IMPLICIT HAMILTON EQUATIONS

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Abstract

A special class of first order differential equations not solved for the derivative, implicit Hamilton equations, is defined as a lagrangian submanifold ot the tangent space of a symplectic manifold, and their typical phase portraits around singularities are studied.

For the one degree of freedom case their generic normal forms are obtained, related to caustics of two dimensional lagrangian submanifolds and to 1-parameter perestroikas of one dimensional wave fronts.

A similar study for one parameter families of these equations is also presented: the generic normal forms are then related to 1-parameter perestroikas of caustics and to 2-parameter perestroikas of one dimensional wave fronts.

Resumo

É definida uma classe de equações diferenciais de primeira ordem não resolvidas em ordem à derivada, equações de Hamilton implícitas, como subvariedades lagrangeanas do espaço tangente de uma variedade simpléctica, e os seus retratos de fase típicos são estudados na vizinhança de singularidades.

São obtidas as formas normais genéricas para problemas com um grau de liberdade, relacionadas com cáusticas de variedades lagrangeanas de dimensão dois e com perestroikas de frentes de onda de dimensão um, dependendo de um parâmetro.

Um estudo semelhante para famílias a um parâmetro destas equações é também apresentado: as formas normais genéricas estão relacionadas com perestroikas de cáusticas, dependendo de um parâmetro, e com perestroikas de frentes de onda de dimensão um, dependendo de dois parâmetros.

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An implicit Hamilton equation is a geometric generalization of a Hamiltonian vector field; it appears in a very natural way as the result of converting Euler equations to their corresponding Hamiltonian form, when the lagrangian is not regular.

They can be viewed as differential equations not solved for the derivative, but in the framework of [4] for studying normal forms, the generic (in the context of all implicit differential equations) assumption made there, namely that the exponent of the critical points is well defined and different from ± 1, is violated: for hamiltonian vector fields the exponent is one for saddles, and is not defined for a centre. The exponent of a nondegenerate critical point of a direction field is defined to be the ratio of the eigenvalue of largest modulus of the linearization of a vector field spanning the direction field to the smallest, for saddles and nodes, and as the modulus of the ratio of the imaginary part to the real, for a focus.

To obtain normal forms and typical phase portraits for the implicit Hamilton equations a different framework is needed.

This is a report on work still in progress, announcing the main results with sketches of their proofs; a more rigorous and detailed account will be published elsewhere.

1. Basic results and definitions

A symplectic manifold is an even dimensional manifold M with a non degenerated closed 2-form ω. As in every case here only the local situation is relevant, M will always be $T^*\mathbf{R}^n$ or $T\mathbf{R}^n$, identified with $\mathbf{R}^{2n}$ with coordinates (x,p) or $(x,\dot{x})$ respectively.

In the situations to be considered, there is a standard way of defining a symplectic form on $T\mathbf{R}^n$ from the one on the cotangent space: assume $M = T^*\mathbf{R}^n$, with coordinates (x,y) and standard symplectic form $\omega_M = dx \wedge dy$; denoting the corresponding coordinates on $N = T^*M = \mathbf{R}^{4n}$ by $((x,y),(p,q))$, the standard symplectic form on N is given by $\omega_N = dp \wedge dx + dq \wedge dy$. Then ω_M

induces a diffeomorphism Φ from TM to T^*M defined by $v = ((x, y), (\dot{x}, \dot{y})) \mapsto ((x, y), (\dot{y}, -\dot{x})) = \omega_M(\,.\,, v)$, and the standard form on TM will be $\omega = d\dot{y} \wedge dx - d\dot{x} \wedge dy$, the pull-back by Φ of the standard form ω_N on $N = T^*M$.

In general, the tangent space of a symplectic manifold M is a new symplectic manifold with a standard 2-form induced by the original symplectic form on M.

A lagrangian submanifold l of (M^{2n}, ω) is a n-dimensional submanifold of M such that the pullback of ω to l by the inclusion map is identically zero, $\omega | l \equiv 0$.

