental dynamical variables, instead of ϕ, and the system has first order.

Once S and $\mathcal{E}$ are defined through constitutive assumptions about the material, Eqs. (2.4)–(2.8), together with initial conditions, determine the motion of the material.

2.2. Strain. Deformation changes the internal energy of the material, which causes stresses within the body. The energy, however, depends only on changes in the distances between points of the body; it is unaffected by spatial rotations. The rotation-independent part of the deformation is the strain.

When the body has undergone a motion with deformation gradient F, material points initially separated by dX have a spatial separation $dx = F\,dX$. Therefore the squared distance between them is $dX^\top F^\top F\,dX$, which can be written as $dX^\top C\,dX$ in terms of the (right) Cauchy-Green strain tensor, defined by

$$C = F^\top F. \tag{2.9}$$

This tensor can be viewed as a metric in the material frame that measures spatial separation. The tensor C is invariant under spatial rotation of the body, for if F is replaced by QF, where Q is orthogonal, then C is unchanged. Moreover, C determines the ratio of the length of $dx = F\,dX$ to the length of dX, so that the internal energy $\mathcal{E}$ depends on F only through C.

An equivalent measure of strain is the Lagrangian strain tensor

$$E = \frac{1}{2}(C - I), \tag{2.10}$$

which vanishes if and only if F is orthogonal, and which reduces to the usual infinitesimal strain tensor (the symmetric part of the displacement gradient) in the small-strain limit. In these terms, the internal energy $\mathcal{E}$ depends on F only through E.

2.3. Plastic strain. When that material is elastic, $\mathcal{E}$ is determined completely by E together with a thermodynamic quantity, such as the entropy or the temperature. For plastic materials, however, another kind of strain affects the energy.

A principal feature of plastic behavior is irrecoverable deformation: whereas an elastic body returns to its undeformed state when all applied forces are relaxed, in general a plastic material does not. On a microscopic level, this behavior is caused by defects in the atomic lattice of the metal. Shear forces cause the displacement of rows of atoms, resulting in the formation of dislocations. Thus plasticity is, on a fundamental level, the result of dislocation dynamics. However, plastic behavior at the continuum level can involve phenomena at many length scales, such as interaction of dislocations, pile-up at grain boundaries, polycrystals, *etc.* For noncrystalline materials, it is difficult to formulate continuum models directly in terms of dislocations.

Instead, phenomenological models are used. A general thermodynamic treatment of such models has been given by Green and Naghdi [5]. In analogy with the elastic strain tensors C and E, they introduced the plastic Cauchy-Green strain tensor C^{p} and the plastic Lagrangian strain tensor $E^{\mathrm{p}} = \frac{1}{2}(C^{\mathrm{p}} - I)$. The internal energy $\mathcal{E}$ depends on E^{p} as well as E, and the dynamics of E^{p} are specified by a plastic flow rule.

The introduction of C^{p} can be motivated by a multiplicative decomposition of the deformation gradient [2, 7, 9]. Imagine cutting out a small volume around a material point and relaxing the forces acting on its surface. The residual deformation is the irrecoverable, or plastic part, of the deformation, characterized by a tensor F_{p}. Thus F can be decomposed as a product

$$F = F_{\mathrm{e}} F_{\mathrm{p}} \tag{2.11}$$

of elastic and plastic parts. In general, neither F_{e} nor F_{p} is the gradient of a map. Nevertheless, one can view F_{p} as taking a material separation dX to a separation $d\overline{X} = F_{\mathrm{p}}\,dX$ in an "intermediate" frame, and similarly view F_{e} as taking $d\overline{X}$ to $dx = F_{\mathrm{e}}\,d\overline{X}$. The squared distance between points separated by $d\overline{X}$ is $dX^{\top} F_{\mathrm{p}}^{\top} F_{\mathrm{p}}\,dX$, which suggests the identification $C^{\mathrm{p}} = F_{\mathrm{p}}^{\top} F_{\mathrm{p}}$. This identification provides a micromechanical interpretation for C^{p}.

Notice that the decomposition (2.11) is not unique, in that F_{e} and F_{p} could just as well be replaced by $F_{\mathrm{e}}\overline{Q}^{-1}$ and $\overline{Q}F_{\mathrm{p}}$, respectively. In general, $\overline{Q}F_{\mathrm{p}}$ and F_{p} represents different plastic deformations, since the separations $\overline{Q}F_{\mathrm{p}}\,dX$ and $F_{\mathrm{p}}\,dX$ can differ in length. However, when $\overline{Q}$ is orthogonal, these lengths are equal for all dX, so that $\overline{Q}F_{\mathrm{p}}$ and F_{p} should be regarded as equivalent. For this reason, a measure of plastic strain should be unaffected by rotations in the intermediate frame. This is true of $F_{\mathrm{p}}^{\top} F_{\mathrm{p}}$ but not of an alternative tensor that we consider below.

We emphasize, however, that the introduction of F_{p}, while useful for intuition and motivation, is not necessary for modeling the plastic behavior of (noncrystalline) materials: since the energy depends on F_{p} only through C^{p}, it suffices to consider C^{p} alone.

2.4. Hyperelastic material. We conclude that the internal energy $\mathcal{E}$ is a function of the strain tensors E and E^{p}, the specific entropy N, and any additional internal variables K for modeling effects such as hardening:

$$\mathcal{E} = \widehat{\mathcal{E}}(E, E^{\mathrm{p}}, N, K). \tag{2.12}$$

Such a material is a generalization of a hyperelastic material, for which $\mathcal{E}$ is a function of E.

Standard arguments in rational thermodynamics [5] show that a hyperelastic energy function completely determines the stress. Specifically, the second Piola-Kirchoff stress tensor is the thermodynamic conjugate of the strain E:

$$S = \rho_0 \frac{\partial \widehat{\mathcal{E}}}{\partial E}. \tag{2.13}$$

Similarly, the temperature $T = \partial\widehat{\mathcal{E}}/\partial N$ is thermodynamically conjugate to the entropy.

