GENERIC SINGULARITIES OF HYPERPLANE SECTIONS OF A SURFACE IN $\mathbb{P}^4$

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Abstract

We obtain formulas for the homology classes of generic singularity loci of hyperplane sections of a surface in $\mathbb{P}^4$.

Resumo

Obtemos fórmulas para classes de homologia suportadas por singularidades genéricas de seções hiperplanas de uma superfície em $\mathbb{P}^4$.

1. Introduction

The literature on singularities of hyperplane sections of plane curves and surfaces in 3-space is very rich. The well known Plücker formulas give the number of bitangent and inflexional lines for plane curves with mild singularities. Cayley and Salmon have pursued the subject of finding numerical invariants for the various types of stationary plane sections of a surface in $\mathbb{P}^3$ (cf. [5]). McCrory et al. [6] have studied the case of hypersurfaces in higher dimensions from the point of view of the singularities of a Gauss map.

Let $M \subset \mathbb{P}^4$ be a smooth, projective surface in complex projective 4-space. In contrast with the codimension one case, now at each $P \in M$, there are ∞^1 tangent hyperplanes. Each tangent hyperplane cuts a curve in M which is singular at P. For most tangent hyperplanes at P, this singularity is quadratic, nondegenerate. The one–parameter family of tangent hyperplanes cuts a pencil of degree 2 divisors on the projective line of tangent directions of M at P. Thus one expects to find, in general, two distinct tangent hyperplanes, each cutting a cusp at P.

*Partially supported by CNPq.

The coalescence of these cuspidal sections is expected to occur along a codimension 1 subset, called the *curve of cuspidal coincidence* cf. §4.1. The *tacnodal locus*, consisting of all points of M where a hyperplane cuts a tacnode is described in §5.

One also expects finitely many tangent hyperplane sections of each of the following types:

The purpose of this note is to derive formulas expressing the classes of each of these loci in terms of Chern classes of the embedding $M \subset \mathbb{P}^4$. We have also included many examples where a precise description of the locus is available. For recent contribution on the fast growing subject of counting curves of low genus in the context of Gromov-Witten invariants, see [1], [3].

In section 1 we give a formula for the locus of tangent coincidences of two families of curves in a family of surfaces. The formula is applied subsequently to express the cuspidal locus of a family of curves (§3) and to the locus of cuspidal coincidences (cf. §4).

We employ the strategy of iteration in order to handle the loci of higher singularities. For instance, a tacnode (resp. higher cusp) is a double point (resp. a cusp) with an infinitely near double point (resp. cusp). As in [7], the idea is to impose the condition that the family of strict transforms possess the appropriate singularity type, cf. §11.

We have not considered the question of whether the loci under consideration are indeed of the right codimension for the general member of a family of surfaces (e.g., complete intersection). Also deferred is the analysis of the local nature of these loci, e.g.,

• what are the singularities of the tacnodal locus and of the locus of cuspidal coincidences?

• what are the intersection multiplicities of the tacnodal locus and the locus of cuspidal coincidences, say at a triple point of a hyperplane section?

• do the higher cusps contribute with weight one to the bicusps locus?

• at how many points are both cuspidal sections tacnodes?

Many thanks are due to M. A. Ruas for bringing my attention to and sharing her insights on the subject. She should undoubtly have accepted this as a joint work.

2. Tangent coincidences

Lemma 2.1. *Let $f : X \to T$ be a smooth map of smooth varieties of relative dimension two. Let Y, $Z \subset X$ be hypersurfaces. Put*

$$\mathcal{TC} = \mathcal{TC}(Y, Z) := \{x \in Y \cap Z \mid \text{ the fiber } Y_{f(x)} \text{ is tangent to } Z_{f(x)}\}.$$

Assume $\mathcal{TC}$ is of codimension 3 in X. Then its class is given by the formula,

$$\mathcal{TC} \;=\; Y \cdot Z \cdot (Y + Z - c_1(\mathcal{T}_{X/T})) \text{ in } A^3(X). \tag{1}$$

Proof. We assume for simplicity Y, Z smooth over T. Consider the diagram of relative tangent/normal bundles restricted to $Y \cap Z$,

$$\begin{array}{ccccc} \mathcal{T}_{Y/T} & \to & \mathcal{T}_{X/T} & \to & N_{Y/X} \\ & & \uparrow & \nearrow & \\ & & \mathcal{T}_{Z/T} & & \end{array}$$

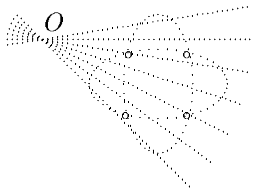
The slant arrow vanishes at a point $s \in Y \cap Z$ if and only if the tangents along the fiber of Y and T are aligned. Thus $\mathcal{TC}$ is the scheme of zeros of a section of $(N_{Y/X} \otimes \mathcal{T}^*_{Z/T})_{|Y \cap Z}$.

Hence we may write

$$\begin{aligned}\mathcal{TC} &= c_1 N_{Y/X|Y\cap Z} - c_1 \mathcal{T}_{Z/T|Y\cap Z}\\ &= c_1 N_{Y/X}\cdot Z - c_1\mathcal{T}_{Z/T}\cdot Y\\ &= Y^2\cdot Z - (c_1\mathcal{T}_{X/T|Z} - c_1 N_{Z/X})\cdot Y\\ &= Y^2\cdot Z - (c_1\mathcal{T}_{X/T}\cdot Z - Z^2)\cdot Y. \qquad \square\end{aligned}$$

Remark 2.2. See [2] for generalizations for the case of tangencies of plane curves.

Example 2.3. Consider a pencil of lines through a point O and a pencil of conics through four points $P_1, \dots, P_4$, such that no three of those five points are aligned. For a general line there are two associated tangent conics in the pencil. For a non degenerated conic there are two tangent lines through O.



Take in the lemma $T = \mathbb{P}^1_\lambda\times\mathbb{P}^1_\mu$ parametrizing the pairs (ℓ_λ, C_μ), ℓ_λ = line through O, C_μ = conic through the four fixed points. Let $F(\lambda,\mu) = 0$ be the tangency correspondence "C_μ tangent to ℓ_λ" in T. Put

$$X = \mathbb{P}^2\times T,\quad Y = \{(P,\lambda,\mu)\in X \mid P\in\ell_\lambda\},\quad Z = \{(P,\lambda,\mu)\in X\mid P\in C_\mu\}.$$

We have $Y \cong Y'\times\mathbb{P}^1$ and $Z\cong Z'\times\mathbb{P}^1$ where Y' is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and Z' is $\mathbb{P}^2$ blownup at four points. We may now compute the various classes occuring in (1). Setting

$$\lambda = c_1\,\mathcal{O}_{\mathbb{P}^1_\lambda}(1),\quad \mu = c_1\,\mathcal{O}_{\mathbb{P}^1_\mu}(1),\quad \ell = c_1\,\mathcal{O}_{\mathbb{P}^2}(1)$$

we get

$$Y = \lambda+\ell,\quad Z = \mu + 2\ell$$

and

$$\begin{aligned}\mathcal{TC} &= (\lambda+\ell)(\mu+2\ell)(\lambda+\mu+3\ell-3\ell)\\ &= 2(\lambda+\mu)\ell^2 + 3\lambda\mu\ell \quad\text{in}\quad A(\mathbb{P}^2\times\mathbb{P}^1_\lambda\times\mathbb{P}^1_\mu).\end{aligned}$$

This class pushes forward to the class $2(\lambda+\mu)$ in the intersection (or cohomology) ring of $\mathbb{P}^1_\lambda \times \mathbb{P}^1_\mu$ and to 3ℓ in that of $\mathbb{P}^2$. Thus it agrees with the facts that: (i) the correspondence $F(\lambda,\mu)=0$ is of bidegree (2,2); (ii) the locus of points of contact is a cubic.