A 1-form α on an odd dimensional M is a contact form if the restriction $d\alpha_x | \{\alpha_x = 0\}$ of $d\alpha$ to the hyperplane $\alpha_x = 0$ is non degenerate for every $x \in M$; a contact structure on M is a family of hyperplanes $H_x \in T_xM$ which can be defined locally by $H_x = \{\alpha_x = 0\}$. In what follows only the local situation is relevant, therefore it will be always assumed that the contact structure is defined by a contact form.

A legendrean submanifold l_e of the contact manifold (M^{2n+1}, α) is a n-dimensional submanifold of M such that the pullback of α to l_e by the inclusion map is identically zero, $\alpha | l_e \equiv 0$.

2. Implicit Hamilton equations

A hamiltonian vector field with hamiltonian H on M is given by

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}.$$

Its graph in TM is a lagrangian submanifold, as the graph of dH is a lagrangian submanifold of T^*M. Thus lagrangian submanifolds of the tangent space generalize hamiltonian vector fields, and the following definition is natural:

An implicit Hamilton equation on a symplectic manifold M is a lagrangian submanifold of its tangent space TM.

The criminant $\mathcal{C}$ of l, or of the implicit Hamilton equation, is the set of critical points for its projection on the base space, and the discriminant $\mathcal{D}$ or caustic is the set of the critical values, the projection of $\mathcal{C}$.

The pull-back to l of the form $\theta = \dot{p}dx - \dot{x}dp$ on $T(T^*N)$ is closed, therefore locally is the differential of a function. When l is a surface (dim l=2) it defines a direction field on l, with singularities at the points where the tangent space of l is contained in $\theta = 0$; these points always belong to the criminant curve of l.

The phase portrait of an implicit Hamilton equation is the projection of the integral curves of that direction field on l, together with the caustic, where all relevant singularities are located: away from the caustic the phase portrait is just the union of integral curves of Hamiltonian vector fields.

The singularities of the phase portrait appear in two ways: as projections of singularities of the direction field in the criminant curve, where the tangent space of l is contained in $\theta = 0$, and as images of critical points of the projection restricted to l.

The phase portraits of two implicit Hamilton equations on $M = T^*N$ are said to be equivalent if there exists a diffeomorphism of M taking one into the other; a similar definition can be made for germs.

3. Hamiltonian form of Euler equations

The main example arises when converting Euler equations to its hamiltonian equivalent: it is usually assumed that the lagrangian $L : TN \longrightarrow \mathbf{R}$ is hyper-regular, i.e. the fibre derivative induces a diffeomorphism $\mathcal{L}$ of the tangent and cotangent spaces of N:

$$(x, \dot{x}) \mapsto (x, p) = \left(x, \frac{\partial L}{\partial \dot{x}}(x, \dot{x})\right).$$

Under these assumptions, to the Euler second order differential equations on N

$$\frac{\partial^2 L}{\partial \dot{x}^2}(x, \dot{x})\ddot{x} + \frac{\partial^2 L}{\partial x \partial \dot{x}}(x, \dot{x})\dot{x} = \frac{\partial L}{\partial x}$$

correspond the Hamilton first order equations on the cotangent space:

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}$$

where H is the Legendre transform of L, defined by:

$$H(x,p) = p\dot{x} - L(x,\dot{x}), \quad (x,\dot{x}) = \mathcal{L}^{-1}(x,p).$$

In a more geometric way, the image of the map $i : TN \longrightarrow T(T^*N)$ defined by:

$$(x,\dot{x}) \mapsto \left(x, \frac{\partial L}{\partial \dot{x}}(x,\dot{x}), \dot{x}, \frac{\partial L}{\partial x}(x,\dot{x})\right)$$

is a lagrangian submanifold l of $T(T^*N)$, the graph of the Hamilton vector field. Then:

$$\theta | l = -dh, \quad h(x,\dot{x}) = \dot{x}\frac{\partial L}{\partial \dot{x}} - L(x,\dot{x}).$$

A point $(x,\dot{x}) \in TN$ is a singular point of the Euler equation if at that point

$$\det \frac{\partial^2 L}{\partial \dot{x}^2}(x,\dot{x}) = 0$$

and therefore the equation cannot be solved for the higher order derivative. In particular, the usual existence and uniqueness theorems for ordinary differential equations do not apply if the initial condition is a singular point [3].