By analogy, we can introduce the plastic stress tensor

$$S_{\mathrm{p}} = -\rho_0 \frac{\partial\widehat{\mathcal{E}}}{\partial E^{\mathrm{p}}} \tag{2.14}$$

conjugate to E^{p} and $\Omega = -\rho_0\partial\widehat{\mathcal{E}}/\partial K$ conjugate to K. (The minus signs here are standard. In the small-strain limit, S_{p} reduces to S.) These quantities are important because they appear in the identity

$$\rho_0 T\,\dot{N} = \mathrm{tr}\left(S_{\mathrm{p}}\,\dot{E}^{\mathrm{p}}\right) + \Omega\,\dot{K} \tag{2.15}$$

for the dissipation of entropy. The entropy dissipation must be nonnegative:

$$\mathrm{tr}\left(S_{\mathrm{p}}\,\dot{E}^{\mathrm{p}}\right) + \Omega\,\dot{K} \geq 0. \tag{2.16}$$

This inequality poses restrictions on the equation of state (2.12).

2.5. Yield. Another key feature of a plastic material is that its behavior is elastic below a certain threshold; only at or beyond this threshold does the material yield. Usually the threshold criterion is phrased in terms of the Cauchy stress. For example, according to the classical von Mises criterion, plastic flow occurs only if

$$\|\mathrm{dev}\sigma\| \geq \sqrt{\frac{2}{3}}\,Y_0(K), \tag{2.17}$$

where Y_0 denotes the static yield stress. (Here we use the notation

$$\|A\| = \left[\mathrm{tr}\left(A^{\top}A\right)\right]^{\frac{1}{2}} \tag{2.18}$$

for the norm and

$$\mathrm{dev}A = A - \frac{1}{3}\mathrm{tr}(A)I \tag{2.19}$$

for the deviator, *i.e.*, trace-free part, of a 3×3 matrix A.) The dependence of the static yield stress on K models effects such as hardening.

If the material reaches the yield threshold, plastic flow can occur. To understand this, picture a small volume of material, represented as a point in stress space. Drawn in this space is the yield surface, defined by equality in the yield criterion (*e.g.*, inequality (2.17)). If the flow carries the point across the yield surface, E^{p} begins to evolve: $\dot{E}^{\mathrm{p}} \neq 0$.

Plastic flow strives to bring the point back to the yield surface (as a consequence of the entropy dissipation inequality (2.16)) at a rate that increases as the yield criterion is exceeded further. In some circumstances, the rate is fast relative to the time-scale of the flow and a rate-independent approximation is appropriate. On a fundamental level, however, plastic flow is rate-dependent. Therefore the evolution equations for the plastic variables take the form $\dot{E}^{\mathrm{p}} = \Lambda$ and $\dot{K} = M$, where the source terms Λ and M involve the state variables but not their derivatives. The source terms vanish unless the yield threshold is exceeded.

2.6. Governing equations. In summary, the equations of motion are

$$\dot{F} - \nabla V = 0, \tag{2.20}$$

$$\rho_0 \dot{V} - \nabla \cdot (FS) = 0, \tag{2.21}$$

$$\rho_0 \left(\frac{1}{2}|V|^2 + \mathcal{E} \right)^{\cdot} - \nabla \cdot \left(V^{\top} F S \right) = 0, \tag{2.22}$$

$$\dot{E}^{\mathrm{p}} = \Lambda, \tag{2.23}$$

$$\dot{K} = M, \tag{2.24}$$

where S is given by Eq. (2.13). The equation of state (2.12) and the source terms Λ and M remain to be specified by a material model.

3. Material Modeling

To complete the equations of motion of a plastic medium we must formulate models for the equation of state and the plastic source terms. In this section, we discuss the motivation for and the derivation of a realistic material model for large-strain plasticity.

Although hardening effects can be very important in plastic flow, a thorough discussion is beyond the scope of the present paper. Accordingly, we shall ignore the hardening variable K in the remainder of the paper.

3.1. Equation of state. The hyperelastic equation of state (2.12) is very general; in practice, a more specialized model is needed. An example is provided by small-strain theory. In this theory, both E and E^{p} are assumed to be small, and the total strain E is decomposed additively:

$$E = E^{\mathrm{e}} + E^{\mathrm{p}}, \tag{3.1}$$

where E^{e} is called the elastic Lagrangian strain. Then the internal energy $\mathcal{E}$ is modeled as a function of $E^{\mathrm{e}} = E - E^{\mathrm{p}}$. (One consequence is that $S_{\mathrm{p}} = S$. See Eqs. (2.13) and (2.14).) Often $\mathcal{E}$ is assumed to be given by the classical St. Venant-Kirchhoff model:

$$\rho_0 \mathcal{E} = \frac{1}{2} K \left(\mathrm{tr} E^{\mathrm{e}}\right)^2 + G \left\| \mathrm{dev} E^{\mathrm{e}} \right\|^2, \tag{3.2}$$

K being the bulk modulus and G being the shear modulus of the material. When the deformation of the material is large, however, such a model proves to be inadequate (for a discussion, see, *e.g.*, Ref. [1]).

To devise a more realistic model, consider again a small volume of material. The internal energy $\mathcal{E}$ stored in this volume is the work done in deforming the volume from its plastic configuration (obtained by relaxing surface forces) to its current configuration. Thus it depends only on the elastic part F_{e} of the deformation gradient [8]. Since $\mathcal{E}$ is unaffected by spatial rotations, it should depend on F_{e} only through the tensor $\overline{C}^{\,\mathrm{e}} = F_{\mathrm{e}}^{\top} F_{\mathrm{e}}$, which is called the elastic Cauchy-Green strain tensor in the intermediate frame. Equivalently, $\mathcal{E}$ would be a function of $\overline{E}^{\,\mathrm{e}} = \frac{1}{2}(\overline{C}^{\,\mathrm{e}} - I)$, the elastic Lagrangian strain tensor in the intermediate frame. These considerations suggest that $\mathcal{E}$ could be modeled by a general function of $\overline{E}^{\,\mathrm{e}}$.

One must be careful, however [6]. The energy function $\mathcal{E}$ must be expressible in terms of E and E^{p}, as required by the thermodynamic arguments leading to the Green-Naghdi form (2.12); but $\overline{E}^{\mathrm{e}}$ cannot be so expressed. Indeed, we have noted nonuniqueness in the decomposition (2.11): if $\overline{Q}$ is orthogonal, then F_{e} can be replaced by $F_{\mathrm{e}}\overline{Q}^{\top}$, so that $\overline{E}^{\mathrm{e}}$ is replaced by $\overline{Q}\,\overline{E}^{\mathrm{e}}\overline{Q}^{\top}$, without affecting $C^{\mathrm{p}} = F_{\mathrm{p}}^{\top} F_{\mathrm{p}}$. To be thermodynamically consistent, an energy function depending only on $\overline{E}^{\mathrm{e}}$ must remain unchanged by such a rotation in the intermediate frame. In other words, $\mathcal{E}$ must be orthogonally invariant, meaning that the elastic response of the material is isotropic [8].

