A-PRIORI ANALYSIS OF FINITE ELEMENT SOLUTIONS OF QUASILINEAR ELLIPTIC DIFFERENTIAL EQUATIONS

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Abstract

When the discrete operator (PDE) is not a Fredhölm operator of some index in the discrete framework, the existence and convergence of sequences of finite element approximations to solutions of strongly nonlinear elliptic partial differential equations are not so clear in general. Usually, this is the case when the exact solutions and/or the boundary of the domain are not smooth enough. We present some new results which give existence and convergence of sequences of discrete solutions. Some numerical results are shown and analysed.

Resumo

Quando o operador discreto (EDP) não é um operador de Fredhölm de determinado índice em uma estrutura discreta, a existência e convergência de sequências de aproximações de soluções de equações diferenciais parciais elípticas não-lineares pelo método do elemento finito não são, em geral, muito fáceis de serem determinadas. Este é o caso, por exemplo, quando as soluções exatas e/ou o contorno do domínio não são suaves o suficiente. Neste trabalho, alguns novos resultados são apresentados para situações que não satisfazem as exigências de uma estrutura de Fredhölm. Alguns resultados numéricos são mostrados e analisados.

1. Introduction to the Problem

Let $W^{k,s}$ be the usual Sobolev spaces with the usual norms, $k, s \in \mathcal{R}$, whose functions are defined over a domain $\Omega \subset \mathcal{R}^n$, $n \geq 1$. Consider $\mathcal{K}$ to be the designation for a given closed convex subset of $W^{1,p}$, for some fixed p, $1 < p < \infty$.

τ_h will designate a partition (mesh) of the given domain Ω into simply connected sub-domains (finite elements). We will say that the size of the mesh is

$h > 0$, when the largest diameter of the smallest spheres circumscribing the finite elements is equal to h. Usually, the finite elements formulations do not provide rich enough geometry representation capabilities in order to exactly represent the boundary $\partial\Omega$ of a general domain Ω. Then, the mathematical representation of the geometry provided by a specific finite element formulation being used would actually generate a new domain $\Omega_h \neq \Omega$. Nevertheless, even restricting the class of domains to be considered (for instance, polygonal domains) we will assume that the finite elements formulations we consider gives $\Omega_h = \Omega$ (a trivial case would be when Ω is a polygonal domain and the element formulation is isoparametric linear). Further restrictions on the mesh τ_h will be considered when needed.

The finite element space (discrete space) $S^h(\tau_h, p, \Omega) = \{v_h \in C^0(\Omega) : v_h|_T \in \mathcal{P}_p(T), \forall T \in \tau_h\}$, where $\mathcal{P}_p(T)$ is a suitable space of polynomials of degree less than or equal to p, defined over the elements $T \in \tau_h$. For instance, the finite element formulation we use for the numerical examples employs bi-quadratic finite elements in the plane, and it can deal with element boundary as complex as lines and arches of circles.

Let $\Omega \subset \mathcal{R}^n$ be open and bounded, $n \geq 1$. Consider the following parameterized functional $J_\lambda : W^{1,p} \longrightarrow \mathcal{R}$, $1 < p < \infty$,

$$J_\lambda[u] = \int_\Omega \Phi(\nabla u, u, \lambda, x)dx - \int_\Omega f(x,\lambda).udx - \int_{\Gamma_n} \Psi(u, \lambda, x)dx \qquad (1)$$

where $u : \Omega \longrightarrow \mathcal{R}^N$, $N \geq 1$, $\lambda \in \mathcal{R}^m$, $m \geq 1$, $f : \Omega \times \mathcal{R}^m \longrightarrow \mathcal{R}^N$, and $\Psi : (\mathcal{R}^N \times \mathcal{R}^m \times \Omega) \longrightarrow \mathcal{R}$, $\Phi : (\mathcal{R}^{N\times n} \times \mathcal{R}^N \times \mathcal{R}^m \times \Omega) \longrightarrow \mathcal{R}$. Here $\Gamma_n \subset \partial\Omega$. Also, we define $\Gamma_d = \partial\Omega - \bar{\Gamma}_n$. For what follows we define

$$\mathcal{K} = \{u \in W^{1,p} \mid u(x) = g(x) \ \text{ for all } \quad x \in \Gamma_d\}$$

where $g : \partial\Omega \longrightarrow \mathcal{R}^N$ is a given function defined in the trace space of functions in $W^{1,p}$. Each one of the sets Γ_n and Γ_d is either empty or a union of nontrivial simply connected subsets of $\partial\Omega$. In order to avoid unnecessary difficulties we will always assume that $\Gamma_d \neq \emptyset$.