Example 2.4. Keep the notation of the previous example. We show that (1) also yields the expected number of ramification points: for precisely four lines in the pencil the two associated conics coincide, to wit, the lines OP_i. For a fixed line ℓ, say $z=0$, the conics C_μ cut a pencil π_ℓ of degree two divisors on ℓ. The complete system of degree two divisors is a projective plane $\mathbb{P}^2_\ell$ associated to the vector space $\langle x^2, xy, y^2\rangle$ of quadratic forms. The discriminant $\Delta_\ell \subset \mathbb{P}^2_\ell$ is a conic. Hence π_ℓ cuts Δ_ℓ in two points. Varying ℓ through O, we ask how often is the line (pencil) π_ℓ tangent to the conic Δ_ℓ.

Take in the lemma $T=\mathbb{P}^1_\lambda$ parametrizing the lines through O, $X=\mathbb{P}(\mathcal{S})$ the $\mathbb{P}^2$-bundle over T parametrizing the complete systems of divisors of degree two on the varying line. Here $\mathcal{S}$ is the vector bundle $/\,T$ given by restrictions of quadratic forms to the varying ℓ. Take $Z=T\times\mathbb{P}^1_\mu$: it consists of pairs (λ,μ), $\ell_\lambda=$ line through O and $C_\mu=$ conic through the four points. Finally take $Y\subset X$ to be the conic bundle over T parametrizing the divisors of the form $2P$, ($P\in\ell$ for the varying $\ell\in T$). We now compute the various classes occurring in the formula. Put $\mathcal{F}=\langle x,y,z\rangle$, the vector space of linear forms in three variables. We have the following exact sequence of vector bundles over the $\mathbb{P}^1_\lambda$ of lines,

$$\mathcal{O}_\lambda(-1)\otimes\mathcal{F} \rightarrowtail \mathrm{Sym}_2\,\mathcal{F} \twoheadrightarrow \mathcal{S}$$

where $\mathcal{O}_\ell(-1)\rightarrowtail\mathcal{F}$ is the tautological rank one subbundle. The fiber of $\mathcal{O}_\ell(-1)$ over a point $\ell\in\mathbb{P}^1$ representing a projective line through O consists of the multiples of a linear form defining ℓ. Hence, setting $\lambda=c_1\mathcal{O}_\ell(1)$, we may compute the Chern class, $c_1\mathcal{S}=3\lambda$. Now $Z\subset\mathbb{P}(\mathcal{S})$ is a projective subbundle. Precisely, let the map of trivial bundles, $\mathcal{O}^{\oplus 2}\rightarrowtail\mathrm{Sym}_2\,\mathcal{F}$ correspond to the choice of two conics through the four points. Compose this inclusion with the surjection $\mathrm{Sym}_2\,\mathcal{F}\twoheadrightarrow\mathcal{S}$. Since no conic in the pencil contains a line through O, we still have the injection $\mathcal{O}^{\oplus 2}\rightarrowtail\mathcal{S}$, whence $Z=\mathbb{P}(\mathcal{O}^{\oplus 2})\subset\mathbb{P}(\mathcal{S})$. Therefore, Z is the scheme of zeros of the composition

$$\mathcal{O}_{\mathcal{S}}(-1)\rightarrowtail\mathcal{S}\twoheadrightarrow\mathcal{S}/\mathcal{O}^{\oplus 2}.$$

Setting $\kappa = c_1\mathcal{O}_{\mathcal{S}}(1)$, this gives

$$Z = c_1((\mathcal{S}/\mathcal{O}^{\oplus 2}) \otimes \mathcal{O}_{\mathcal{S}}(1)) = \kappa + 3\lambda.$$

Put $\mathcal{E} = \mathcal{F}/(\mathcal{O}_\lambda(-1))$, the bundle of restrictions of linear forms to the varying $\ell \in T$. We have $\mathcal{S} = \mathrm{Sym}_2\mathcal{E}$. The divisor of double points $\Delta \subset X$ is the image of the Veronese embedding, $\Delta = \mathbb{P}(\mathcal{E}) \hookrightarrow \mathbb{P}(\mathrm{Sym}_2\mathcal{E})$. Now $\mathcal{O}_{\mathcal{S}}(-1) \rightarrowtail \mathcal{S}$ corresponds to the universal symmetric map $\mathcal{E}^* \to \mathcal{E}(1)$ (via the usual correspondence quadratic form $\leftrightarrow$ symmetric map). Thus Δ is the divisor of zeros of $\bigwedge^2 \mathcal{E}^* \to \bigwedge^2(\mathcal{E}(1))$ and we may write

$$\Delta = c_1((\bigwedge^2 \mathcal{E})^{\otimes 2}(2))) = 2\lambda + 2\kappa. \tag{2}$$

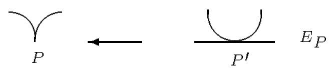
Substituting $Y = \Delta$ in (1), we get

$$2(\lambda + \kappa)(3\lambda + \kappa)(5\lambda + 3\kappa - (3\lambda + 3\kappa)) \ = \ 4\lambda\kappa^2.$$

Since $\lambda\kappa^2$ is the class of a point in X, we retrieve the expected number, 4, of ramification points.

3. Cuspidal locus

We apply (1) to get the class of the **cuspidal locus of a family of curves** on a surface M. Note that $P \in C \subset M$ is a cusp of the curve C if the strict transform of C in the blow up $M'_P \to M$ of the surface M at P intersects the exceptional line E_P in a divisor containing a divisor of the form $2P'$.



So we are after the tangencies of two families of curves: one given by the exceptional lines and the other by the strict transform of a singular member of the family of curves at each point of M.

We may as well let M also vary so that our initial data will consist of

- a smooth map $f : \mathbb{M} \to S$ of relative dimension 2 and

- a relative Cartier divisor $\mathbb{D} \subset \mathbb{M}$.

The fibers $\mathbb{D}_s \subset \mathbb{M}_s$ should be thought of as a family of curves in the family of surfaces. We assume the parameter space S is a local complete intersection.