Conversely, if $\mathcal{E}$ is an orthogonally invariant function of $\overline{E}^{\mathrm{e}}$, then it can be expressed in terms of E and E^{p}. The reason is as follows [11]. An invariant function of a symmetric tensor can be written as a function of the principal invariants of the tensor, *i.e.*, the elementary symmetric functions of its eigenvalues [14]. Since $\overline{C}^{\mathrm{e}} = F_{\mathrm{p}}\,(C^{\mathrm{p}})^{-1} C\,F_{\mathrm{p}}^{-1}$, the eigenvalues of $\overline{C}^{\mathrm{e}}$ and $(C^{\mathrm{p}})^{-1}C$ coincide. Therefore invariants of $\overline{E}^{\mathrm{e}}$ can be related to invariants of $(C^{\mathrm{p}})^{-1}C$. In particular, $\mathcal{E}$ can be expressed in terms of E and E^{p}.

Notice that

$$(C^{\mathrm{p}})^{-1}C = \left(I + 2E^{\mathrm{p}}\right)^{-1}(I + 2E) \approx I + 2\left(E - E^{\mathrm{p}}\right) \tag{3.3}$$

when E^{p} is small. In other words, the tensor $\frac{1}{2}\,[(C^{\mathrm{p}})^{-1}C - I]$ reduces to $E^{\mathrm{e}} = E - E^{\mathrm{p}}$ in the small-strain limit. Therefore it can be regarded as a large-strain generalization of the elastic strain.

When $\mathcal{E}$ is an orthogonally invariant function of $\overline{E}^{\mathrm{e}}$, and more generally when $\mathcal{E}$ is expressible as a function of $(C^{\mathrm{p}})^{-1}C$, there is a simple formula relating the stress tensors S and S_{p} [11]:

$$S_{\mathrm{p}} = SC(C^{\mathrm{p}})^{-1} = (C^{\mathrm{p}})^{-1}CS. \tag{3.4}$$

In particular, S_{p} and S are approximately equal in the small-strain limit. A proof of formula (3.4) is presented in the appendix.

3.2. Isotropic material. Because metals are usually modeled as having isotropic elastic response, we assume that the internal energy $\mathcal{E}$ is an orthogonally invariant function of $\overline{E}^{\mathrm{e}}$. By the foregoing arguments, $\mathcal{E}$ can be written as a function of the three principal invariants of $\overline{E}^{\,\mathrm{e}}$, or equivalently of $\overline{C}^{\,\mathrm{e}}$ or $(C^{\mathrm{p}})^{-1}C$.

It is useful to distinguish two kinds of the elastic deformations: dilations and shear deformations. In a dilation, volume is changed but there is no change of shape, meaning that F_{e} is proportional to I. By contrast, a shear deformation preserves volume: $\det F_{\mathrm{e}} = 1$. The dependence of $\mathcal{E}$ on dilations gives rise to pressure forces, whereas its dependence on shear deformations causes shear forces.

An elastic deformation with gradient F_{e} can be decomposed into a dilation followed by a shear deformation: $F_{\mathrm{e}} = \widetilde{F}_{\mathrm{e}} \cdot J_{\mathrm{e}}^{1/3} I$, where the Jacobian

$$J_{\mathrm{e}} = \det F_{\mathrm{e}} = \left(\det \overline{C}^{\,\mathrm{e}}\right)^{1/2} \tag{3.5}$$

is orthogonally invariant. Correspondingly, we can decompose the strain as

$$\overline{C}^{\,\mathrm{e}} = J_{\mathrm{e}}^{2/3}\, \widetilde{C}^{\,\mathrm{e}}, \tag{3.6}$$

where $\widetilde{C}^{\,\mathrm{e}}$ represents the shear strain, since $\det \widetilde{C}^{\,\mathrm{e}} = 1$. In these terms, $\mathcal{E}$ can be written as a function of J_{e}, which measures dilation, and two independent invariants of $\widetilde{C}^{\,\mathrm{e}}$, which measure shear.

There are several choices for these invariants. In the context of one-dimensional flow, which is the focus of later sections of the present paper, it is convenient to choose invariants of the logarithmic strain

$$\overline{L}^{\,\mathrm{e}} = \frac{1}{2} \log \overline{C}^{\,\mathrm{e}} \tag{3.7}$$

(as is done, *e.g.*, in Refs. [4, 13]). The decomposition of $\overline{L}^{\,\mathrm{e}}$ into its trace and its deviatoric (*i.e.*, trace-free) parts separates dilation and shear. Indeed,

$$\mathrm{tr}\overline{L}^{\,\mathrm{e}} = \frac{1}{2} \log \det \overline{C}^{\,\mathrm{e}} = \log J_{\mathrm{e}} \tag{3.8}$$

is independent of shear, whereas

$$\mathrm{dev}\overline{L}^{\,\mathrm{e}} = \mathrm{dev}\left[\frac{1}{3}\left(\log J_{\mathrm{e}}\right) I + \frac{1}{2}\log \widetilde{C}^{\,\mathrm{e}}\right] = \frac{1}{2}\mathrm{dev}\log \widetilde{C}^{\,\mathrm{e}} \tag{3.9}$$

is independent of dilation.