Let us assume first that

Hyp. 0.1 *: $J_\lambda[.]$ is Gâteaux-differentiable with respect to the function in $W^{1,p}$, for all $\lambda \in \mathcal{R}$.*

□

Thus, we may define the following problem (exact problem):

Pr.0 : find $(u, \lambda) \in (\mathcal{K} \cap W^{1,p}) \times \mathcal{R}^m$, such that u is a stationary point for $J_\lambda[.] : W^{1,p} \longrightarrow \mathcal{R}$ □

or, equivalently,

Pr.1 : Find $(u_0, \lambda_0) \in (\mathcal{K} \times \mathcal{R}^m)$ such that

$$F(u_0, \lambda_0) = 0 \quad \text{ on } \Omega$$

□

where $F(u_0, \lambda_0) = 0$, for $F : W^{1,p} \times \mathcal{R}^m \longrightarrow W^{-1,q}$, $\frac{1}{p} + \frac{1}{q} = 1$, means, in the distributional sense, the boundary value problem:

$$F = \begin{bmatrix} F_0 \\ G_0 \end{bmatrix} \tag{2}$$

$$\begin{aligned} F_0(u_0, \lambda_0) &= -\nabla.[\frac{\partial \Phi}{\partial \nabla u}(\nabla u_0, u_0, \lambda_0, x)] + \frac{\partial \Phi}{\partial u}(\nabla u_0, u_0, \lambda_0, x) \\ &\quad -f(x, \lambda_0) = 0 \end{aligned}$$

$$G_0(u_0, \lambda_0) = \frac{\partial \Phi}{\partial \nabla u}(\nabla u_0, u_0, \lambda_0, x).\mathbf{n}(x) - \frac{\partial \Psi}{\partial u}(u_0, \lambda_0, x) = 0 \quad \text{ on } \Gamma_n$$

$$\partial\Omega = \bar{\Gamma}_d \cup \bar{\Gamma}_n, \qquad \Gamma_d \cap \Gamma_n = \emptyset$$

and $g(x)$ is in the trace space of $W^{1,p}$, as well as $\frac{\partial \Psi(u_0(x), \lambda_0, x)}{\partial u}$ is in the dual of the trace space of $W^{1,p}$ over the boundary Γ_n. The vector $\mathbf{n}$ is the outward normal of the boundary, and $f \in W^{-1,q}$.

Since we are going to deal with a-priori analysis, we will assume that the following hypothesis is satisfied

Hyp. 0.2 : *There exists a nonempty set* $\Lambda \subset \mathcal{R}^m$ *such that, for all* $\lambda_0 \in \Lambda$, $J_{\lambda_0}[.]$ *has at least one stationary point* $u_0(\lambda_0)$ *in some given admissible closed convex set* $\mathcal{K} \subset W^{1,p}$, *i.e.,* $(u_0(\lambda_0), \lambda_0)$ *is a solution of Pr.0 for each* $\lambda_0 \in \Lambda$.

□

Then, it makes sense to define the discrete problem (finite element problem) as

DPr.0 : find $(u_h, \lambda_h) \in (S^h(\tau_h) \cap \mathcal{K}) \times \Lambda_h$, such that u_h is a stationary point for $J_{\lambda_h}[.] : S^h(\tau_h) \longrightarrow \mathcal{R}^m$ □

Where $\Lambda \subset \Lambda_h$. Equivalently, we may cast the following alternative versions to Pr.1 and DPr.0:

Pr.2 : Find $(u_0, \lambda_0) \in (\mathcal{K} \times \mathcal{R}^m)$ such that

$$B_{\lambda_0}(u_0, v) = L_{\lambda_0}(v) \quad \text{for all } v \in W_d^{1,p}$$

□

where $W_d^{1,p} = \{u \in W^{1,p} \mid u(x) = 0 \forall x \in \Gamma_d\}$. We define the parameterized form $B_{(.)} : (W^{1,p} \times W^{-1,q}) \times \mathcal{R}^m \longrightarrow \mathcal{R}$; and the functional $L : (W^{1,p} \times W^{-1,q}) \times \mathcal{R}^m \longrightarrow \mathcal{R}$ by

$$\begin{aligned} B_\lambda(u, v) &= \int_\Omega [\frac{\partial \Phi}{\partial \nabla u}(\nabla u, u, \lambda, x) : \nabla v + \frac{\partial \Phi}{\partial u}(\nabla u, u, \lambda, x).v] dx \\ L_\lambda(u, v) &= \int_\Omega f(\lambda, x).v dx + \int_{\Gamma_n} \frac{\partial \Psi}{\partial u}(u, \lambda, x).v d\Gamma \end{aligned}$$

Now we set the discrete problem as

DPr.1 : Find $(u_h, \lambda_h) \in (S^h(\tau_h, p, \Omega) \cap \mathcal{K}) \times \mathcal{R}^m$ such that

$$B_{\lambda_h}(u_h, v_h) = L_{\lambda_h}(u_h, v_h), \quad \text{for all } v_h \in S^h(\tau_h, p, \Omega) \cap W_d^{1,p}$$

□

We also remark that it is assumed that the Dirichlet boundary condition can be satisfied exactly by functions in $S^h(\tau_h, p, \Omega)$.