We take in (1)

$$T = \{P \in \mathbb{M} \,|\, P \text{ is singular in } \mathbb{D}_{f(P)}\}.$$

It carries a natural structure of closed subscheme of $\mathbb{M}$, locally defined by equations $F = F_x = F_y = 0$, where F stands for a local equation of $\mathbb{D}$ in $\mathbb{M}$ and the partial derivatives are taken with respect to local parameters along a fiber of f. Globally, there is a section $\sigma^1_{\mathbb{D}}$ of the sheaf of relative principal parts defined in the diagram,

$$\begin{array}{ccccc} & & \mathcal{O} & & \\ & & \sigma^1_{\mathbb{D}} \downarrow & \searrow \sigma_{\mathbb{D}} & \\ \Omega^1_{\mathbb{M}/S} \otimes \mathcal{O}(\mathbb{D}) & \rightarrowtail & \mathcal{P}^1_{\mathbb{M}/S}(\mathcal{O}(\mathbb{D})) & \twoheadrightarrow & \mathcal{O}(\mathbb{D}), \end{array}$$

where the slant arrow $\sigma_{\mathbb{D}}$ is the section defining $\mathbb{D}$ in $\mathbb{M}$.

Assuming T is of the expected codimension 3 in $\mathbb{M}$, we may write the formula for its class,

$$\begin{aligned} T &= c_3 \mathcal{P}^1_{\mathbb{M}/S}(\mathcal{O}(\mathbb{D})) \\ &= \mathbb{D} \cdot (\mathbb{D}^2 + \mathbb{D}K + \chi), \end{aligned}$$

where we have set for short

$$K = c_1 \Omega^1_{\mathbb{M}/S} \qquad \text{and} \qquad \chi = c_2 \Omega^1_{\mathbb{M}/S}. \tag{3}$$

Let

$$b : \mathbb{M}^{(2)} \to \mathbb{M} \times_S \mathbb{M}$$

be the blowup of the diagonal $\Delta_{\mathbb{M}}$ in $\mathbb{M} \times_S \mathbb{M}$ and set $E = b^{-1}\Delta_{\mathbb{M}}$. Let $p_i : \mathbb{M} \times_S \mathbb{M} \to \mathbb{M}$ be the projection. Note that the fiber $\mathbb{M}^{(2)}_P = b^{-1}p_1^{-1}P$ is the blow up of $\mathbb{M}_s$ at the point P, where $s = f(P)$. Put

$$X = \mathbb{M}^{(2)} \underset{\mathbb{M}}{\times} T.$$

Thus X is the set of (P, P') with $P' \in \mathbb{M}^{(2)}_P$ such that P a singularity of $\mathbb{D}_s$. Put $Z' = b^{-1}p_2^{-1}\mathbb{D}$. This is the total space of the family of total transforms of the members of the family of curves. Let $Y \subset X$ be the pullback of the exceptional divisor via the natural map $X \to \mathbb{M}^{(2)}$. By construction, Z' and Y are Cartier divisors in X and

$Z := Z' - 2Y$ is effective. The fiber of Z over a point $P \in T$ is the divisor $b_P^{-1}\mathbb{D}_s - 2E_P$. It coincides with the strict transform of $\mathbb{D}_s$ if P is a point of multiplicity two in $\mathbb{D}_s$. Thus, the locus of tangencies of Y and Z along the fibers of $X \to T$ accounts for the cuspidal locus. We compute first the relevant classes in X. We have

$$c_1 \mathcal{T}_{X/T} = (p_2 b)^*(-K) - E,$$
$$Y = E, \qquad Z = (p_2 b)^*(\mathbb{D}) - 2E.$$

Hence (1) now reads $E \cdot ((p_2 b)^*(\mathbb{D}) - 2E) \cdot ((p_2 b)^*(\mathbb{D} + K))$. This class pushes forward to the cycle

$$\kappa = \kappa(\mathbb{D}) := 2(\mathbb{D} + K) \cdot \mathbb{D} \cdot (\mathbb{D}^2 + \mathbb{D}K + \chi) \tag{4}$$

in $\mathbb{M}$. It is supported by the set of points $P \in \mathbb{M}$ such that P is singular in $\mathbb{D}_s$ ($s = f(P)$) and the transform $\mathbb{D}'_s = b^*\mathbb{D}_s - 2E_P$ cuts in the exceptional line a subscheme containing twice a point. Note that if P is a point of multiplicity ≥ 3 in $\mathbb{D}_s$ then $\mathbb{D}'_s$ contains E_P and P is in the cuspidal locus. Generically, however, this behaviour is not expected for families of curves of dimension ≤ 3.

3.1 Example. Consider a linear system of dimension n on a surface M. We take $S = \check{\mathbb{P}}^n$, $\mathbb{M} = M \times \check{\mathbb{P}}^n$. Writing C for (the class of) a member of the system and $\check{H}$ for (the class of) a hyperplane in $\check{\mathbb{P}}^n$, we substitute $\mathbb{D} = C + \check{H}$ in (4). We find the class in $M \times \check{\mathbb{P}}^n$,

$$2\check{H}^4 + (4K + 8C)\check{H}^3 + 2(6CK + 6C^2 + K^2 + \chi)\check{H}^2.$$

If $n = 4$ (e.g., for the system of hyperplane sections of a surface $M \subset \mathbb{P}^4$), it pushes forward to twice the fundamental class of M. This is in agreement with the fact that a general point of M is a cusp for two members of the system.

For $n = 3$ (e.g., for the system of plane sections of a surface $M \subset \mathbb{P}^3$), it pushes forward to the class of a curve in M given by

$$4(K + 2C).$$

For $n = 2$, it pushes forward to a zero cycle in M with class given by

$$2(6CK + 6C^2 + K^2 + \chi).$$

For instance, if $M = \mathbb{P}^2$, we may write

$$\ell = \text{ class of a line}, \qquad K = -3\ell, \qquad \chi = 3\ell^2, \qquad C = d\ell \ (d\text{=degree } C).$$

We find $2(6d(-3)+6d^2+9+3)=12(d-1)(d-2)$ for the number of cuspidal members of a general net of plane curves of degree d, cf. [5].

