# REAL SINGULARITIES AND DIHEDRAL REPRESENTATIONS

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**Abstract**

The Euler characteristics of some *real* algebraic hypersurfaces determine and are determined by natural dihedral representations on the cohomology of their associated *complex* Milnor fibers.

## 1. Introduction

This paper can be seen as one in a series of papers, see for instance [P], [DHP], in which we intend to investigate the topology of real algebraic varieties by using the extensive existing knowledge about complex singularities. Let $f \in \mathbb{R}[x_0, \cdots, x_n]$ be a weighted homogeneous polynomial of degree $d$ with respect to the weights $\deg(x_j) = w_j$, where $w_j$ are strictly positive integers for $j = 0, \cdots, n$.

Consider the smooth affine hypersurfaces

$$F_{\mathbb{R}} = F_{\mathbb{R}}(f) = \{x \in \mathbb{R}^{n+1}; f(x) = 1\}$$

$$F = F(f) = \{x \in \mathbb{C}^{n+1}; f(x) = 1\}$$

which we call the *real* (resp.*complex) Milnor fiber* of the polynomial $f$.

The monodromy homeomorphism

$$h : F \to F,\ h(x) = (t^{w_0} x_0, \cdots, t^{w_n} x_n)$$

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where $t = \exp(2\pi i/d)$ and the induced *monodromy operator* $h_* : \widetilde{H}_*(F) \to \widetilde{H}_*(F)$, where $\mathbb{C}$-coefficients are used for (co)homology if not explicitly mentioned otherwise, are familiar objects of study, see for instance [M], [M Or], [AGV], [D2]. In particular, when $f$ has an isolated singularity at the origin, then the pair $(\widetilde{H}_*(F), h_*)$ is completely determined by the weighted homogeneity type $(\underline{w}; d) = (w_0, \cdots, w_n; d)$, see [M Or].

On the other hand, the topology of the real Milnor fiber $F_{\mathbb{R}}$ is much more complicated to describe, e.g. cannot be derived just from the type $(\underline{w}; d)$.

**Example 1.** Any quadratic polynomial $f$ ($w_0 = \cdots = w_n = 1$, $d = 2$) is linearly equivalent over $\mathbb{R}$ to one of the normal forms $f_p$, $p = -1, 0, \cdots, n$, where $f_p(x) = x_0^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2$, see [D1] p.46. It is easy to show that $F_{\mathbb{R}}(f_p)$ has the homotopy type of the sphere $S^p$, compare to Remark 11 below. Here we use the convention $S^{-1} = \emptyset$.

The aim of this paper is to show that there is a natural dihedral representation on the reduced homology $\widetilde{H}_*(F)$ of the complex Milnor fiber of $f$ combining the action of the monodromy with the complex conjugation.

Let

$$K_{\mathbb{R}} = K_{\mathbb{R}}(f) = \{x \in S^n \subset \mathbb{R}^{n+1}; f(x) = 0\}$$

be the *real link* of the polynomial $f$. When $f$ is a homogeneous polynomial (i.e. $w_0 = \cdots = w_n = 1$) we consider also the real projective hypersurface defined by $f$, namely

$$V_{\mathbb{R}} = V_{\mathbb{R}}(f) = \{x \in \mathbb{R}P^n; f(x) = 0\}.$$

The understanding of the topology of such hypersurfaces $V_{\mathbb{R}}$ is a central problem in mathematics, going back to Hilbert $16^{th}$ problem, see for instance the excellent survey by Gudkov [G] and also [BCR], [Si].

Assuming the zeta function of the monodromy operator $h_*$ known, as is often the case by [AC], [MOr], [D2] $\cdots$, it turns out that the equivariant Euler characteristic of $F$ with respect to the dihedral representation introduced here determines (and is determined by) the Euler numbers $\chi(F_{\mathbb{R}})$ and $\chi(K_{\mathbb{R}})$, see Cor. 19.

In the case of an isolated singularity $f$, the equivariant Euler characteristic of $F$ satisfies some additional properties and, as a result, is determined just by $\chi(F_{\mathbb{R}})$.