With these considerations as motivation, we are led to introduce the logarithmic elastic strain

$$L^{\mathrm{e}} = \frac{1}{2}\log\left[(C^{\mathrm{p}})^{-1}C\right] \tag{3.10}$$

in the Lagrangian frame and define the following associated invariants: with $\tau_0 = \rho_0^{-1}$,

$$\tau = \tau_0 \exp \mathrm{tr} L^{\mathrm{e}} = \tau_0 J_{\mathrm{e}}, \tag{3.11}$$

$$\gamma = \mathrm{tr}\left[(\mathrm{dev} L^{\mathrm{e}})^2\right] = \left\|\mathrm{dev}\overline{L}^{\mathrm{e}}\right\|^2, \tag{3.12}$$

$$\delta = \mathrm{tr}\left[(\mathrm{dev} L^{\mathrm{e}})^3\right]. \tag{3.13}$$

The invariant τ measures dilation and γ and δ measure shear strain. In these terms, the equation of state takes the form

$$\mathcal{E} = \widetilde{\mathcal{E}}(\tau, \gamma, \delta, N). \tag{3.14}$$

From this equation of state we can calculate the Piola-Kirchhoff stress:

$$S = J_{\mathrm{e}}\left\{\frac{\partial\widetilde{\mathcal{E}}}{\partial\tau}I + \frac{2}{\tau}\frac{\partial\widetilde{\mathcal{E}}}{\partial\gamma}\mathrm{dev} L^{\mathrm{e}} + \frac{3}{\tau}\frac{\partial\widetilde{\mathcal{E}}}{\partial\delta}\mathrm{dev}\left[(\mathrm{dev} L^{\mathrm{e}})^2\right]\right\} C^{-1}. \tag{3.15}$$

(See, *e.g.*, Refs. [4,13].) The Cauchy stress is therefore given by

$$\sigma = \frac{\partial\widetilde{\mathcal{E}}}{\partial\tau}I + \frac{2}{\tau}\frac{\partial\widetilde{\mathcal{E}}}{\partial\gamma}\mathrm{dev}\ell^{\mathrm{e}} + \frac{3}{\tau}\frac{\partial\widetilde{\mathcal{E}}}{\partial\delta}\mathrm{dev}\left[(\mathrm{dev}\ell^{\mathrm{e}})^2\right], \tag{3.16}$$

where $\ell^{\mathrm{e}} = FL^{\mathrm{e}}F^{-1}$. (For simplicity, we have assumed here that $J = J_{\mathrm{e}}$, *i.e.*, $\det C^{\mathrm{p}} = 1$. This assumption is clarified below.)

Notice that, at this stage, the conceptual product decomposition (2.11) is no longer needed; the internal energy is expressed in terms of E and E^{p}.

3.3. Small anisotropy. As we shall see in discussing the plastic flow rule, the plastic behavior of metals entails that the shear strain remains quite small, even when the dilation is permitted to be large. In quantitative terms, γ and δ remain less than about $(0.002)^2$ and $(0.002)^3$, respectively. Therefore anisotropy caused by shear strain is small, and an approximation for the internal energy is warranted. In deriving such an approximation, we combine ideas from Refs. [15, 4, 3, 13].

In the small-anisotropy approximation, $\mathcal{E}$ is expanded about states of pure dilation, for which $\gamma = 0$ and $\delta = 0$ but $\tau > 0$ is arbitrary. Technically, we make the assumption that $\widetilde{\mathcal{E}}$ depends smoothly on γ and δ and then expand the energy to first order in these strain variables. Since $\delta = O\left(\gamma^{3/2}\right)$, as follows from definitions (3.12) and (3.13), the effect of δ on $\mathcal{E}$ is regarded as negligible compared to the effect of γ. Therefore we ignore δ and write

$$\widetilde{\mathcal{E}}(\tau, \gamma, \delta, N) = \mathcal{E}_0(\tau, N) + \tau G(\tau, N)\gamma. \tag{3.17}$$

The first term represents the elastic energy corresponding to dilations; the second term is the correction for shear strain.

According to formulae (3.15) and (3.16), the Piola-Kirchhoff is

$$S = J_{\mathrm{e}} \left\{ -\overline{p}\, I + 2G \mathrm{dev} L^{\mathrm{e}} \right\} C^{-1} \tag{3.18}$$

and the Cauchy stress is

$$\sigma = -\overline{p}\, I + 2G \mathrm{dev} \ell^{\mathrm{e}}, \tag{3.19}$$

where

$$\overline{p} = -\frac{1}{3}\mathrm{tr}\sigma = -\frac{\partial \mathcal{E}_0}{\partial \tau} - \frac{\partial(\tau G)}{\partial \tau}\gamma. \tag{3.20}$$

is the mean pressure. In particular, $\|\mathrm{dev}\,\sigma\| = 2G\sqrt{\gamma}$. Since the plastic flow strives to keep $\|\mathrm{dev}\,\sigma\|$ smaller than $\sqrt{2/3}\,Y_0$ (by virtue of the yield condition (2.17)), and since $Y_0/G \approx 0.005$ for a typical metal, we conclude that $\sqrt{\gamma}$ remains smaller than about 0.002. Thus the small anisotropy approximation is justified.

In the small-strain regime, $L^{\mathrm{e}} \approx E^{\mathrm{e}}$ by virtue of Eq. (3.3), so that $\tau \approx \tau_0[1 + \mathrm{tr}E^{\mathrm{e}}]$ and $\gamma \approx \|\mathrm{dev}E^{\mathrm{e}}\|^2$. Therefore (suppressing the variable N for simplicity of notation)

$$\rho_0 \widetilde{\mathcal{E}} \approx \rho_0 \mathcal{E}_0(\tau_0) + \mathcal{E}_0'(\tau_0)\mathrm{tr}E^{\mathrm{e}} + \frac{1}{2}\tau_0 \mathcal{E}_0''(\tau_0)\left(\mathrm{tr}E^{\mathrm{e}}\right)^2 + G(\tau_0)\left\|\mathrm{dev}E^{\mathrm{e}}\right\|^2. \tag{3.21}$$

We may, without loss of generality, assume that $\mathcal{E}_0(\tau_0) = 0$; moreover, we assume that the pressure vanishes in the reference state, so that $\mathcal{E}_0'(\tau_0) = 0$. Comparing approximation (3.21) to the energy (3.2) for a St. Venant-Kirchhoff material, we can identify $\tau_0\mathcal{E}_0''(\tau_0)$ as the bulk modulus and $G(\tau_0)$ as the shear modulus.

More generally, $\mathcal{E}_0(\tau, N)$ would be the energy of the material if it were a fluid (*i.e.*, if it had no shear strength), and $G(\tau, N)$ is the shear modulus when the material has specific volume τ and entropy N.

3.4. Plastic flow rule. It remains to specify the plastic flow rule $\dot{E}^{\mathrm{p}} = \Lambda$. In small-strain theory, plastic flow of metals is described by the von Mises yield condition and by the Lévy-St. Venant-Prandtl-Reuss plastic flow rule, which posits that $\dot{E}^{\mathrm{p}}$ is normal to the yield surface, *i.e.*, parallel to $\mathrm{dev}\sigma$. Before generalizing this theory to large-strain plasticity, we make two remarks.