4. Curve of cuspidal coincidence

We now apply (1) to the study of singular hyperplane sections of a surface $M \subset \mathbb{P}^4$. Let $\check{\mathbb{P}}^4$ be the dual space. Set

$$M' \;=\; \{(P,h) \in M \times \check{\mathbb{P}}^4 \,|\, h \supset \mathcal{T}_P M\}. \tag{5}$$

This is often called the *conormal* variety of $M \subset \mathbb{P}^4$. We have that M' is a $\mathbb{P}^1$-bundle over M. Precisely, setting for short $\mathcal{M} = \mathcal{O}_{\mathbb{P}^4}(1)_{|M}$, recall there are natural maps of ith. order Taylor expansion evaluation,

$$H^0(M,\mathcal{M}) \otimes \mathcal{O}_M \longrightarrow \mathcal{P}^i_M(\mathcal{M})$$

where $\mathcal{P}^i_M(\mathcal{M})$ is the sheaf of principal parts. We have $\mathcal{P}^0_M(\mathcal{M}) = \mathcal{M}$ and there are exact sequences,

$$\mathrm{Sym}_i\,\Omega^1_M \otimes \mathcal{M} \rightarrowtail \mathcal{P}^i_M(\mathcal{M}) \twoheadrightarrow \mathcal{P}^{i-1}_M(\mathcal{M}).$$

Assume henceforth $\operatorname{rank} H^0(M,\mathcal{M}) = 5$. (A well known result of Severi asserts that the Veronese surface is the only non linearly normal surface in $\mathbb{P}^4$.) We have $H^0(M,\mathcal{M}) \otimes \mathcal{O}_M \longrightarrow \mathcal{P}^1_M(\mathcal{M})$ surjective. This amounts to saying that there are enough hyperplane sections to fill in all tangent directions at each point $P \in M$. Put

$$\mathcal{F} \;=\; \ker\Big(H^0(M,\mathcal{M}) \otimes \mathcal{O}_M \longrightarrow \mathcal{P}^1_M(\mathcal{M})\Big).$$

Thus $\mathcal{F}$ is a vector bundle of rank 2 and one checks that $M' = \mathbb{P}(\mathcal{F})$. Furthermore, we have the following commutative diagram of vector bundles / M,

$$\begin{array}{ccccc}
\mathcal{S} := \mathrm{Sym}_2\Omega^1_M \otimes \mathcal{M} & \rightarrowtail & \mathcal{P}^2_M(\mathcal{M}) & \twoheadrightarrow & \mathcal{P}^1_M(\mathcal{M}) \\
\vdots\uparrow & & \uparrow & & \| \\
\mathcal{F} & \rightarrowtail & \mathcal{O}_M^{\oplus 5} & \twoheadrightarrow & \mathcal{P}^1_M(\mathcal{M})
\end{array}$$

The induced map $\mathcal{F} \longrightarrow \mathcal{S}$ fails to be injective at the *triple* points. There are in general only finitely many such. We postpone their count to (10).

We get the following diagram of projective bundles over M,

$$\begin{tikzcd} \mathbb{P}(\mathcal{F}) \arrow[rr, dotted] \arrow[dr] & & \mathbb{P}(\mathcal{S}) \arrow[dl] \\ & M & \end{tikzcd} \tag{6}$$

where the top map is rational. It is a linear embedding off the triple points. If $P \in M$ is not triple, the fiber $\mathbb{P}(\mathcal{F})_P$ is a pencil of singular hyperplane sections, each displaying a quadratic singularity at P. The image of $\mathbb{P}(\mathcal{F})_P$ in $\mathbb{P}(\mathcal{S})_P$ is a pencil π_P of degree two divisors, each corresponding to the (quadratic) tangent cone of a hyperplane section at P. The pencil π_P cuts the discriminant at two points, each representing a cuspidal section. Thus, a "cuspidal coincidence" occurs when the pencil is tangent to the discriminant. We're again in the setup of (1).

Let Z be the closure of the image of $\mathbb{P}(\mathcal{F})$ in $\mathbb{P}(\mathcal{S})$. The class of Z lives in $A_3(\mathbb{P}(\mathcal{S}))$. Since the fibers of $\mathbb{P}(\mathcal{S}) \to M$ are two dimensional, restricting Z over the complement of the finetely many triple points causes no harm as far as computing that class goes. Hence, setting

$$\kappa = c_1\mathcal{O}_{\mathcal{S}}(1), \qquad C = c_1\mathcal{M}, \qquad K = c_1\Omega^1_M, \tag{7}$$

we may write

$$\begin{aligned} Z &= c_1((\mathcal{S}/\mathcal{F}) \otimes \mathcal{O}_{\mathcal{S}}(1)) \\ &= \kappa + 6C + 4K. \end{aligned}$$

As in (2), the class of the Veronese $Y = \mathbb{P}(\Omega^1_M) \subset \mathbb{P}(\mathrm{Sym}_2\Omega^1_M \otimes \mathcal{M}) = \mathbb{P}(\mathcal{S})$ may be computed as

$$\begin{aligned} Y &= 2(K + c_1(\mathcal{O}_{\mathcal{S}}(1) \otimes \mathcal{M})) \\ &= 2(K + C + \kappa). \end{aligned}$$

Substituting in (1) and pushing forward to M, we have shown the following.

Proposition 4.1. *The class of the curve of cuspidal coincidence of a general surface $M \subset \mathbb{P}^4$ is equal to*

$$2(5C + 3K).$$

□

Example 4.2. Take M to be a Del Pezzo surface in $\mathbb{P}^4$. This is $\mathbb{P}^2$ blown up at five points, embedded by the system of cubics through the five points. It is the intersection of two quadrics in $\mathbb{P}^4$. We have $K = -3\ell + e_1 + \cdots + e_5$, $C = 3\ell - (e_1 + \cdots + e_5)$, where ℓ is the class of a line in $\mathbb{P}^2$ and the e_i are the exceptional lines. Substituting in (4.1), we find $2(5C + 3K) = 2(6\ell - 2e_1 - \cdots) = 4C$, class of a curve of degree $4C^2 = 16$.

5. Iteration: tacnodes, etc.

We say a point P in a curve $C \subset M$ is a **tacnode** if P is a double point with an infinitely near double point. Precisely, let

$$b_P : M'_P \longrightarrow M$$

be the blow up at P. Put $E_P = b_P^{-1}P$. The total transform of C is the Cartier divisor $b_P^{-1}C = 2E_P + \widetilde{C}$, where $\widetilde{C}$ denotes the strict transform. The double point $P \in C$ is a tacnode if there is a double point of $\widetilde{C}$ lying on the exceptional line E_P.

We define the **tacnodal locus of a surface** $M \subset \mathbb{P}^4$ as the set of points $P \in M$ that are a tacnode of some hyperplane section. It will shortly be given a global description.

In order to handle the imposition of several (possibly infinitely near) singularities we recast the construction given in (5) for M' in a slightly more general setting, cf. [7].

Our starting point is as in §3: a smooth map $f : \mathbb{M} \to S$ together with a section $\sigma^{\mathbb{D}} : \mathcal{O}_{\mathbb{M}} \to \mathcal{L}$ of an invertible $\mathcal{O}_{\mathbb{M}}$−module. We write $\mathbb{D}$ for the zeros of $\sigma^{\mathbb{D}}$. This is an effective Cartier divisor wherever $\sigma^{\mathbb{D}}$ is locally a nonzero divisor. The extra bit of generality is required to allow $\mathbb{M}$ to be possibly reducible as well as that some fibers of $\mathbb{D}$ to be the whole fiber of f. In the iteration procedure described in (11), even if at the outset $\mathbb{M}$ is integral, the next ambient space may be reducible. We assume throughout S is a local complete intersection. We think of S as a space of parameters and the fibers $\mathbb{M}_s = f^{-1}s$, $s \in S$ of f as a family of smooth varieties together with a family of hypersurfaces $\mathbb{D}_s \subseteq \mathbb{M}_s$. Some care must be taken to ensure the set where $\mathbb{D}_s = \mathbb{M}_s$ is not too big.