One of the main advantages of our method is that this dihedral representation behaves well under the Thom-Sebastiani construction, in a similar way to $\chi(F)$, $\chi(F_{\mathbb{R}})$ but unlike $\chi(K_{\mathbb{R}})$ or $\chi(V_{\mathbb{R}})$, see Example 24 for some explicit computations along this line.

## 1. Review of dihedral representations

In this section we recall the basic facts on dihedral groups and their representations following Serre [Se], pp. 36-38. The dihedral group $D_m$ is a finite group of order $2m$ and has the following presentation

$$D_m = \langle r, s; r^m = s^2 = srsr = 1 \rangle.$$

To discuss the irreducible representations of the dihedral group $D_m$ we have to consider two cases according to the parity of $m$.

**Case A:** ($m$ even $\geq 2$)

In this case there are 4 irreducible representations of degree 1, given by the following obvious table :

**Table 2**

| | $r$ | $s$ |
|---|---|---|
| $\psi_1$ | 1 | 1 |
| $\psi_2$ | 1 | $-1$ |
| $\psi_3$ | $-1$ | 1 |
| $\psi_4$ | $-1$ | $-1$ |

Next there are $m/2 - 1$ irreducible representations of degree 2 given by

$$\rho^h(r) = \begin{pmatrix} \lambda^h & 0 \\ 0 & \overline{\lambda}^h \end{pmatrix}, \ \rho^h(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $0 < h < m/2$ and $\lambda = \exp(2\pi i/m)$.

Actually the above formulas define a $D_m$-representation for any $h \in \mathbb{Z}$ such that

$$\begin{cases} \rho^h = \rho^{m-h} = \rho^{h+m} \\ \rho^0 = \psi_1 \oplus \psi_2 \\ \rho^{m/2} = \psi_3 \oplus \psi_4 \end{cases} \tag{3}$$

where "=" means isomorphic representations and " $\oplus$ " denotes the direct sum of representations.

**Case B:** ($m$ odd)

In this case there are only 2 irreducible representations of degree 1, namely $\psi_1$ and $\psi_2$ from Table 2.

And there are $(m-1)/2$ irreducible representations of degree 2, given by $\rho^h$ above for $0 < h < m/2$.

The following result has a trivial proof using Ex. 5.2 in (5.3) from [Se]. Here $\otimes$ denotes the tensor product of two representations.

**Lemma 4.**

(i) $\psi_j \otimes \psi_j = \psi_1, \;\; \psi_1 \otimes \psi_j = \psi_j$ *for* $\;\; j = 1, 2, 3, 4$

$\psi_2 \otimes \psi_3 = \psi_4, \;\; \psi_2 \otimes \psi_4 = \psi_3, \;\; \psi_3 \otimes \psi_4 = \psi_2$

(ii) $\rho^h \otimes \rho^k = \rho^{h+k} \oplus \rho^{h-k}$ *for any* $h, k \in \mathbb{Z}$ ;

(iii) *If $V$ and $W$ are irreducible $G$-representations and* $\dim V = 1$ *then $V \otimes W$ is an irreducible $G$-representation for any group $G$.*

## 2. Dihedral representations on the Milnor fiber

Since $f$ is a polynomial with real coefficients, we can consider the complex conjugation map

$$c : F \to F, c(x) = \overline{x}.$$

The monodromy homeomorphism $h$ and the complex conjugation homeomorphism $c$ satisfy the following easy-to-check relations

$$h^d = c^2 = chch = 1.$$

In other words, there is a group homomorphism $\psi : D_d \to Homeo(F)$ defined by $\psi(r) = h$, $\psi(s) = c$ where $D_d$ is the $d$-dihedral group with the presentation given in section 1, and $Homeo(F)$ is the group of all the homeomorphisms of the complex Milnor fiber $F$.

Using the natural group homomorphism $Homeo(F) \to Aut(\widetilde{H}_*(F))$ we get thus the following basic result.

**Proposition/Definition 5.** The complex Milnor fiber $F$ of a real weighted homogeneous polynomial $f$ of degree $d$ has a natural $D_d$-action. This induces a dihedral representation on the graded vector space $\widetilde{H}_*(F)$ called the *dihedral Milnor representation*.