First, it is observed experimentally that plastic flow in metals preserves volume. Therefore we assume that the flow rule maintains $\det C^{\mathrm{p}} = 1$. One consequence is that $J = J_{\mathrm{e}}$, so that $\tau = \rho^{-1}$ is simply the specific volume. Another is that

$$\mathrm{tr}\left[\left(C^{\mathrm{p}}\right)^{-1}\dot{E}^{\mathrm{p}}\right] = \frac{1}{2}\mathrm{tr}\left[\left(C^{\mathrm{p}}\right)^{-1}\dot{C}^{\mathrm{p}}\right] = \frac{1}{2}\left(\det C^{\mathrm{p}}\right)^{-1}\left(\det C^{\mathrm{p}}\right)^{\cdot} = 0. \qquad (3.22)$$

Therefore, in order to preserve plastic volume, $(C^{\mathrm{p}})^{-1}\dot{E}^{\mathrm{p}}$ must be deviatoric.

Second, thermodynamics requires the dissipation of entropy to be nonnegative. Since we ignore hardening, inequality (2.16) requires that

$$\mathrm{tr}\left(S_{\mathrm{p}}\,\dot{E}^{\mathrm{p}}\right) \geq 0. \qquad (3.23)$$

Notice that if $(C^{\mathrm{p}})^{-1}\dot{E}^{\mathrm{p}} = \mathrm{dev}[(C^{\mathrm{p}})^{-1}\dot{E}^{\mathrm{p}}]$, then this inequality can be written

$$\mathrm{tr}\left(\mathrm{dev}\left[S_{\mathrm{p}}\,C^{\mathrm{p}}\right]\mathrm{dev}[\left(C^{\mathrm{p}}\right)^{-1}\dot{E}^{\mathrm{p}}]\right) \geq 0. \qquad (3.24)$$

Therefore it is natural to assume that $(C^{\mathrm{p}})^{-1}\dot{E}^{\mathrm{p}}$ is a nonnegative multiple of $\mathrm{dev}\left[S_{\mathrm{p}}\,C^{\mathrm{p}}\right]$. In the small-strain limit, we recover the Lévy-St. Venant-Prandtl-Reuss rule that $\dot{E}^{\mathrm{p}}$ is proportional to $\mathrm{dev}\sigma$.

To express the flow rule more precisely, we shall use the following notation: if A is a 3×3 matrix and B is a metric (a symmetric, positive definite, 3×3 matrix), then the norm of A with respect to B is

$$\|A\|_B = \left[\mathrm{tr}\left(A^{\top}BAB\right)\right]^{\frac{1}{2}} \qquad (3.25)$$

and the deviator of A with respect to the B is

$$\mathrm{dev}_B A = A - \frac{1}{3}\mathrm{tr}(AB)B^{-1}. \tag{3.26}$$

In this notation, $\left(\mathrm{dev}_{C^{\mathrm{p}}} S_{\mathrm{p}}\right) C^{\mathrm{p}} = \mathrm{dev}\left[S_{\mathrm{p}}\, C^{\mathrm{p}}\right]$ and, because of formula (3.4),

$$\left\|\mathrm{dev}_{C^{\mathrm{p}}} S_{\mathrm{p}}\right\|_{C^{\mathrm{p}}}^2 = \mathrm{tr}\left(\left[\mathrm{dev}\left(S_{\mathrm{p}}\, C^{\mathrm{p}}\right)\right]^2\right) = \mathrm{tr}\left([\mathrm{dev}(SC)]^2\right) = J^2\|\mathrm{dev}\sigma\|^2. \tag{3.27}$$

These considerations motivate the following large-strain plastic flow model. Let Y_0 denote the static yield strength, and let

$$Y = \sqrt{\frac{3}{2}}\,\left\|\mathrm{dev}_{C^{\mathrm{p}}} S_{\mathrm{p}}\right\|_{C^{\mathrm{p}}} \tag{3.28}$$

denote the dynamic yield strength. Then the plastic strain is governed by

$$\dot{E}^{\mathrm{p}} = \Lambda := \sqrt{\frac{3}{2}}\,\lambda\, C^{\mathrm{p}} \frac{\mathrm{dev}_{C^{\mathrm{p}}} S_{\mathrm{p}}}{\left\|\mathrm{dev}_{C^{\mathrm{p}}} S_{\mathrm{p}}\right\|_{C^{\mathrm{p}}}} C^{\mathrm{p}}, \tag{3.29}$$

where λ, the equivalent plastic strain rate, is a nonnegative, nondecreasing function of Y that vanishes if

$$Y \le Y_0. \tag{3.30}$$

Remark. The flow rule (3.29) and the yield criterion (3.30) differ from that of Ref. [11], in that S_{p} has been used in place of S. The present formulation corresponds more closely to the classical infinitesimal theory for an isotropic material.

4. One-Dimensional Flow

In this section we specialize the general three-dimensional flow equations to a particular one-dimensional flow, uniaxial strain. This kind of one-dimensional flow occurs, for example, in the head-on collision of thin metal plates, and is it important in the experimental determination of material properties under conditions of high pressure and high strain rate. The defining feature of this flow configuration is that, since the transverse dimension of the metal plate is large compared to its thickness, there is effectively no motion of material in the transverse direction.

Flow with uniaxial strain is to be distinguished from flow with uniaxial stress, which arises in modeling the elongation of a long thing metal rod. Since the transverse dimension of a rod is small, the material can move in the transverse direction. In fact, the material must move transversely in such a way that the forces normal to the surface of the rod vanish; thus an elongated metal rod forms a neck. The structure of the governing equations and their solutions in uniaxial stress are quite different from those in uniaxial strain. For instance, whereas shock waves are an important feature of flow with uniaxial strain, they do not necessarily occur under conditions of uniaxial stress [12].

4.1. Uniaxial strain. In uniaxial strain (see, *e.g.*, Ref. [9, 4, 16]), aligned in the X-direction:

(1) all variables are independent of Y and Z;

(2) the particle velocity takes the form
$$V = (v, 0, 0)^{\top}; \tag{4.1}$$

(3) the total deformation gradient F is diagonal with yY- and zZ-components equal to 1;

(4) the plastic deformation gradient F_{p} is diagonal with equal yY- and zZ-components.