Example 5.1. In our setup for studying singular hyperplane sections of a surface $M \subset \mathbb{P}^4$ we take $\mathbb{M} = M \times \check{\mathbb{P}}^4$, $S = \check{\mathbb{P}}^4$, $\mathcal{L} = \mathcal{M} \otimes \mathcal{O}_{\check{\mathbb{P}}^4}(1)$, and $\mathbb{D} = \{(P,h) \in M \times \check{\mathbb{P}}^4 \mid P \in h\}$.

Let $b : \mathbb{M}^{(2)} \longrightarrow \mathbb{M} \underset{S}{\times} \mathbb{M}$ be the blowup of the diagonal $\Delta_{\mathbb{M}}$. Let b_i be the composition of b with the projection $p_i : \mathbb{M} \underset{S}{\times} \mathbb{M} \to \mathbb{M}$. The restriction of $(p_2 b)$ to the fiber $b_1^{-1} P$ of $\mathbb{M}^{(2)}$ over $P \in \mathbb{M}$ is equal to the blowup $b_P : \mathbb{M}_s^{(1)}{}_P \to \mathbb{M}_s$ at P of the fiber of $\mathbb{M}$ over $s = f(P)$. Thus $\mathbb{M}^{(1)}$ is the total space of the family of all blowups of fibers of $\mathbb{M} \to S$ at each of its points. We write $\mathbb{E} = b^{-1}\Delta_{\mathbb{M}}$ for the exceptional divisor of $\mathbb{M}^{(2)}$ and $\mathbb{E}_P = b_P^{-1} P$ for the exceptional divisor of $\mathbb{M}_s^{(2)}{}_P$. We consider the following diagram

$$
\begin{tikzcd}
 & \mathbb{E} \subset \mathbb{M}^{(2)} \supset (p_2 b)^{-1}\mathbb{D} \arrow[ddl, "p_1 b"'] \arrow[d, "b"] \arrow[ddr, "p_2 b"] & \\
 & \mathbb{M} \underset{S}{\times} \mathbb{M} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathbb{M} \arrow[dr, "f"'] & & \mathbb{M} \supset \mathbb{D} \arrow[dl, "f"] \\
 & S &
\end{tikzcd}
$$

Now P is a point of multiplicity $\geq m$ in the divisor $\mathbb{D}_s \subset \mathbb{M}_s$ if and only if the Cartier divisor $b_P^{-1}\mathbb{D}_s - m\mathbb{E}_P$ is effective. We call $b_P^{-1}\mathbb{D}_s - m\mathbb{E}_P$ the $m-$**virtual transform** of $\mathbb{D}_s$ at P. It is equal to the strict transform if P is a point of multiplicity m in $\mathbb{D}_s$.

We define

$$
\begin{aligned}
\mathbb{D}(m) \; &:= (p_2 b)^{-1}\mathbb{D} - m\mathbb{E} \\
\Sigma(m;\mathbb{D}) \; &:= \; \{P \in \mathbb{M} \mid \mathbb{D}(m)_P \; = \; b_P^{-1}\mathbb{D}_{f(P)} - m\mathbb{E}_P \text{ is effective}\}.
\end{aligned} \tag{8}
$$

Next we endow $\Sigma(m;\mathbb{D})$ with a closed subscheme structure so that it is precisely the locus in $\mathbb{M}$ over which $\mathbb{D}(m)$ is an effective Cartier divisor. This is achieved by describing $\Sigma(m;\mathbb{D})$ in two different ways as a scheme of zeros of a section of a suitable sheaf. Put

$$
\sigma_m^{\mathbb{D}} : \; \mathcal{O}_{\mathbb{M}^{(2)}} \longrightarrow \mathcal{O}_{m\mathbb{E}} \otimes (p_2 b)^* \mathcal{L}. \tag{9}
$$

This is the section induced by pulling back the section $\mathcal{O} \to \mathcal{L}$ defining $\mathbb{D}$ and restricting it to "infinitesimal neighborhood" $m\mathbb{E} \subset \mathbb{M}^{(2)}$. By construction, $\sigma_m^{\mathbb{D}}$ vanishes over the fiber of b_1 over P iff the virtual transform $\mathbb{D}(m)_P$ is effective iff $P \in \Sigma(m; \mathbb{D})$. As in [7] one checks that

1. *the scheme of zeros of* $\sigma_m^{\mathbb{D}}$ *along the fibers of* b_1 *exists;*
2. *it coincides with the scheme of zeros of the section of the direct image sheaf* $b_{1*}(\mathcal{O}_{m\mathbb{E}} \otimes (p_2 b)^*\mathcal{L})$;*;*
3. *the latter sheaf is equal to the sheaf of relative principal parts of order* $m-1$, $\mathcal{P}^{m-1}_{\mathbb{M}/S}(\mathcal{L})$.

The first assertion is crucial for the iteration process that allows us to impose further singularities, cf. (11) below. The second one is useful for the computations of homology classes. Indeed, we get that $\Sigma(m; \mathbb{D})$ represents the top Chern class of the sheaf of principal parts,

$$\Sigma(m; \mathbb{D}) = c_r \mathcal{P}^{m-1}_{\mathbb{M}/S}(\mathcal{L})$$

provided $\Sigma(m; \mathbb{D})$ is of the right codimension $r = \operatorname{rank} \mathcal{P}^{\mathrm{m}-1}_{\mathbb{M}/\mathrm{S}}(\mathcal{L})$ in $\mathbb{M}$.

Example 5.2. Notation as in (5.1) and (5), we have $\Sigma(\mathbb{D}, 2) = M'$. Set $\check{H} = c_1 \mathcal{O}_{\check{\mathbb{P}}^4}(1)$. Keeping the notation of (7), we have $\mathcal{L} = \mathcal{M} \otimes \mathcal{O}_{\check{\mathbb{P}}^4}(1)$ and we may compute,

$$\begin{aligned} M' &= c_3(\mathcal{P}^1_M(\mathcal{L})) \\ &= c_2(\mathcal{P}^1_M)(C + \check{H}) + c_1(\mathcal{P}^1_M)(C + \check{H})^2 + (C + \check{H})^3 \\ &= \check{H}^3 + (K + 3C)\check{H}^2 + (2CK + 3C^2 + \chi)\check{H} \end{aligned}$$

for the class of M' in $A(M \times \check{\mathbb{P}}^4)$. Note *en passant* that its image in $A(\check{\mathbb{P}}^4)$ is $\check{H}$ times the degree $\int_M (2CK + 3C^2 + \chi)$. This retrieves the well known formula for the **degree of the dual variety of** M.