Next, we will define other equivalent problems of the preceding ones, but requiring an alternative way of locally parameterizing the solution set. This is important if one wants to eliminate the negative aspects of critical points such as turning points, where it is not possible to define local charts based on the natural parameters λ.

Set

$$F(u,\lambda) = \nabla_u J_\lambda[u], \quad \text{for all } (u,\lambda) \in W_0^{1,p} \times \mathcal{R}^m$$

Let $V : (W_0^{1,p} \times \mathcal{R}^m) \times \mathcal{R}^m \longrightarrow W^{-1,p'} \times \mathcal{R}^m$ be as

$$V(u,\lambda;t) = \begin{bmatrix} F(u,\lambda) \\ g(u,\lambda) - t \end{bmatrix} \tag{3}$$

The function $g : W^{1,p} \times \mathcal{R}^m \longrightarrow \mathcal{R}^m$ represents a change in parameterization for dealing with regular and turning points [1].

Then, for some $s > 2$, as close to 2 as necessary, and for some given isomorphism $K : W_0^{1,s} \times \mathcal{R}^m \longrightarrow W^{-1,s} \times \mathcal{R}^m$, we define $V_h : \tilde{W}_0^{1,s} \times \mathcal{R}^m \times \mathcal{R}^m \longrightarrow \tilde{W}^{1,s} \times \mathcal{R}^m$ as

$$V_h(v_h,\mu;t) = P_h K^{-1} V(v_h,\mu;t) \quad \text{for all } (v_h,\mu) \in \tilde{W}_0^{1,s} \times \mathcal{R}^m \tag{4}$$

where

$$\tilde{W}_0^{1,s} \equiv S^h;$$

$$\tilde{W}^{-1,s} \equiv KS^h;$$

$$\|.\|_{\tilde{W}^{1,s}} = h^\gamma \|.\|_{W^{1,s}};$$

$$\|.\|_{\tilde{W}^{-1,s}} = h^\gamma \|.\|_{W^{-1,s}},$$

$$\gamma = -\frac{2}{s}.$$

and $P_h : H_0^1 \times \mathcal{R}^m \longrightarrow H^{-1} \times \mathcal{R}^m$ is a suitably defined projection, which is dependent on the operator K and is bounded as an operator from $W_0^{1,s}$ into itself, for some fixed $s > 2$.

Pr.3 : Find $(u,\lambda;t) \in (\mathcal{K} \times \mathcal{R}^m) \times \mathcal{R}^m$ such that

$$V(u,\lambda;t) = 0$$

□

DPr.2 : With $s > 2$ fixed, find $(u, \lambda; t) \in (\tilde{W}^{1,s} \cap \mathcal{K}) \times \mathcal{R}^m) \times \mathcal{R}^m$ such that

$$V_h(u, \lambda; t) = 0$$

□

2. Comments on the Problem

This work is intended to analyse certain aspects of the existence, uniqueness and convergence of sequences of finite element solutions to problem DPr.0 (or its variations). Such a problem has been the subject of several previous works [1], [2], [3], [4], [5], [6], [7], [8], but many important and basic questions still remain open. The class of problems we are dealing with is used in the mathematical formulation of several applied problems in many different areas, e.g., elasticity, elasto-plasticity, chemical reactions, thermal analysis of fluids and solids, etc.

Since the solution set of Pr.1 can be very complex [9], it is important to state the subset of the set we are considering. However, this question demands some elaboration to be answered properly, and we are not going into the details of it. For a more complete consideration we refer to [11]. Here we just state that we are considering the subsets of regular and simple turning points. So, we will say that $\mathcal{M}_0$ is the solution set of Pr.1 and $\mathcal{M}_r$ and $\mathcal{M}_{st}$ are the set of regular points and the set of simple turning poins of Pr.1, respectively.

When the solution and the operator are smooth enough, it is usually possible to cast the problem Pr.1 within a suitable space framework, such that the operator $F(.,..)$ is a Fredhölm operator of some index. Several previous works have used that structure as the center piece of the development of the theory for the numerical analysis of such problems as we have stated in the previous section. However, such smoothness is frequently an unrealistic requirement. For instance, if the PDE was not of the semi-linear type, then strong smoothness is to be required for the solution points (e. g., $W^{2,s}$ $s > 2$) and for the domain ($C^{1,\alpha}$, $\alpha > 0$, or local convexity at corners). It is possible to observe that the Fredhölm structure requires such restrictions on the problem [11]. So, we are left with the problem of defining the appropriate considerations in order to take the smoothness of the solution, of the domain and of the operator into consideration in a separate manner. It is important to point out that unsmoothness of the solution set and of the boundary of the domain is an important issue in engineering, where that aspect is common place.