The map $\mathcal{L}$ is not a diffeomorphism at a singular point and consequently the corresponding hamiltonian is not well defined. Then l does not correspond to a vector field: around the image of the singular points, the submanifold l does not project diffeomorphically onto the base $M = T^*N$.

4. Caustics and normal forms

A map $F : \mathbf{R}^k \times \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}$ is the generating family of a lagrangian submanifold l of TM, where $M = T^*N$ and $N = \mathbf{R}^n$, if:

$$l = \left\{(x,p,\dot{x},\dot{p}) : \ \dot{x} = \frac{\partial F}{\partial p}(\lambda,x,p), \dot{p} = -\frac{\partial F}{\partial x}(\lambda,x,p), \frac{\partial F}{\partial \lambda}(\lambda,x,p) = 0\right\}.$$

If the lagrangian submanifold l corresponds to a Hamiltonian vector field, the generating family is just the Hamiltonian of that vector field and $k = 0$.

Two generating families F_1 and F_2 are R^+-equivalent if there exists a function $S : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}$ and a change of coordinates $(\lambda, x, p) \mapsto (\Lambda(\lambda, x, p),$

$X(x,p), P(x,p))$ such that:

$$F_1(\lambda, x, p) = F_2(\Lambda(\lambda, x, p), X(x,p), P(x,p)) + S(x,p)$$

and R-equivalent if in addition $S \equiv 0$. They are stably equivalent if they become (R, R^+)-equivalent after adding quadratic forms Q_i on new variables λ:

$$F_1(\lambda_1, \dots, \lambda_k, x, p) + Q_1(\lambda_{k+1}, \dots, \lambda_s) \sim F_2(\lambda_1, \dots, \lambda_l, x, p) + Q_2(\lambda_{l+1}, \dots, \lambda_s).$$

A lagrangian map $l \longrightarrow M$ is the composition of the embedding of the lagrangian submanifold l in TM with the projection $\pi : TM \longrightarrow M$; a caustic is the set of critical values of a lagrangian map.

Two lagrangian maps (germs) are Lagrange equivalent if there exist a map (germ) $S : M \longrightarrow \mathbf{R}$ and (the germ of) a diffeomorphism $g : M \longrightarrow M$ such that the (germ of a) diffeomorphism $g_* + \pi^* dS : TM \longrightarrow TM$ transforms one lagrangian submanifold into the other. They are strictly equivalent if $S \equiv 0$. Unless otherwise stated, all germs are taken at the origin.

Theorem 1 (**[1]**). *All lagrangian submanifolds l can be constructed from a generating family, and two lagrangian germs are (lagrangian, strictly) equivalent if the germs of their generating families are stably (R^+, R)-equivalent. If the dimension of l is two, the germs of the generic lagrangian maps are Lagrange equivalent to the germs of the projection on M of the lagrangian submanifolds defined by the generating families:*

- $F(\lambda, x, p) = \lambda^3 + p\lambda$
- $F(\lambda, x, p) = \pm\lambda^4 + x\lambda^2 + p\lambda$

with $\lambda \in \mathbf{R}$.

Given two implicit Hamilton equations, if the generating families F_1 and F_2 of the corresponding lagrangian submanifolds are R-equivalent, their phase portraits are equivalent, but that is not necessary, F_1 can be just R-equivalent to a multiple of F_2, for instance: only a difference in the parametrization of the integral curves is involved.

From theorem 1 it follows that then the possible generating families of a generic (as will always be assumed) implicit Hamilton equation are:

- $F(\lambda, x, p) = \lambda^3 + p\lambda + S(x, p)$
- $F(\lambda, x, p) = \pm\lambda^4 + x\lambda^2 + p\lambda + S(x, p)$

for some smooth function S on $M = T^*N$.