Example 5.3. Planar points. For a linear system of dimension 4, one expects finitely many members displaying a triple point. Their number is given by

$$c_2 \mathcal{P}^2_M(\mathcal{M}) = 5K^2 + 20CK + 15C^2 + 5\chi. \tag{10}$$

For instance, take the system of plane cubics through five points. The triple points are given by the configurations of three concurrent lines containing the five points. There are just 15 of these, in agreement with $5 \cdot 9 - 20 \cdot 9 + 15 \cdot 9 + 5 \cdot 3$ prescribed by (10).

For a Del Pezzo surface in $\mathbb{P}^4$ (cf. 4.2), the formula gives 40. There are 15 due to the configurations of cubics with a triple point. Five more from the conic through the five points with the tangent line at each. The remaining 20 are accounted for by the choice of line through two points and conic through the remaining three and tangent to the line at either marked point.

Continuing with the description of the iteration procedure, note that the section $\sigma_m^{\mathbb{D}}$ defined in (9) fits into the following diagram,

$$\begin{tikzcd}
 & & \mathcal{O}_{\mathbb{M}^{(2)}} \arrow[d] \arrow[dll, dotted, "\widetilde{\sigma}"'] \arrow[dr, "\sigma_m^{\mathbb{D}}"] & \\
0 \to \mathcal{O}_{\mathbb{M}^{(2)}}(\mathbb{D} - m\mathbb{E}) & \longrightarrow & \mathcal{O}_{\mathbb{M}^{(2)}}(\mathbb{D}) \arrow[r] & \mathcal{O}_{m\mathbb{E}}(\mathbb{D}) \to 0
\end{tikzcd}$$

with horizontal sequence exact. Restricting it over $\Sigma(m;\mathbb{D})$ kills the section $\sigma_m^{\mathbb{D}}$ thereby inducing the dotted arrow in the above diagram.

Put $S' = \Sigma(m;\mathbb{D})$ and $\mathbb{M}' = b_1^{-1}\Sigma(m;\mathbb{D})$. Let $f' : \mathbb{M}' \to S'$ be induced by the restriction of b_1. Let $\mathbb{D}'$ be the zeros of $\widetilde{\sigma}$. All in all, we got again a smooth family together with a section of an invertible sheaf. The fibers of the new family are blowups of fibers of the initial family $\{\mathbb{M}_s\}_{s\in S}$ at each point where a member of the family $\{\mathbb{D}_s\}_{s\in S}$ is of multiplicity $\geq m$.

Now, if we specify another multiplicity m', we may set

$$\Sigma(m, m';\mathbb{D}) \ := \ \Sigma(m';\mathbb{D}').$$

By the same token, more generally, given a sequence $\underline{m} = m_1, \ldots, m_r$, we may construct inductively

$$\Sigma(\underline{m};\mathbb{D}) \ := \ \Sigma(m_2, \ldots, m_r;\mathbb{D}'). \tag{11}$$

Note that $\Sigma(m, m';\mathbb{D})$ is a closed subscheme of $\mathbb{M}^{(2)}$. Similarly, $\Sigma(m, m', m'';\mathbb{D})$ is a closed subscheme of the scheme obtained by blowing up the diagonal of $\mathbb{M}^{(2)} \underset{\mathbb{M}}{\times} \mathbb{M}^{(2)}$ along $\Delta_{\mathbb{M}^{(2)}}$ and so on. A point in $\Sigma(\underline{m};\mathbb{D})$ may be thought of as sequence $P_1, P_2, \ldots$ such that P_1 lies in $\mathbb{M}$ and is at least m_1–fold in $\mathbb{D}_s$, $(s = f(P_1))$, P_2 lies in $\mathbb{M}^{(2)}_{P_1}$ (the blowup of $\mathbb{M}_s$ at P_1) and is at least m_2–fold in the effective divisor $b_{P_1}^{-1}\mathbb{D}_s - m_1\mathbb{E}_{P_1}$, and so on.

Notation 5.4. Set $\mathbb{M}^{(0)} = S$, $\mathbb{M}^{(1)} = \mathbb{M}$, $f_1 = f : \mathbb{M}^{(1)} \to \mathbb{M}^{(0)}$. For $r \geq 1$ denote by

$$b_{r+1} : \mathbb{M}^{(r+1)} \to \mathbb{M}^{(r)} \times_{f_r} \mathbb{M}^{(r)} \quad \text{and} \quad p_{r+1,i} : \mathbb{M}^{(r)} \times_{f_r} \mathbb{M}^{(r)} \to \mathbb{M}^{(r)}$$

respectively the blowup of the diagonal and the projection. Set $f_{r+1,i} = p_{r+1,i} \circ b_{r+1}$. We think of each $\mathbb{M}^{(r)}$ as a scheme over $\mathbb{M}^{(r-1)}$ with structure map $f_r = f_{r,1}$. Write $\mathbb{E}_{1,r}$ for the exceptional divisor of b_r. For $2 \leq j < r$ set $\mathbb{E}_{r-j+1,j} = f^*_{r,2} \cdots f^*_{j+1,2}\mathbb{E}_{1,j}$. By abuse, still denote by the same symbol pullbacks of $\mathbb{E}_{r-j+1,j}$ via compositions of the structure maps $f_3, f_4, \ldots$. We also define the divisor on $\mathbb{M}_{r+1}$,

$$\underline{m}\mathbb{E} = m_r\mathbb{E}_{1,r+1} + \cdots + m_2\mathbb{E}_{r-1,3} + m_1\mathbb{E}_{r,2}.$$

We set

$$\mathcal{L}(\underline{m}) = f^*_{r+1,2} \cdots f^*_{2,2}\mathcal{L} \otimes \mathcal{O}_{\mathbb{M}_{r+1}}(-\underline{m}\mathbb{E}).$$

We quote from [7] the following.

Proposition 5.5. *There are locally free $\mathcal{O}_{\mathbb{M}^{(r)}}-$modules denoted $\mathcal{E}(\underline{m};\mathcal{L})$ endowed with a section $\sigma^{\mathbb{D}}_{\underline{m}}$ such that the following holds.*

1. $\Sigma(\underline{m};\mathbb{D})$ *is the scheme of zeros of* $\sigma^{\mathbb{D}}_{\underline{m}}$.

2. *There are exact sequences,*

$$0 \longrightarrow \mathcal{E}(m_r;\mathcal{L}(\underline{m}')) \longrightarrow \mathcal{E}(\underline{m};\mathcal{L}) \longrightarrow f_r^*\mathcal{E}(\underline{m}';\mathcal{L}) \longrightarrow 0,$$

   where $\underline{m}'$ *denotes the truncated sequence* $m_1, \ldots, m_{r-1}$.