In this work we present two different main results related to the existence and convergence of sequences of solutions to DPr.1 when an exact solution point is assumed to exist. We are not considering the continuation properties for those solutions points. The first result requires very little from the smoothness of the operator, of the solution set and of the domain, but it considers only solution points which are strict minima of the functional $J_\lambda[.]$, for some $\lambda \in \mathcal{R}^m$. The second result concerns stronger requirements over the smoothness of the solution set, of the operator and of the boundary of the domain. Neither of these results imply the Fredhölm structure.

3. Main Results

In this section we will show the main results of this work, which concern two different aspects of the problem described above. We are not going to present any proof for them, since they are very long and technical. Nevertheless, the interested reader may find them in [11]. We start this section by stating results concerning the equivalence between the problems DPr.2 and Pr.3 and DPr.0 and Pr.0, respectively.

Lemma. 1 *: $V_h(u, \lambda; t) = 0$ if and only if $(u, \lambda) \in S^h \times \mathcal{R}^m$ satisfies $< F(u, \lambda), v_h >= 0$ for all $v_h \in S^h$, (which means that (u, λ) solves DPr.0); and $g(u, \lambda) = t$.*

$\square$

Lemma. 2 *: $V(u, \lambda; t) = 0$ if and only if (u, λ) solves problem Pr.0 and $g(u, \lambda) = t$.*

$\square$

Next, we provide a sequence of definitions and a sequence of two theorems, one lemma and one corollary, which together compose the main results.

Def. 0.1 (Palais-Smale sequences) *: Let $J : W^{1,p} \longrightarrow \mathcal{R}$ be Gâteaux-differentiable, and let a sequence $\{u_k\} \subset W^{1,p}$ be defined such that*

$$|J[u_k]| < C \neq C(k)$$

for some $C < \infty$, and for all k's; and

$$\|\nabla_u J[u_k]\|_{W^{1,p}} \longrightarrow 0$$

as $k \longrightarrow 0$. Then $\{u_k\}$ is called a Palais-Smale sequence.

$\square$

Def. 0.2 (Palais-Smale condition) *: Let $\{u_k\} \subset W^{1,p}$ be a Palais-Smale sequence for some given functional $J : W^{1,p} \longrightarrow \mathcal{R}$. $\{u_k\}$ is said to satisfy the Palais-Smale condition (PS) if and only if it is possible to extract a subsequence of $\{u_k\}$ which is strongly convergent in $W^{1,p}$ (PS sequences are sequentially compact).*

$\square$

Def. 0.3 (Class-I) *: Let $J_\lambda : W^{1,p} \longrightarrow \mathcal{R}$ be a parameterized functional, for which $\lambda \in \mathcal{R}^m$, $m \geq 0$. Then, J_λ is said to be in the class-I if and only if*

i) It is Fréchet-differentiable in $W^{1,p} \times \mathcal{R}^m$;

ii) It satisfies the P-S condition.

$\square$

Def. 0.4 *We define $\Omega \in$ class $\mathcal{D}^q$, $2 < q < \infty$, if the equation*

$$\Delta u = \nabla . f$$

has a unique solution $u \in W_0^{1,q}$ *for every vector valued function in* $[L^q]^n$, $\Omega \in \mathcal{R}^n$, *and*

$$\|u\|_{W^{1,q}} \leq K_q \||f|\|_{L^q}$$

holds for a constant K_q *independent of* f.

$\square$

Theorem. 3 : *Let* $J_{(..)}[.] : W_0^{1,p} \times \mathcal{R}^m \longrightarrow \mathcal{R}$, $1 < p < \infty$, *be a functional of class-I type. Suppose* $\Omega \in \mathcal{D}^t$ $(t = \min\{p, p'\})$. *Let* $u_0 \in W_0^{1,p}$ *be a local strict minimum point of* $J_{\lambda_0}[.] : W_0^{1,p} \longrightarrow \mathcal{R}$, *for some* $\lambda_0 \in \mathcal{R}^m$. *Then, there exists* $h_0 > 0$, *such that, for each* $0 < h \leq h_0$, *there exists at least one solution* $u_h \in S^h(\tau_h)$, *which solves DPr.0, and is a local minimum point for* $J_{\lambda_0}[.]$ *over* $S^h(\tau_h)$.