The origin is a fold point of an implicit Hamilton equation if it is a regular point of the caustic and is not the projection of a singular point of the direction field on l; the corresponding generating family is:

$$F(\lambda, x, p) = \lambda^3 + p\lambda + S(x, p), \quad \frac{\partial S}{\partial x}(0, 0) \neq 0.$$

Since $\lambda^3 + p\lambda + x$ is a R-miniversal unfolding of λ^3 [1], it is possible to find an admissible change of coordinates such that F becomes:

$$F(\lambda, x, p) = \lambda^3 + p\lambda + x.$$

In general the condition on the derivative of S fails at discrete points on the caustic: the origin is said to be a folded critical point if it is a regular point of the caustic and the projection of a singularity; it corresponds to:

$$F(\lambda, x, p) = \lambda^3 + p\lambda + S(x, p), \quad \frac{\partial S}{\partial x}(0, 0) = 0, \quad \frac{\partial^2 S}{\partial x^2}(0, 0) \neq 0, \quad \frac{\partial S}{\partial p}(0, 0) \neq 0$$

and it is a centre or a saddle as S_p and S_{xx} have, or have not, the same sign at the origin.

The origin is a pleat point of an implicit Hamilton equation if it is a critical value of the restriction of the projection to the criminant curve; it corresponds to:

$$F(\lambda, x, p) = \pm\lambda^4 + x\lambda^2 + p\lambda + S(x, p), \quad \frac{\partial S}{\partial x}(0, 0) \neq 0$$

and it is elliptic or hyperbolic as $S_x > 0$ or $S_x < 0$ respectively.

5. Perestroikas of fronts and phase portrait models

A map $F : \mathbf{R}^k \times \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}$ is the generating family of a legendrean submanifold l_e of $\mathbf{R}^{4n+1} = T(T^*N) \times \mathbf{R}$ with $N = \mathbf{R}^n$ if:

$$\begin{aligned} l_e &= \{(x,p,z,\dot{x},\dot{p}) : \frac{\partial F}{\partial \lambda}(\lambda,x,p) = 0, z = F(\lambda,x,p), \\ &\qquad \dot{x} = \frac{\partial F}{\partial p}(\lambda,x,p), \dot{p} = -\frac{\partial F}{\partial x}(\lambda,x,p)\}. \end{aligned}$$

This is in fact a particular case of taking as generating family $G(\lambda,x,p,z) = F(\lambda,x,p) - z$.

Two generating families F_1 and F_2 are V-equivalent if there exists a function $M : \mathbf{R}^k \times \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}$ and a change of coordinates $(\lambda,x,p) \mapsto (\Lambda(\lambda,x,p), X(x,p), P(x,p))$ such that:

$$F_1(\lambda,x,p) = M(\lambda,x,p) F_2(\Lambda(\lambda,x,p), X(x,p), P(x,p))$$

They are stably equivalent if they become V-equivalent after adding convenient quadratic forms Q_i on new variables λ.

Proposition 1 ([1,i]). *All legendrean submanifolds can be constructed from a generating family, and two legendrean germs are legendrean equivalent if the germs of their generating families are stably V-equivalent.*

A legendrean fibration of a contact manifold E is a fibration $\pi_e : E \longrightarrow B$ such that the fibres are legendrean submanifolds of E.

A legendrean map $l_e \longrightarrow B$ is the composition of the embedding of the legendrean submanifold l_e in E with the projection $\pi_e : E \longrightarrow B$; a front is the image of a legendrean map.

Given a map $F(\lambda,x,p,\tau)$, $\tau \in \mathbf{R}^m$, it defines both a legendrean submanifold l_e of $T(T^*\mathbf{R}^n \times \mathbf{R}^m) \times \mathbf{R}$ and a family of legendrean submanifolds l_e^τ of $T(T^*\mathbf{R}^n) \times \mathbf{R}$, fixing τ; the corresponding fronts are the big front and the instantaneous fronts.