3. $$\mathcal{E}(m;\mathcal{L}) = \mathcal{P}^{m-1}_{\mathbb{M}/S}(\mathcal{L}).$$

4. $$\operatorname{rank}\mathcal{E}(\underline{m};\mathcal{L}) = \Sigma\binom{dim f + m_i - 1}{dim f}.$$

5. $\Sigma(\underline{m};\mathbb{D})$ *is of codimension* $\leq \rho := \operatorname{rank}\mathcal{E}(\underline{m};\mathcal{L})$ *in* $\mathbb{M}^{(r)}$ *and if equality holds then*

$$\Sigma(\underline{m};\mathbb{D}) = c_\rho\,\mathcal{E}(\underline{m};\mathcal{L}) \quad \text{in } A(\mathbb{M}^{(r)}).$$

Tacnodes 5.6. We may now describe the **scheme of tacnodes** as

$$\mathcal{T}_{\mathcal{AC}} := \mathbb{E} \cap \Sigma(2,2,\mathbb{D}) \subset \mathbb{M}^{(2)}$$

Its class in $A(\mathbb{M}^{(2)})$ is given by

$$\mathcal{T}_{\mathcal{AC}} := \mathbb{E} \cdot c_3 \mathcal{P}^1_{\mathbb{M}^{(2)}/\mathbb{M}}(\mathcal{L}(2)) \cdot c_3 \mathcal{P}^1_{\mathbb{M}/S}(\mathcal{L}) \tag{12}$$

We register for later use the following formulas for the Chern classes:

$$c_3 \mathcal{P}^1_{\mathbb{M}/S}(\mathcal{O}(\mathbb{D})) = \mathbb{D}^3 + K\mathbb{D}^2 + \mathbb{D}\chi,$$

with K, χ as in (3). In the case of a linear system on a surface M, the parameter space $S = \check{\mathbb{P}}^m$ and $\mathbb{M} = M \times \check{\mathbb{P}}^m$, with $\mathcal{L} = \mathcal{M} \otimes \mathcal{O}_{\check{\mathbb{P}}^m}(1) = \mathcal{O}(C + \check{H})$.

$$\begin{aligned} c_t \mathcal{P}^1_{\mathbb{M}/S}(\mathcal{M}) = \;& (2CK + 3C^2 + \chi)t^2 + (K + 3C)t + 1 \\ c_t \mathcal{P}^1_{\mathbb{M}^{(2)}/\mathbb{M}}(\mathcal{M}(2)) = \;& ((7C + 6K)\mathbb{E}^2 - (4CK + 5C^2)\mathbb{E})t^3 + \\ & ((p_2 b)^*(\chi + 2CK + 3C^2) - \mathbb{E}(10C + 4K) + 7\mathbb{E}^2)t^2 + \\ & (-5\mathbb{E} + 3(p_2 b)^*(C + K))t + 1 \end{aligned} \tag{13}$$

Example 5.7. Tacnodal members of a web. We apply (12) to a web, e.g., a system of plane sections of a surface $M \subset \mathbb{P}^3$. Notation as in (5.2), using (13) we may find the pushforward of $\mathcal{T}_{\mathcal{AC}}$ under $M^{(2)} \times \check{\mathbb{P}}^4 \xrightarrow{p} M^{(2)} \xrightarrow{b_1} M$:

$$\begin{aligned} \overline{\mathcal{T}_{\mathcal{AC}}}_0 &= b_{1*}\Big(\mathbb{E} \cdot p_* c_6(\mathcal{P}^1_{\mathbb{M}/S}(\mathcal{L}) + \mathcal{P}^1_{\mathbb{M}^{(2)}/\mathbb{M}}(\mathcal{L}(2)))\Big) \\ &= b_{1*}\Big(\mathbb{E} \cdot c_3(\mathcal{P}^1_M(\mathcal{M}) + \mathcal{P}^1_{M^{(2)}/M}(\mathcal{M}(2)))\Big) \\ &= 64CK + 50C^2 + 17K^2 + 5\chi. \end{aligned}$$

For a web of plane cubics passing through six points we have $C = 3\ell$, $K = -3\ell$, $\chi = 3\ell$. We find the number $64(-9) + 50 \cdot 9 + 17 \cdot 9 + 5 \cdot 3 = 42$. Note that a general plane cubic is a tacnode if and only if it is the union of a conic with a tangent line. Thus, the possible configurations for tacnodal cubics through 6 general points are given by

1. $\binom{6}{5}$ (conics through 5 of the points) times 2 (tangent lines from the sixth point) =12;

2. $\binom{6}{2}$ (lines through 2 of the points) times 2 (tangent conics through the remaining four points) =30

total=42.

For a cubic surface, we find 54, twice the number of lines. For each line, the pencil of planes it defines cuts the cubic in a residual conic. Two of these are tangent to the fixed line, as it can be seen from (1). Indeed, take the parameter space $S = \mathbb{P}^1$, $L \subset M$ a line in a cubic surface, $\mathbb{M} = M \times S$, $Y = L \times S$ and $Z =$ total space for the family of residual conics. Put $\lambda = c_1 \mathcal{O}_{\mathbb{P}^1}(1)$. We may write $Y + Z = C + \lambda =$ class in $\mathbb{M}$ of the total space for the pencil of plane sections. We also have $c_1(\mathcal{T}_{\mathbb{M}/S}) = c_1 \mathcal{T}_M = -K = C$. Thus (1) now reads

$$L \cdot (C + \lambda - L) \cdot (C + \lambda - C) = \lambda \cdot L \cdot (C - L)$$

which is a zero cycle of degree 2 because $L \cdot C = 1$ and $L^2 = -1$.

Example 5.8. Tacnodal line. We apply (12) to a linear system of dimension 4, e.g., the hyperplane sections of a surface $M \subset \mathbb{P}^4$. Notation as in the previous example, we now compute

$$\begin{aligned} \overline{\mathcal{T}_{A}}c_1 &= b_{1*}\Big(\mathbb{E} \cdot c_2(\mathcal{P}^1_M(\mathcal{M}) + \mathcal{P}^1_{M^{(2)}/M}(\mathcal{M}(2)))\Big) \\ &= 16K + 25C. \end{aligned} \tag{14}$$

Example 5.9. Consider the system of plane cubics through five general points. A tacnode is a configuration consisting of a conic together with a tangent line. So, the tacnodal line will consist of the following.

1. The conic through the five points;

2. For each choice of four among the five points, the tangency locus for the pencil of conics and the pencil of lines through the remaining point. This gives five cubics.

3. For each choice of three of the five points, ask for the curve of tangency coincidences for the net of conics and the fixed line through the remaining two points. This gives ten lines.

We retrieve the right degree $27 = 2 + 5 \cdot 3 + 10 (= 25 \cdot 3 - 3 \cdot 16)$ for the tacnodal line as prescribed by (14). Just take $K = -3\ell$, $C = 3\ell$.

Note that the curve of tangency coincidences referred to in the last item comes out neatly from (1). Just take $S = \check{\mathbb{P}}^2$, the parameter space for the net of conics, $\mathbb{M} = \mathbb{P}^2 \times S$, $Y = L \times S$ (L= fixed line) and Z = total space for the net of conics. We may compute $L \cdot (2L + \check{H}) \cdot (3L + \check{H} - 3L)$. This class pushes forward to L in $\mathbb{P}^2$.