Since $\det F = J = \rho_0 \tau$, assumption (3) means that
$$F = \mathrm{diag}(\rho_0 \tau, 1, 1). \tag{4.2}$$

Recalling that $\det F_{\mathrm{p}} = 1$, by virtue of the flow rule, assumption (4) entails that F_{p} takes the form
$$F_{\mathrm{p}} = \mathrm{diag}\left(e^{2\epsilon_{\mathrm{p}}/3}, e^{-\epsilon_{\mathrm{p}}/3}, e^{-\epsilon_{\mathrm{p}}/3}\right). \tag{4.3}$$

Here ϵ_{p} is a scalar measure of the plastic strain. It proves convenient to let

$$\epsilon = \log(\rho_0 \tau) \tag{4.4}$$

and to define the elastic strain ϵ_{e} through the additive decomposition

$$\epsilon = \epsilon_{\mathrm{e}} + \epsilon_{\mathrm{p}}. \tag{4.5}$$

In terms of ϵ and ϵ_{e}, the elastic deformation gradient is easily calculated to be

$$F_{\mathrm{e}} = e^{\epsilon/3} \mathrm{diag}\left(e^{2\epsilon_{\mathrm{e}}/3}, e^{-\epsilon_{\mathrm{e}}/3}, e^{-\epsilon_{\mathrm{e}}/3}\right). \tag{4.6}$$

From the definition of the logarithmic elastic strain we find that

$$L^{\mathrm{e}} = \frac{1}{3}\epsilon I + \mathrm{dev} L^{\mathrm{e}} \tag{4.7}$$

with

$$\mathrm{dev} L^{\mathrm{e}} = \epsilon_{\mathrm{e}} \mathrm{diag}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right). \tag{4.8}$$

Consequently,

$$\gamma = \frac{2}{3}{\epsilon_{\mathrm{e}}}^2. \tag{4.9}$$

These kinematic formulae give the following picture of uniaxial motion. As argued above, γ, and therefore $|\epsilon_{\mathrm{e}}|$, is small; as a result, ϵ and ϵ_{p} are nearly equal, but they can be large in absolute value. Suppose, for definiteness, that $\epsilon < 0$ and $\epsilon_{\mathrm{p}} < 0$. Then each slab of material is squeezed in the X-direction by the factor $J = \exp \epsilon$; there is no motion in the Y- or Z-direction. In response, the material deforms plastically by contracting in the X-direction and expanding in the Y- or Z- directions, in a manner that preserves volume and symmetry. The residual elastic deformation consists of a pure dilation by the factor $J^{1/3} = \exp(\epsilon/3)$ and a small shear deformation, characterized by ϵ_{e}.

4.2. Equation of state. For uniaxial flow, the small anisotropy expansion (3.17) of the equation of state is

$$\mathcal{E} = \mathcal{E}_0(\tau, N) + \frac{2}{3}\tau G(\tau, N)\, {\epsilon_{\mathrm{e}}}^2. \tag{4.10}$$

We shall use τ and ϵ_{p} as fundamental dynamical variables, so that ϵ_{e} here should be replaced by $\log(\rho_0\tau) - \epsilon_{\mathrm{p}}$, yielding $\mathcal{E} = \overline{\mathcal{E}}(\tau, \epsilon_{\mathrm{p}}, N)$. Associated to this functional form of the energy are the conjugate quantities

$$\sigma = \frac{\partial \overline{\mathcal{E}}}{\partial \tau} \tag{4.11}$$

and

$$\sigma_{\mathrm{p}} = -\rho_0 \frac{\partial \overline{\mathcal{E}}}{\partial \epsilon_{\mathrm{p}}}, \tag{4.12}$$

which appear naturally in the equations of motion. Indeed, the Piola-Kirchhoff stress tensor is

$$S = J \mathrm{diag}\left(\sigma, \sigma - \frac{3}{2}J^{-1}\sigma_{\mathrm{p}}, \sigma - \frac{3}{2}J^{-1}\sigma_{\mathrm{p}}\right) C^{-1}, \tag{4.13}$$

as one verifies easily. In other words, the xx (*i.e.*, longitudinal) component of the Cauchy stress tensor is σ, the mean pressure is $\overline{p} = -\sigma + J^{-1}\sigma_{\mathrm{p}}$, and the deviator (*i.e.*, shear part) of the Cauchy stress is $J^{-1}\sigma_{\mathrm{p}}\mathrm{diag}\left(1, -\frac{1}{2}, -\frac{1}{2}\right)$, so that the dynamic yield strength is $Y = \frac{3}{2}|\sigma_{\mathrm{p}}|$.

The properties of the material are now encoded in the "fluid" energy $\mathcal{E}_0$ and the shear modulus G. Sophisticated physical models for these functions can, of course, be quite complicated. In the remainder of the paper, we consider a very simple model, obtained as follows.

1. Ignore thermal effects. Therefore we suppress the entropy variable N and omit the conservation of energy equation. In effect, the flow is isentropic, so that the entropy is conserved rather than generated according to the identity (2.15). In compensation, energy must be lost; in fact, the energy equation acquires a source term equal to the negative of the source term in the entropy dissipation identity.

2. Take the shear modulus $G(\tau)$ and the bulk modulus $K = \tau\mathcal{E}_0''(\tau)$ to be independent of specific volume. Using the additional conditions $\mathcal{E}_0(\tau_0) = 0$ and $\mathcal{E}_0'(\tau_0) = 0$ discussed in Sec. 3.3, we can determine $\mathcal{E}_0$.

For this simple model, we have the following explicit formulae:

$$\mathcal{E}_0(\tau) = \rho_0^{-1} K \left\{ \rho_0 \tau \log(\rho_0 \tau) - \rho_0 \tau + 1 \right\}, \tag{4.14}$$

$$\overline{p} = -K \log(\rho_0 \tau) - \frac{2}{3} G {\epsilon_{\mathrm{e}}}^2, \tag{4.15}$$

$$\sigma = K \log(\rho_0 \tau) + \frac{4}{3} G \epsilon_{\mathrm{e}} + \frac{2}{3} G {\epsilon_{\mathrm{e}}}^2, \tag{4.16}$$

$$\sigma_{\mathrm{p}} = \frac{4}{3} \rho_0 \tau G \epsilon_{\mathrm{e}}. \tag{4.17}$$

4.3. Equations of motion. It follows that the governing equations reduce to

$$\rho_0 \dot{\tau} - v_X = 0, \tag{4.18}$$

$$\rho_0 \dot{v} - \sigma_X = 0, \tag{4.19}$$

$$\dot{\epsilon}_{\mathrm{p}} = \operatorname{sgn}(\sigma_{\mathrm{p}}) \mu. \tag{4.20}$$