Two families of fronts Σ_1^τ and Σ_2^τ, corresponding to families of legendrean submanifolds l_1^τ and l_2^τ, have equivalent perestroikas if there exists a diffeomorphism $\Phi = (\Phi^\tau, T)$, T depending only on the parameters τ, such that:

$$\Phi^\tau(\Sigma_1^\tau) = \Sigma_2^{T(\tau)}.$$

Two generating families $F_1(\lambda, x, p, \tau)$ and $F_2(\lambda, x, p, \tau)$, depending on parameters $\tau \in \mathbf{R}^m$, are parametrized V-equivalent if there exists a function $M : \mathbf{R}^k \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}$ and a change of coordinates $(\lambda, x, p, \tau) \mapsto (\Lambda(\lambda, x, p, \tau), X(x, p, \tau), P(x, p, \tau), T(\tau))$ such that:

$$F_1(\lambda, x, p, \tau) = M(\lambda, x, p, \tau) F_2(\Lambda(\lambda, x, p, \tau), X(x, p, \tau), P(x, p, \tau), T(\tau)).$$

Theorem 2 *The phase portrait at a point in the caustic of the germ of a generic implicit Hamilton equation on $M = T^*N$ is equivalent to the phase portrait of the germ of the implicit Hamilton equation defined by one of the generating families:*

- $F(\lambda, x, p) = \lambda^3 + p\lambda + x$ *at a fold point.*
- $F(\lambda, x, p) = \lambda^3 + p\lambda + p + x^2$ *at a folded centre point.*
- $F(\lambda, x, p) = \lambda^3 + p\lambda + p - x^2$ *at a folded saddle point.*
- $F(\lambda, x, p) = \lambda^4 + x\lambda^2 + p\lambda + x$ *at an elliptic pleat point.*
- $F(\lambda, x, p) = \lambda^4 + x\lambda^2 + p\lambda - x$ *at a hyperbolic pleat point.*

Proof. In the context of implicit Hamilton equations, to the lagrangian submanifold l corresponds a legendrean submanifold l_e with the same generating family F, or more precisely $G(\lambda, x, p, z) = F(\lambda, x, p) - z$; since $\theta|l = dF|l$ it follows that the integral curves of the direction field in l are contained in the level sets of F: l_e is the big front and the instantaneous fronts l_e^z, obtained fixing $z = F(\lambda, x, p)$, give the individual integral curves.

Considering the generating function $G(\lambda, x, p, z)$, z should be interpreted as a one dimensional parameter; now two germs of perestroikas of fronts are equivalent if the germs of their generating families are stably parametrized V-equivalent [5,6]. The list above follows from the lists established in [5,6] for the generic generating families with $n = 2$.

To avoid moduli in the classification, weaker notions of equivalence can be used, and the following is then natural: instead of changes of coordinates depending only on (x, p), changes of coordinates (x, p, z) preserving the big front

$$z = F(\lambda, x, p), \quad \frac{\partial F}{\partial \lambda}(\lambda, x, p) = 0$$

are considered. The stable functions on $\mathbf{R}^3$ for the equivalence based on changes of coordinates preserving the generic fronts are [1]:

generating family	function
$\lambda^3 + p\lambda - z$	$(x, p, z) \mapsto x$
	$(x, p, z) \mapsto p \pm x^2$
$\pm\lambda^4 + x\lambda^2 + p\lambda - z$	$(x, p, z) \mapsto x$

These are essentially the functions $S(x, p)$, defined in the previous section, involved in the list of theorem 2.

6. Families of implicit Hamilton equations and perestroikas of caustics

A family of implicit Hamilton equations depending on parameters $\tau \in \mathbf{R}^m$ is just a family of lagrangian submanifolds of $T(T^*N)$ depending on those parameters. It can also be viewed as a lagrangian submanifold of $T(T^*N^n \times \mathbf{R}^m)$.