Furthermore,

$$\eta(\|u_h - u_0\|_{W^{1,p}}) \leq \inf_{w_h \in B_\varepsilon(u_0) \cap S^h} \{J_{\lambda_0}[w_h] - J_{\lambda_0}[u_0]\}$$

and

$$\eta'(\|u_h - u_0\|_{W^{1,p}}) \leq \|\nabla_u J_{\lambda_0}[u_h]\|_{W^{-1,p'}}.$$

If $J_{(..)}[.]$ *satisfies (G.iii) and (H.iii) (see below), then the above estimates can have the form*

$$\eta(\|u_h - u_0\|_{W^{1,p}}) \leq C \inf_{w_h \in B_\varepsilon(u_0) \cap S^h} \{\|w_h - u_0\|_{W^{1,p}}^{1+\beta}\}$$

where η, η' : $\mathcal{R}^+ \longrightarrow \mathcal{R}^+$ *are strictly increasing continuous functions, and* $\eta(0) = 0 = \eta'(0)$, $C = C(\|u_0\|_{W^{1,p}}, |\lambda_0|)$, *and* $0 < \beta \leq 1$.

$\square$

The class of functionals will be further restricted into the following category:

$$J_{(..)}[.] : W_0^{1,p} \times \mathcal{R}^m \longrightarrow \mathcal{R}$$

$$J_\lambda[u] = G_\lambda[u] - H_\lambda[u] - L_\lambda[u] \tag{5}$$

where $G_{(..)}[.]$ satisfies

G.i) It is Fréchet-differentiable in $W_0^{1,p} \times \mathcal{R}^m$;

G.ii) $< \nabla_u G_\lambda[u_1] - \nabla_u G_\lambda[u_2], u_1 - u_2 > \geq \alpha_0 \|u_1 - u_2\|_{W^{1,p}}^{\theta_0}$, for all u_1, $u_2 \in W_0^{1,p}$ where α_0 may be a decreasing function of the norms of u_1 and u_2 in $W_0^{1,p}$ and λ; furthermore, $1 < \theta \leq \max\{p, 2\}$ and $\lambda \in \mathcal{R}^m$;

G.iii) $\|\nabla_u G_\lambda[u_1] - \nabla_u G_\lambda[u_2]\|_{W^{-1,q}} \leq \alpha_1 \|u_1 - u_2\|_{W^{1,p}}^{\theta_1}$, for some $0 < \theta_1 \leq \min\{1, p-1\}$; for all u_1, $u_2 \in W_0^{1,p}$; and where α_1 may be an increasing function of the norms of u_1 and u_2 in $W_0^{1,p}$;

G.iv) $G_\lambda[0] = 0$ and $\nabla_u G_\lambda[0] = 0$ for all $\lambda \in \mathcal{R}^m$;

G.v) It is two times Fréchet-differentiable in $W^{1,\infty} \times \mathcal{R}^m$, and

$$\|\nabla_u^2 G_\lambda[u + \varphi] - \nabla_u^2 G_\lambda[u]\|_{\mathcal{L}(W_0^{1,s}, W^{-1,s})} \leq C_0 \|\varphi\|_{W^{1,\infty}}^{\gamma_0}$$

where $s \in (1, \infty)$ is any number; $C_0 = C_0(\|u\|_{W^{1,\infty}}, \|\varphi\|_{W^{1,\infty}}, |\lambda|)$ is an increasing function of its arguments; $u, \varphi \in W^{1,\infty}$, $\lambda \in \mathcal{R}^m$; and $1 \geq \gamma_0 > 0$. Furthermore, the coefficients of the linearized operator are determined by the regularity of the first derivative of its argument; and $1 \geq \gamma_0 > 0$;

G.vi) $< \nabla_u^2 G_\lambda[u]\varphi, \varphi > \geq \beta_0 \|\varphi\|_{H^1}^2$, for all $u \in W^{1,\infty}$, $\varphi \in H^1$, and where $\beta_0 > 0$ may depend decreasingly on the norm of the function u in the norm of $W_0^{1,\infty}$;

G.vii) $G_{(..)}[u] : \mathcal{R}^m \longrightarrow \mathcal{R}$, $\nabla_u G_{(..)}[u] : \mathcal{R}^m \longrightarrow W^{-1,q}$, $\nabla_u^2 G_{(..)}[u] : \mathcal{R}^m \longrightarrow \mathcal{L}(W^{1,p}, W^{-1,q}$, are at least two times continuously differentiable for all (u, λ) in bounded sets of $W_0^{1,\infty} \times \mathcal{R}^m$;

and $H_{(..)}[.]$ satisfies:

H.i) It is Fréchet-differentiable in $W_0^{1,p} \times \mathcal{R}^m$;