Here μ, which is $\frac{3}{2}$ times the equivalent plastic strain rate λ, is a nonnegative, nondecreasing function of $|\sigma_{\mathrm{p}}|$ that vanishes if

$$|\sigma_{\mathrm{p}}| \le \frac{2}{3} Y_0. \tag{4.21}$$

Because we have ignored thermal effects, the entropy dissipation identity is replaced by the energy dissipation identity

$$\rho_0 \left(\frac{1}{2} v^2 + \mathcal{E} \right)^{\cdot} - (v\sigma)_X = -|\sigma_{\mathrm{p}}| \mu. \tag{4.22}$$

This system of equations has the same form as does the system governing reactive gas dynamics in Lagrangian coordinates: ϵ_{p} is analogous to a reaction progress variable, since it appears in the constitutive relations (4.10)–(4.11) and its governing equation (4.20) is an ordinary differential equation along particle paths. There is an important difference, however, between plasticity and reactive gas dynamics. In reactive flow, the source term in the equation governing the reaction progress variables vanishes at the equilibrium states, which are isolated points. In plasticity, by contrast, the source term of Eq. (4.20) vanishes on the so-called elastic range, defined by the yield condition (4.21), which has a nonempty interior. This feature of the source term leads to phenomena, such as hysteresis, that are characteristic of plasticity.

4.4. Characteristic speeds. The four characteristic speeds for system (4.18)–(4.20) are $-J^{-1}c$, 0, 0, and $J^{-1}c$, where the (elastic, or frozen) sound speed c is defined by

$$\rho^2 c^2 = \frac{\partial \sigma}{\partial \tau} = \frac{\partial^2 \overline{\mathcal{E}}}{\partial \tau^2}. \tag{4.23}$$

The analogy with reactive gas dynamics suggests that another speed is important as well, *viz.*, the speed of sound in equilibrium. In the present context, "equilibrium" means the boundary of region where $\mu \neq 0$, *i.e.*, the yield surface $|\sigma_{\mathrm{p}}| = \frac{2}{3} Y_0$. Thus we are led to define the plastic, or equilibrium, sound speed c_{p} by

$$\rho^2 {c_{\mathrm{p}}}^2 = \left.\frac{\partial \sigma}{\partial \tau}\right|_{\mathrm{yield}}, \tag{4.24}$$

the derivative being taken with $|\sigma_{\mathrm{p}}|$ held fixed at $\frac{2}{3} Y_0$:

$$\rho^2 {c_{\mathrm{p}}}^2 = \frac{\partial \sigma}{\partial \tau} - \frac{\partial \sigma}{\partial \epsilon_{\mathrm{p}}} \frac{\partial |\sigma_{\mathrm{p}}| / \partial \tau}{\partial |\sigma_{\mathrm{p}}| / \partial \epsilon_{\mathrm{p}}} = \frac{\partial^2 \overline{\mathcal{E}}}{\partial \tau^2} - \left(\frac{\partial^2 \overline{\mathcal{E}}}{\partial \tau \partial \epsilon_{\mathrm{p}}} \right)^2 \left(\frac{\partial^2 \overline{\mathcal{E}}}{\partial {\epsilon_{\mathrm{p}}}^2} \right)^{-1}. \tag{4.25}$$

In order for the system (4.18)–(4.20) to be hyperbolic, c must be real. Furthermore, experience with reactive gas dynamics suggests that the equilibrium sound speed should not exceed the frozen sound speed; this is the subcharacteristic condition [10]. The formulae for the sound speeds show that the following two statements are equivalent:

1. the frozen sound speed c and the equilibrium sound speed c_{p} are real and satisfy $c > c_{\mathrm{p}} > 0$;

2. the internal energy $\mathcal{E}$ is strictly convex as a function of τ and ϵ_{p}.

In this way, hyperbolicity and the subcharacteristic condition relate to a thermodynamic condition, convexity of the energy.

For this particular equation of state,

$$\frac{\partial^2 \overline{\mathcal{E}}}{\partial \tau^2} = \tau^{-1} \left\{ K + \frac{4}{3} G \left(1 + \epsilon_{\mathrm{e}}\right) \right\}, \tag{4.26}$$

$$\frac{\partial^2 \overline{\mathcal{E}}}{\partial \tau \partial \epsilon_{\mathrm{p}}} = -\frac{4}{3} G \left(1 + \epsilon_{\mathrm{e}}\right), \tag{4.27}$$

$$\frac{\partial^2 \overline{\mathcal{E}}}{\partial {\epsilon_{\mathrm{p}}}^2} = \frac{4}{3} \tau G, \tag{4.28}$$

and

$$\frac{\partial^2 \overline{\mathcal{E}}}{\partial \tau^2} \frac{\partial^2 \overline{\mathcal{E}}}{\partial {\epsilon_{\mathrm{p}}}^2} - \left(\frac{\partial^2 \overline{\mathcal{E}}}{\partial \tau \partial \epsilon_{\mathrm{p}}} \right)^2 = \frac{4}{3} G \left\{ K - \frac{4}{3} G \epsilon_{\mathrm{e}} \left(1 + \epsilon_{\mathrm{e}}\right) \right\}. \tag{4.29}$$

Also,

$$\frac{\partial}{\partial \tau} \rho^2 c^2 = \frac{\partial^2 \sigma}{\partial \tau^2} = \frac{\partial^3 \overline{\mathcal{E}}}{\partial \tau^3} = -\tau^{-2} \left(K + \frac{4}{3} G \epsilon_{\mathrm{e}} \right). \tag{4.30}$$

Consequently, there is a constant ${\epsilon_{\mathrm{e}}}^{\max}$ (of order 1 if K/G is of order 1) such that, in the region where $|\epsilon_{\mathrm{e}}| < {\epsilon_{\mathrm{e}}}^{\max}$: (a) the internal energy $\mathcal{E}$ is strictly convex as a function of τ and ϵ_{p} and (b) the stress σ is strictly concave as a function of τ.