Taking the phase portrait of one equation as sections $z = c$, $c \in \mathbf{R}$, of a big front, then sections $(z, \tau) = c$, $c \in \mathbf{R}^{m+1}$, should be considered for a family of implicit Hamilton equations; this approach will be developed elsewhere, here an alternative treatment will be discussed.

Given a map $F(\lambda, x, p, \tau)$, $\tau \in \mathbf{R}^m$, it defines both a lagrangian submanifold l of $T(T^*\mathbf{R}^n \times \mathbf{R}^m)$ and a family of lagrangian submanifolds l^τ of $T(T^*\mathbf{R}^n)$, fixing τ; the corresponding caustics are the big caustic and the instantaneous caustics.

Two families of caustics Σ_1^τ and Σ_2^τ, corresponding to families of lagrangian submanifolds l_1^τ and l_2^τ, have equivalent perestroikas if there exists a diffeomorphism $\Phi = (\Phi^\tau, T)$, T depending only on the parameters τ, such that:

$$\Phi^\tau(\Sigma_1^\tau) = \Sigma_2^{T(\tau)}$$

Theorem 3 ([6]). *All perestroikas of caustics depending on one parameter τ can be constructed from a generating family defining a big caustic. In dimension two, the germs of the generic perestroikas of caustics are equivalent to the germs of the caustics of the generating families:*

- $F(\lambda, x, p, \tau) = \lambda^3 + p\lambda$
- $F(\lambda, x, p, \tau) = \lambda^4 + x\lambda^2 + p\lambda$
- $F(\lambda, x, p, \tau) = \lambda^4 + (\tau \pm x^2)\lambda^2 + p\lambda$
- $F(\lambda, x, p, \tau) = \lambda^5 + \tau\lambda^3 + x\lambda^2 + p\lambda$
- $F(\lambda_1, \lambda_2, \tau, x, p) = \lambda_1^3 \pm \lambda_1\lambda_2^2 + (\tau \pm x + mp)\lambda_1^2 + x\lambda_1 + p\lambda_2$

with $\lambda, \lambda_1, \lambda_2, \tau, m \in \mathbf{R}$.

As before, we add to these normal forms an adequate function $S_\tau(x,p)$, depending now on the real parameter τ, to obtain:

Theorem 4 *The phase portrait at a point in the caustic of the germ of a generic one parameter family of implicit Hamilton equations on $M = T^*N$ is equivalent to the phase portrait of the germ of one of families of theorem 2 or one of the following:*

- $F(\lambda, x, p, \tau) = \lambda^3 + p\lambda + \tau p + x^2$
- $F(\lambda, x, p, \tau) = \lambda^3 + p\lambda + p + \tau x + x^3$
- $F(\lambda, x, p, \tau) = \lambda^4 + x\lambda^2 + p\lambda + p + \tau x + x^2$
- $F(\lambda, x, p, \tau) = \lambda^4 + (\tau \pm x^2)\lambda^2 + p\lambda + x$

- $F(\lambda, x, p, \tau) = \lambda^5 + \tau\lambda^3 + x\lambda^2 + p\lambda + x$.
- $F(\lambda_1, \lambda_2, \tau, x, p) = \lambda_1^3 \pm \lambda_1\lambda_2^2 + (\tau \pm x + mp)\lambda_1^2 + x\lambda_1 + p\lambda_2 \pm x + mp$

with $\lambda, \lambda_1, \lambda_2, \tau, m \in \mathbf{R}$.

Proof. If the phase portrait of one equation can be interpreted as a one parameter perestroika, it is natural now to interpret the phase portrait of an equation depending on one parameter as a two parameter perestroika of fronts.