H.ii) $\nabla_u H_\lambda[.] : W_0^{1,p} \longrightarrow W^{-1,q}$, and $\nabla_u H_\lambda[.] : C^{1,\alpha} \longrightarrow W_\alpha$, $0 < \alpha < 1$, are compact for all $\lambda \in \mathcal{R}^m$;

H.iii) $\|\nabla_u H_\lambda[u_1] - \nabla_u H_\lambda[u_2]\|_{W^{-1,q}} \leq C_1\|u_1 - u_2\|^{\gamma_2}_{W^{1,p}}$, for some $0 < \gamma_2 \leq 1$; for all u_1, $u_2 \in W_0^{1,p}$; and where C_1 may be an increasing function of the norms of u_1 and u_2 in $W_0^{1,p}$ and of $|\lambda|$, $\lambda \in \mathcal{R}^m$;

H.iv) $H_\lambda[0] = 0$ and $\nabla_u H_\lambda[0] = 0$ for all $\lambda \in \mathcal{R}^m$;

H.v) It is two times Fréchet-differentiable in $W^{1,\infty} \times \mathcal{R}^m$, and

$$\|\nabla_u^2 H_\lambda[u+\varphi] - \nabla_u^2 H_\lambda[u]\|_{\mathcal{L}(W_0^{1,s}, W^{-1,s})} \leq C_0\|\varphi\|^{\gamma_1}_{W^{1,\infty}}$$

where $s \in (1,\infty)$ is any number; $C_0 = C_0(\|u\|_{W^{1,\infty}}, \|\varphi\|_{W^{1,\infty}}, |\lambda|)$ is an increasing function of its arguments; $u, \varphi \in W^{1,\infty}$, $\lambda \in \mathcal{R}^m$; and $1 \geq \gamma_1 > 0$. Furthermore, the regularity of the coefficients of the linearized operator is determined by the regularity of the first derivative of its argument;

H.vi) $< \nabla_u^2 H_\lambda[u]\varphi, \varphi > \geq C_2\|\varphi\|^2_{L^2}$, and $\|\nabla_u^2 H_\lambda[u]\varphi\|_{L^2} \leq C_3\|\varphi\|_{L^2}$, for all $u \in W^{1,\infty}$,$\varphi \in L^2$, where C_2 and C_3 may be a decreasing, respectively increasing, function of $\|u\|_{W^{1,\infty}}$ and $|\lambda|$;

H.vii) $H_{(..)}[u] : \mathcal{R}^m \longrightarrow \mathcal{R}$, $\nabla_u H_{(..)}[u] : \mathcal{R}^m \longrightarrow W^{-1,q}$, $\nabla_u^2 H_{(..)}[u] : \mathcal{R}^m \longrightarrow \mathcal{L}(W^{1,p}, W^{-1,q})$, are at least two times continuously differentiable for all (u,λ) in bounded sets of $W_0^{1,\infty} \times \mathcal{R}^m$; furthermore, $\frac{\partial^\alpha \nabla_u H_{(\lambda)}[.]}{\partial \lambda^\alpha} : W^{1,p} \longrightarrow W^{-1,q}$ is a nonlinear compact operator and $\frac{\partial^\alpha \nabla_u^2 H_{(\lambda)}[u]}{\partial \lambda^\alpha} : H_0^1 \longrightarrow H_0^1$ is a linear compact operator; $|\alpha| = 0, 1, 2$;

and $L_{(..)[.]}$ satisfies

L.i) $L_{(..)}[.] : W_0^{1,p} \longrightarrow \mathcal{R}$ is a bounded linear functional for all $\lambda \in \mathcal{R}^m$, and is denoted by

$$L_\lambda[u] = < f, u > \quad \text{for all } u \in W_0^{1,p}$$

where $f \in W^{-1,p'}$ is as smooth as necessary.

Lemma. 4 *: Let the projection $P_h : W_0^{1,s} \longrightarrow S^h$ be defined based on the bilinear form*

$$B(u,v) = < \nabla_u^2 G_{\lambda_0}[u_0]u, v >, \quad \textit{for all } u \in W_0^{1,s} \textit{ and } v \in W_0^{1,s'}$$

where $\lambda_0 \in \mathcal{R}^m$, $u_0 \in W_0^{1+r,\bar{s}}$, $r > \frac{2}{\bar{s}}$, $\bar{s} > 2$. *Then,* $P_h : W_0^{1,s} \longrightarrow W_0^{1,s}$ *is a bounded operator for all* s *close enough to* 2.