It is reasonable to assume that $|\epsilon_{\mathrm{e}}|$ is kept much smaller than ${\epsilon_{\mathrm{e}}}^{\max}$ by the plasticity of the material. Thus the sound speeds, which are given by the formulae

$$\rho c^2 = K + \frac{4}{3} G \left(1 + \epsilon_{\mathrm{e}}\right) \tag{4.31}$$

and

$$\rho {c_{\mathrm{p}}}^2 = K - \frac{4}{3} G \epsilon_{\mathrm{e}} \left(1 + \epsilon_{\mathrm{e}}\right), \tag{4.32}$$

satisfy $c > c_{\mathrm{p}} > 0$; moreover, ρc is strictly decreasing as a function of τ.

4.5. Wave structure. In elastic flow, during which ϵ_{p} remains constant, the governing equations essentially reduce to the system for isentropic gas dynamics. The foregoing calculations show that $-\sigma$, which is the analogue of the pressure, is strictly convex as a function of specific volume τ. Therefore elastic flow is qualitatively the same as flow of an ideal gas: expansion leads to elastic rarefaction waves, and elastic shock waves form under compression.

In plastic flow, the source term in the flow rule (4.20) is nonzero and competes with the nonlinearity of the flux functions. A balance between these effects is set up by traveling wave solutions. Such solutions, which are called plastic waves, are prominent features in uniaxial collision experiments.

For a traveling wave solution, the flow variables τ, v, and ϵ_{p} depend on X and t only through the combination $\xi = X - \tau_0 m t$, where $m > 0$ is the mass flux through the wave. (We assume, for definiteness, that the wave is right-facing.) The conservation laws show that the quantities $m\tau + v$ and $mv + \sigma$ are constant across the wave, so that τ and σ lie along the Rayleigh line, $m^2\tau - \sigma = \text{const.}$ On the other hand, the variation of ϵ_{p} across the wave is determined by the differential equation

$$-\tau_0 m \frac{d\epsilon_{\mathrm{p}}}{d\xi} = \operatorname{sgn}(\sigma_{\mathrm{p}})\mu. \tag{4.33}$$

The right-hand side is nonzero if and only if the yield stress is exceeded: $|\sigma_{\mathrm{p}}| > \frac{2}{3}Y_0$. Therefore the profile of a traveling wave solution, when projected into the (τ, σ)-plane: (1) lies along the Rayleigh line; (2) lies in the region $|\sigma_{\mathrm{p}}| > \frac{2}{3}Y_0$; and (3) begins and ends where the Rayleigh line intersects the yield curve $|\sigma_{\mathrm{p}}| = \frac{2}{3}Y_0$. Along the yield curve, the stress is

$$\sigma|_{\text{yield}} = K \log(\rho_0\tau) + \frac{2}{3}Y_0 \operatorname{sgn}(\sigma_{\mathrm{p}})\,(\rho_0\tau)^{-1} + \frac{Y_0^2}{6G}(\rho_0\tau)^{-2}. \tag{4.34}$$

One verifies easily that this curve is concave except in a region of extreme compression, with $\rho_0\tau$ on the order of Y_0/K. Except in this extreme, a Rayleigh line lies outside the yield surface if its endpoints lie on the compression ($\sigma_{\mathrm{p}} < 0$) branch of the yield curve (but not if they lie on the expansion branch). Therefore plastic waves occur in compression.

Notice that the thickness of a plastic wave is set by the μ^{-1}, which has units of time. If μ^{-1} is much smaller than other time scales in the flow, then the rate-independent limit $\mu^{-1} \to 0$ is a reasonable approximation. In this limit, the thickness of a plastic wave shrinks to zero, becoming a plastic shock wave.

Thus discontinuous waves arise in rate-independent flow with uniaxial strain.

Acknowledgments. The author thanks the organizers of the IVth Workshop on Partial Differential Equations in July, 1995 for an enjoyable and productive visit to IMPA. He is also grateful to Tai-Ping Liu for his hospitality during a visit to Stanford University in August, 1994, when an initial version of this paper was written.

Appendix A. Stress Calculations.

For hyperelastic materials, stresses are related to derivatives of the internal energy. Recall that if f is a function of a tensor argument A, then its derivative $\partial f/\partial A$ can be calculated as follows [14]: we imagine that A depends smoothly on a parameter α and calculate $df(A)/d\alpha$; then

$$\frac{d}{d\alpha}f(A) = \mathrm{tr}\left[\left(\frac{\partial f}{\partial A}\right)^{\top}\frac{dA}{d\alpha}\right]. \tag{A.1}$$

Based on this identity, we verify formula (3.4) for the plastic stress.

Lemma A.1. *If* $\mathcal{E} = \check{\mathcal{E}}((C^{\mathrm{p}})^{-1}C)$, *then*

$$S_{\mathrm{p}} = SC(C^{\mathrm{p}})^{-1} = (C^{\mathrm{p}})^{-1}CS. \tag{A.2}$$

Proof. Let A denote $(C^{\mathrm{p}})^{-1}C$. Imagine that C^{p} is fixed but that C varies smoothly with a parameter α. Then

$$\frac{d}{d\alpha}\check{\mathcal{E}}((C^{\mathrm{p}})^{-1}C) = \mathrm{tr}\left[\left(\frac{\partial \check{\mathcal{E}}}{\partial A}\right)^{\top}(C^{\mathrm{p}})^{-1}\frac{dC}{d\alpha}\right], \tag{A.3}$$

so that $\partial\mathcal{E}/\partial C = (C^{\mathrm{p}})^{-1}\partial\check{\mathcal{E}}/\partial A$. On the other hand, if C is fixed and C^{p} varies with β, then

$$\frac{d}{d\beta}\check{\mathcal{E}}((C^{\mathrm{p}})^{-1}C) = -\mathrm{tr}\left[\left(\frac{\partial \check{\mathcal{E}}}{\partial A}\right)^{\top}(C^{\mathrm{p}})^{-1}\frac{dC^{\mathrm{p}}}{d\beta}(C^{\mathrm{p}})^{-1}C\right]. \tag{A.4}$$

Using the cyclicity of the trace, we see that $\partial\mathcal{E}/\partial C^{\mathrm{p}} = -(C^{\mathrm{p}})^{-1}\left(\partial\check{\mathcal{E}}/\partial A\right)C(C^{\mathrm{p}})^{-1}$. By virtue of Eqs. (2.13) and (2.14), $S_{\mathrm{p}} = SC(C^{\mathrm{p}})^{-1}$. Since S_{p} is symmetric, $S_{\mathrm{p}} = (C^{\mathrm{p}})^{-1}CS$ as well.

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Received December 19, 1995
Revised June 10, 1996