The classification of the corresponding genetrating families depending on the two parameters (τ_1, τ_2) can be obtained from the singularities:

- A_2: $F(\lambda, a) = \lambda^3 + a_1\lambda + a_2$
- A_3: $F(\lambda, a) = \lambda^4 + a_1\lambda^2 + a_2\lambda + a_3$
- A_4: $F(\lambda, a) = \lambda^5 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$
- $D_4^{\pm}$: $F(\lambda, a) = \lambda_1^3 \pm \lambda_1\lambda_2^2 + a_1\lambda_1^2 + a_2\lambda_1 + a_3 + a_4\lambda_2$

by the path formulation method, the paths $a = a(x, p, \tau_1, \tau_2)$ depending on two variables (x, p) and having codimension up to 2, as summarized in the following:

singularity	path	unfolding terms
A_2	(p, x)	
	$(p, p \pm x^2)$	(0,1)
	(p, x^2)	$(0, 1), (0, p)$
	$(p, p + x^3)$	$(0, 1), (0, x)$
A_3	$(x, p, \pm x)$	$(0, 0, 1)$
	$(x, p, p + x^2)$	$(0, 0, 1), (0, 0, x)$
	$(\pm x^2, p, x)$	$(0, 0, 1), (1, 0, 0)$
A_4	$(0, x, p, x)$	$(0, 0, 0, 1), (1, 0, 0, 0)$
D_4^+	$(x + mp, x, x + mp, p)$	$(1, 0, 0, 0), (0, 0, 1, 0)$
D_4^-	$(x + mp, x, x + mp, p), \lvert m\rvert \neq 1$	$(1, 0, 0, 0), (0, 0, 1, 0)$

The generic 2-parameter perestroikas in the plane, corresponding to generating families depending on two parameters, are given by:

singularity	generating family
A_2	$\lambda^3 + p\lambda + x$ $\lambda^3 + p\lambda + p \pm x^2 + \tau_1$ $\lambda^3 + p\lambda + x^2 + \tau_1 + \tau_2 p$ $\lambda^3 + p\lambda + p + x^3 + \tau_1 + \tau_2 x$
A_3	$\lambda^4 + x\lambda^2 + p\lambda \pm x + \tau_1$ $\lambda^4 + x\lambda^2 + p\lambda + p + x^2 + \tau_1 + \tau_2 x$ $\lambda^4 + (\tau_2 \pm x^2)\lambda^2 + p\lambda + x + \tau_1$
A_4	$\lambda^5 + \tau_1\lambda^3 + x\lambda^2 + p\lambda + x + \tau_2$
D_4^+ D_4^-	$\lambda_1^2\lambda_2 + \lambda_2^3 + (x + mp + \tau_1)\lambda_2^2 + x\lambda_2 + (x + mp + \tau_2) + p\lambda_1$ $\lambda_1^2\lambda_2 - \lambda_2^3 + (x + mp + \tau_1)\lambda_2^2 + x\lambda_2 + (x + mp + \tau_2) + p\lambda_1,$ $\|m\| \neq 1$

From here it is easy to obtain the list in the statement of the theorem, taking $z = -\tau_1$.

The new two A_2 cases correspond to two different saddle-centre bifurcations: in the first one a critical point on the criminant curve crosses at $\tau = 0$ another critical point of different type, that for different values of τ is not on that curve, and both change type; in the other case two critical points of different types, both on the criminant curve, meet and disappear.

In the first new A_3 case, a saddle on the criminant curve passes through a pleat point; a similar situation involving a centre is impossible for generic one parameter families.

The second new A_3 and the A_4 cases correspond to the three different ways two pleat points, or two cusp points in the caustic, coalesce.

In the D_4^- case, three pleat points, or three cusp points in the caustic, come together and separate again. In the D_4^+ a cusp point crosses a different (at the local level) branch of the caustic, the corresponding pleat point in the front crossing a fold edge.

A special situation is also important: if a family of lagrangian submanifolds depends on one parameter identified with time, the corresponding family of implicit Hamilton equations is just a non autonomous equation. Its phase portrait cannot be obtained through sections, as it is not longer true that the parameter, and also the generating family, are constant along the trajectories. The above description is no longer valid; the phase portrait has to be obtained by different methods, integrating a non autonomous vector field corresponding to the time-dependent generating family.