□

Theorem. 5 *: Let* $J_{(.,.)}[.] : W_0^{1,p} \times \mathcal{R}^m \longrightarrow \mathcal{R}$ *satisfy (G.i-vii), (H.i-vii) (but not H.vi) and (L.i). Let* $(u_0, \lambda_0) \in \mathcal{M}_r \cup \mathcal{M}_{st}$ *be given, and assume that* $\mathcal{M}_0 \subset W_0^{1+r,\bar{s}} \times \mathcal{R}^m$, *where* $r > \frac{2}{\bar{s}}$, $\bar{s} > 2$. *Let the domain* $\Omega \in \mathcal{D}^t$, *for some* $2 < t < \infty$, *and let* $\bar{s}$ *be as close to* 2 *as needed.*

Then, for any given but fixed q, $0 < q < 1$, *there exists* $h_0 > 0$, *and* $\delta > 0$, *such that, for all* $0 < h \leq h_0$, *there exists a unique* $(u_h, \lambda_h) \in S^h \times \mathcal{R}^m \in B_\delta(P_h u_0, \lambda_0)$, *which solves DPr.0 (the discrete problem). Furthermore, we have the following two-sided estimates*

$$
\begin{aligned}
& \frac{\|[D_{(u,\lambda)}V_h(P_h u_0, \lambda_0; t_0)]^{-1} V_h(P_h u_0, \lambda_0; t_0)\|_{W^{1,s}\times\mathcal{R}^m}}{1+q} \leq \\
\leq \quad & (\|u_h - P_h u_0\|_{W^{1,s}} + |\lambda_h - \lambda_0|) \leq \\
\leq \quad & \frac{\|[D_{(u,\lambda)}V_h(P_h u_0, \lambda_0; t_0)]^{-1} V_h(P_h u_0, \lambda_0; t_0)\|_{W^{1,s}\times\mathcal{R}^m}}{1-q}.
\end{aligned}
$$

for all $s \geq \bar{s}$ close enough to 2.

□

Corollary. 6 *Let the hypothesis of the Theorem* **??** *be true. Then,*

$$\|u_h - u_0\|_{W_0^{1,s}} + |\lambda_h - \lambda_0| \leq C h^{r + \frac{2}{s} - \frac{2}{\bar{s}}}$$

for all $s \in [\bar{s}, \infty)$ *and*

$$\|u_h - u_0\|_{W^{1,\infty}} + |\lambda_h - \lambda_0| h^\gamma \leq C h^{r - \frac{2}{\bar{s}}}$$

for all $h > 0$ *small enough. Here* $C = C(\|u_0\|_{W_0^{1+r,\bar{s}}}, |\lambda_0|)$.

□

OBS.: 1. Theorem **??** is important in the sense that very little is required about the functional and the exact solution point besides that it is a strict minimum point for $J_\lambda[.]$. It does not provide uniqueness for the discrete solution, but it shows that any discrete sequence converges when $h \longrightarrow 0$. This is substantial, since there are algorithms built to choose one unique minimizing sequence from the set of all possible minimizing sequences. It is interesting to note that Theorem **??** can be specialized to functionals whose gradient are monotone coercive operators and be compared with the results obtained in [9]. This will give interesting interpretations for the functions $\eta(.)$ and $\eta'(.)$.

2. Theorem **??** provides a way of thinking separately about the effect of the regularity of the solution points, of the boundary of the domain and of the operator, and the role each one of these aspects plays in the existence issue for the discrete solution.

□

Numerical Examples

In this section we present an example of strongly nonlinear problem (i.e., the growth of the functional can not be located between two quadratic functionals). It has a smooth solution set but the boundary of the domain is unsmooth, and then it can not be cast into a Fredhölm structure. The program NFEARS was used, which means that the finite element space is the space defined by bi-quadratic finite elements with possible irregular nodes for graded meshes.

The basic functional is:

$$J[u] = \int_\Omega [\frac{(1+|\nabla u|^2)^2}{4} - \mu\frac{u^3}{3} - f_{(\lambda,\mu)}u]d\Omega - \int_{\Gamma_n} g_\lambda u d\gamma$$

where the domain is the L-shaped region as in Fig.1. We will refer to the problems by the letter L (for L-shaped region). The convex set where we will seek the solution set for the stationarity condition of the above functional will depend on the desired smoothness and on the domain; and so will the definition of Γ_n, the part of the boundary where we define Neumann boundary conditions. The problem depends on the vector of real parameters (μ,λ), and

$$f_{(\mu,\lambda)}(r,\theta,\alpha) = -2\lambda^3\alpha^3(\alpha-1)r^{3\alpha-4}\sin(\alpha\theta) - \mu\lambda^2 r^{2\alpha}\sin^2(\alpha\theta)$$

and

$$g_\lambda(r,\theta,\alpha) = (1+\lambda^2\alpha^2 r^{2(\alpha-1)})\lambda\alpha r^{\alpha-1}\{n_1\sin[\theta(\alpha-1)] + n_2\cos[\theta(\alpha-1)]\}$$

on Γ_n, where $(n_1, n_2) = (n_1(r,\theta), n_2(r,\theta))$ is the normal to the boundary $\partial\Omega$ at (r,θ). The parameter α is fixed, in order to define the smoothness and boundary conditions of the problem. As usual, (r,θ) is the representation of a position in $\mathcal{R}^2$ in polar coordinates. We will set $(\mu,\lambda) = (1.0, 0.01)$, even though some comments will be necessary in some cases.