Example 5.10. Node-tacs. We compute the tacnodes of the family obtained by (virtually) resolving one node. We find, setting $d = C^2$,

$$150d^2 + (51(K^2) + 292(CK) - 1260 + 65\chi)d + 5\chi^2 +$$
$$(17(K^2) + 74(CK) - 60)\chi + 128(CK)^2 + (34(K^2) - 1820)(CK) - 596(K^2).$$

For a linear system of plane curves of degree m, this reduces to

$$6(m-3)(25m^3 - 71m^2 - 122m + 280).$$

Example 5.11. Higher cusps. A higher cusp is a cusp with an infinitely near cusp on top of it. Thus one must impose that the strict transform at the cusp be again a cusp at a point in the exceptional line.

Let a point P be a double point of a curve $C \subset M$. Let $b : M'_P \to M$ be the blowup of P end let E_P be the exceptional line. Write $C' = b^*C - 2E_P$ for the strict transform. Suppose a point $P' \in C'$ is a double point lying on E_P. Then P *must* be a cusp since an ordinary double point is resolved by a single blowup. A similar reasoning shows that a triple point does not contribute to the higher cusp locus defined in the sequel.

The above observation leads to the following candidate for the scheme of higher cusps. With the setup as in §5, first form $\Sigma(2, \mathbb{D}) \subset \mathbb{M}$. Then take

$$\mathbb{K} := \kappa(\mathbb{D}') \cap \mathbb{E} \subset \mathbb{M}^{(2)}{}_{|\Sigma(2,\mathbb{D})}.$$

Setting

$$K' = c_1\Omega_{\mathbb{M}^1/\mathbb{M}} = (p_2 b)^* K + \mathbb{E} \qquad \text{and} \qquad \chi' = c_2\Omega_{\mathbb{M}^1/\mathbb{M}} = (p_2 b)^*\chi - \mathbb{E}^2$$

and recalling (4), we may compute

$$\mathbb{K} = 2\mathbb{E}(\mathbb{D}' + K')\mathbb{D}'(\mathbb{D}'^2 + \mathbb{D}'K' + \chi')\mathbb{D}(\mathbb{D}^2 + \mathbb{D}K + \chi)$$

for the class of the higher cusp locus.

When the given family comes from a linear system of dimension 4, we find that the class above pushes forward to the zero cycle

$$\overline{\mathbb{K}} = 20(14CK + 9C^2 + 5K^2).$$

For a system of plane curves of degree m this becomes,

$$\overline{\mathbb{K}}(m) = 60(3m-5)(m-3).$$

For a complete intersection surface in $\mathbb{P}^4$ of bidegree m_1, m_2 we find

$$20m_1m_2(5m_1 + 5m_2 - 16)(m_1 + m_2 - 4).$$

It is comforting to check that for a K3 surface with a polarization of genus 4 (whence $C^2 = 6$), the formula for $\overline{\mathbb{K}}$ gives 1080, the same as the previous one with $m_1 = 2, m_2 = 3$.

Example 5.12. Binode-cusps. We find the cusp locus for the family obtained by the imposition of two double points: take the family of binodes and impose a cusp. Precisely, with the notation as in (5.4) we employ the formula (4) to the following data:

- $\widehat{S} = \Sigma(2, 2, \mathbb{D})$ (replaces S);
- $\widehat{\mathbb{M}} = \mathbb{M}^{(3)}{}_{|\widehat{S}}$ (in place of $\mathbb{M}$) and
- $\widehat{\mathbb{D}} = (f_{3,2}^*\mathbb{D} - 2\mathbb{E}_{1,3} - 2\mathbb{E}_{2,2})_{\widehat{\mathbb{M}}}$ (replaces $\mathbb{D}$).

We find the class

$$2(\widehat{\mathbb{D}} + \widehat{K}) \cdot \widehat{\mathbb{D}} \cdot (\widehat{\mathbb{D}}^2 + \widehat{\mathbb{D}}\widehat{K} + \widehat{\chi}) \qquad \text{in } A(\widehat{\mathbb{M}})$$

where $\widehat{K} = c_1\Omega^1_{\widehat{\mathbb{M}}/\widehat{S}}$ and $\widehat{\chi} = c_2\Omega^1_{\widehat{\mathbb{M}}/\widehat{S}}$.

Employing the formula for $\widehat{S}$ in $\mathbb{M}^{(2)}$ and using MAPLE, we get the following formula for the degree of the zero cycle of binode–cusps:

$$\begin{gathered}
54d^3 + (9(K^2) + 126(CK) - 972 + 45\chi)d^2 + \\
(12\chi^2 + (6(K^2) + 72(CK) - 396)\chi + 96(CK)^2 + \\
(12(K^2) - 1830)(CK) + 4500 - 294(K^2))d + \chi^3 + ((K^2) - 31 + 10(CK))\chi^2 + \\
(28(CK)^2 + (4(K^2) - 417)(CK) - 85(K^2) + 228)\chi + 24(CK)^3 \\
+(4(K^2) - 810)(CK)^2 + (6180 - 219(K^2))(CK) - 6(K^2)^2 + 1932(K^2).
\end{gathered}$$

In particular, for a family of plane curves of degree m passing through the appropriate number of points, we get

$$18(m-3)(3m^5-12m^4-30m^3+125m^2+82m-280).$$

For $m=4$, we retrieve a number first found by Zeuthen. It appears also in [4].

Example 5.13. Bicusps.

We impose a new cusp to the family $\mathbb{D}-2\mathrm{E}$ restricted to $\kappa(\mathbb{D})$. Note that a triple point gives rise to three double points in the family of transforms, hence it does not contribute to the locus of bicusps. A higher cusp however does count as a bicusp, presumably with multiplicity one.

We need to compute the class

$$2(\widetilde{\mathbb{D}}+\widetilde{K})\cdot\widetilde{\mathbb{D}}\cdot(\widetilde{\mathbb{D}}^2+\widetilde{\mathbb{D}}\widetilde{K}+\widetilde{\chi}) \qquad \text{in } A(\widetilde{\mathbb{M}})$$

where

$$\widetilde{\mathbb{D}}=\mathbb{D}-2\mathrm{E}, \quad \widetilde{K}=b_2^*K, \quad \widetilde{\mathbb{M}}=\mathbb{M}^{(2)}_{\mathcal{K}(\mathbb{D})}, \quad \widetilde{K}=c_1\Omega^1_{\widetilde{\mathbb{M}}/\widetilde{S}}, \quad \widetilde{\chi}=c_2\Omega^1_{\widetilde{\mathbb{M}}/\widetilde{S}}.$$

We get

$$72d^2+(24\chi+144(CK)+24(K^2)-450)d+2\chi^2+(4(K^2)+24(CK)-24)\chi$$
$$+72(CK)^2+(24(K^2)-620)(CK)+2(K^2)^2-194(K^2).$$

For a family of plane curves of degree $m\geq 3$ this gives

$$6(m-3)(12m^3-36m^2-27m+85).$$

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