7. Lagrangian models

Coming back to the main example, $TN \oplus T^*N$ with coordinates $(x, \dot{x}, p)$ and the 1-form $\theta = \frac{\partial L}{\partial x}(x, \dot{x})dx - \dot{x}dp$ is a contact manifold (with singularities), and

$$\mathcal{H}(x, \dot{x}, p) = c, \quad p = \frac{\partial L}{\partial \dot{x}}(x, \dot{x})$$

define a legendrean submanifold l_e^c. As the projection $TN \oplus T^*N \longrightarrow T^*N$ is a legendrean fibration, the trajectory in T^*N corresponding to $\mathcal{H} = c$ is a front.

$T(T^*N) \times \mathbf{R}$ with coordinates $(x, p, \dot{x}, \dot{p}, z)$ and the 1-form $\alpha = dz - \dot{p}dx + \dot{x}dp$ is a contact manifold, and

$$z = \mathcal{H}(x, \dot{x}, p), \quad \dot{p} = \frac{\partial L}{\partial x}(x, \dot{x}), \quad p = \frac{\partial L}{\partial \dot{x}}(x, \dot{x})$$

define a legendrean submanifold L_e. As the projection $T(T^*N) \times \mathbf{R} \longrightarrow T^*N \times \mathbf{R}$ is a legendrean fibration, the image of L_e is a front.

Also $L_e = \cup l_e^c$ and l_e^c depends smoothly on c and is obtained from L_e by taking $z = c$; thus L_e is the big front of the family l_e^c. The family of projections of l_e^c is a perestroika of fronts.

Assuming $N = \mathbf{R}$, a straightforward computation shows that the list of situations considered in theorem 2 can be obtained from Euler equations with suitable lagrangians $L(x, \dot{x})$, the corresponding generating families being $\mathcal{H}(x, \dot{x}, p) = p\dot{x} - L(x, \dot{x})$:

- Fold point: $L(x, \dot{x}) = -\dot{x}^3 - x$, $\mathcal{H}(x, \dot{x}, p) = \dot{x}^3 + p\dot{x} + x$.
- Folded centre: $L(x, \dot{x}) = -(\dot{x} - 1)^3 - x^2$, $\mathcal{H}(x, \dot{x}, p) = (\dot{x} - 1)^3 + p\dot{x} + x^2$.
- Folded saddle: $L(x, \dot{x}) = -(\dot{x} - 1)^3 + x^2$, $\mathcal{H}(x, \dot{x}, p) = (\dot{x} - 1)^3 + p\dot{x} - x^2$.

- Elliptic pleat: $L(x, \dot{x}) = -\dot{x}^4 - x\dot{x}^2 - x$, $\mathcal{H}(x, \dot{x}, p) = \dot{x}^4 + x\dot{x}^2 + p\dot{x} + x$.
- Hyperbolic pleat:$L(x, \dot{x}) = -\dot{x}^4 - x\dot{x}^2 + x$, $\mathcal{H}(x, \dot{x}, p) = \dot{x}^4 + x\dot{x}^2 + p\dot{x} - x$.

Remark: for the folded centre and saddle, the relevant point is $x = 0$, $\dot{x} = 1$ (taking $\lambda = \dot{x} - 1$), and the origin in the remaining ones ($\lambda = \dot{x}$); at $\dot{x} = 0$ the critical points are always degenerate.

For families depending on one parameter, the lagrangians are (keeping the order of theorem 4):

- $L(\tau, x, \dot{x}) = -(\dot{x} - \tau)^3 - x^2$.
- $L(\tau, x, \dot{x}) = -(\dot{x} - 1)^3 - \tau x - x^3$.
- $L(\tau, x, \dot{x}) = -(\dot{x} - 1)^4 - x(\dot{x} - 1)^2 - \tau x - x^2$.
- A_3^1: $L(\tau, x, \dot{x}) = -\dot{x}^4 - (\tau \pm x^2)\dot{x}^2 - x$.
- A_4: $L(\tau, x, \dot{x}) = -\dot{x}^5 - \tau\dot{x}^3 + x\dot{x}^2 - x$.

As now the generating families depend only on one variable $\dot{x}$, the D_4 cases can not be realized in this context.

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