It can be proved that the above functional satisfies the conditions (G.i-vii); (H.i-vii) (with the exception of H.vi) [11], where

$$G_{(\mu,\lambda)}[u] = \int_\Omega \frac{(1+|\nabla u|^2)^2}{4} d\Omega$$

$$H_{(\mu,\lambda)}[u] = \mu \int_\Omega \frac{u^3}{3} d\Omega$$

and

$$L_{(\mu,\lambda)}(u) = \int_\Omega f_{(\mu,\lambda)} u d\Omega + \int_{\Gamma_n} g_\lambda u d\Gamma,$$

for all $u \in W^{1,4}$.

Let us set $\alpha = \frac{8}{3}$. A branch of smooth exact solution points is

$$u_0((r,\theta),(\lambda,\mu)) = \lambda r^{\alpha} \sin(\alpha\theta)$$

for all $(\lambda,\mu) \in \mathcal{R}^2$. The solution points are in $W^{\frac{11}{3},2}$, which means that they are as smooth as we may want. However, the domain is not smooth, and the Fredhölm setting breaks down. Here, the convex set where we look for solutions is

$$\mathcal{K} = \{u \in W^{1,4} \mid u(r,\theta) = 0, \quad \text{for all} \quad (r,\theta) = (r,0), \text{ and } (r,\theta) = (r,\tfrac{3\pi}{2}),$$
$$0 < r < 1\}$$

Remark : Actually, this problem has been classified as unsmooth, since we can not use the Fredhölm structure, which means that we do not know for sure if one can extend the solution in a continuous way. Another important remark is related to the smoothness of the domain as required in Theorem **??**, that is, we should have $\Omega \in \mathcal{D}^s$, for some $s > 2$. Checking the Definition ?? we observe that it is difficult, sometimes, to have a finer control of the domain smoothness by using that definition. However, we know that that is true for some $s > 2$ (very close to 2, but still bigger than 2) [11]. In Theorem ?? we should have a constant $2 < \bar{s} < s$, such that $r > \frac{2}{\bar{s}}$, where, in the present case, r can be any number such that $r < \frac{5}{3} + \frac{2}{\bar{s}}$, which means that we are in good shape. Hence, we conclude that, for this case, the importance of the unsmoothness of the domain was damped out by the high smoothness of the solution points.

$\square$

The main tools to analyze this problem are Theorem ?? and Corollary ??, in what concerns a-priori estimates. Following Corollary ?? the rates of convergence should be the best possible, that is, 1 with respect to the norms of the spaces H^1, $W^{1,4}$, because $u_0 \in W^{3,s}$, for all $1 < s < 6$. So, we can not expect to have the best possible rate of convergence with respect to $W^{1,\infty}$, but, as a consequence of Theorem ??, we may expect, as in Corollary ??, to have a rate not worse than $\frac{5}{6}$, because $u_0 \in W^{2+\frac{5}{6},\bar{s}}$, for all $\bar{s} > 1$, as big as we wish.

Fig. 2 and Table 1 present the result for the true relative error for a sequence of uniformly refined meshes. The observed rates of convergence for the two last meshes are ≈ 1.0, ≈ 1.0, and 0.84, for the norms of H^1, $W^{1,4}$ and $W^{1,\infty}$, respectively.

mesh	d.o.f.	L^∞	$W^{1,\infty}$	L^p	H^1	$W^{1,p}$
1	56	0.26414E-02	0.12247E-01	0.30561E-02	0.15225E-01	0.15996E-01
2	208	0.38320E-03	0.36767E-02	0.39196E-03	0.38316E-02	0.37673E-02
3	800	0.58052E-04	0.11311E-02	0.49790E-04	0.96061E-03	0.92049E-03
4	3136	0.89700E-05	0.35209E-03	0.62854E-05	0.24043E-03	0.22759E-03
5	12416	0.13994E-05	0.11025E-03	0.79045E-06	0.60137E-04	0.56542E-04

Table 1: True relative error - Strongly nonlinear smooth - L-uniform - $\alpha = \frac{8}{3}$

More numerical examples may be seen in [11].

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