**Example 6.**

(i) $f(x) = x_0^d, (n = 0)$. It is well-known that the monodromy operator $h_*$ has in this case the following characteristic polynomial

$$\Delta(t) = \det(tI - h_*) = (t^d - 1)/(t - 1) \tag{7}$$

see for instance [M], [M Or].

To describe the corresponding dihedral Milnor representation on $\widetilde{H}_*(F) = \widetilde{H}_0(F) = \mathbb{C}^{d-1}$ we have to consider two cases.

(a) $d$ odd, $d \geq 3$.

Using (7) it follows that the dihedral Milnor representation in this case is (isomorphic to) the direct sum

$$\rho^1 \oplus \rho^2 \oplus \cdots \oplus \rho^{(d-1)/2}$$

(b) $d$ even, $d \geq 2$.

In this case $h_*$ has a real eigenvalue, namely $-1$. The corresponding dihedral Milnor representation is the sum

$$\psi_3 \oplus \rho^1 \oplus \cdots \oplus \rho^{d/2-1}.$$

More precisely, let $\epsilon_k = \exp(2\pi ik/d)$ be the $d$-th roots of unity. Then a basis for the vector space $\widetilde{H}_0(F)$ is given by $\alpha_k = \epsilon_k - \epsilon_{k+1}$ for $k = 0, 1, \cdots, d-2$.

The vector corresponding to the representation $\psi_3$ above is

$$\alpha = \alpha_0 + \alpha_2 + \cdots + \alpha_{d-2}.$$

Indeed, one has $h_*(\alpha) = -\alpha$ and $c_*(\alpha) = \alpha$.

(ii) $f(x) = -x_0^d$

As in (i) above, there two cases to consider

(a) $d$ odd, $d \geq 3$. The corresponding dihedral Milnor representation is again

$$\rho^1 \oplus \cdots \oplus \rho^{(d-1)/2}.$$

This can be proved as above, or by using the real homeomorphism

$$\sigma : F(x_0^d) \to F(-x_0^d), \sigma(x) = -x$$

(b) $d$ even, $d \geq 2$.

The corresponding dihedral Milnor representation is now

$$\psi_4 \oplus \rho^1 \oplus \cdots \oplus \rho^{d/2-1}.$$

Indeed, the elements $\beta_k = \tilde{\epsilon}_k - \tilde{\epsilon}_{k+1}$ form a basis for $\widetilde{H}_0(F)$ in this case, where $\tilde{\epsilon}_k = \exp((2k+1)\pi i/d)$. The vector

$$\beta = \beta_0 + \beta_2 + \cdots + \beta_{d-2}$$

satisfies the relations $h_*(\beta) = c_*(\beta) = -\beta$. This explains why the representation $\psi_4$ occurs here.

Let $f \in \mathbb{R}[x_0, \cdots, x_n]$ and $g \in \mathbb{R}[y_0, \cdots, y_m]$ be two weighted homogeneous polynomials of the same degree $d$ (note that this last property can always be achieved just by changing the weights of $f$ and $g$ by suitable factors!).

Then Oka's version [O] of the Thom-Sebastiani construction [ST] tells us that the Milnor fiber $F(f+g)$ is homotopy equivalent to the join $F(f)*F(g)$ of the "partial" Milnor fibers $F(f)$ and $F(g)$.

Passing to reduced homology, this gives the following formula (up-to a shift in grading!)

$$\widetilde{H}_*(F(f+g)) \simeq \widetilde{H}_*(F(f)) \otimes \widetilde{H}_*(F(g)) \tag{8}$$

Using the explicit form of the homotopy equivalence $j : F(f)*F(g) \to F(f+g)$ given by Oka [O], it follows that $j_*$ commutes with both the monodromy operators (fact already stated in [O]) and complex conjugation morphisms $c_*$. This gives us the following

**Proposition 9.** The dihedral Milnor representation on $\widetilde{H}_*(F(f+g))$ is isomorphic to the tensor product of the dihedral Milnor representations on $\widetilde{H}_*(F(f))$ and $\widetilde{H}_*(F(g))$.

**Example 10.** Consider again the polynomial $f_p$ from Example 1. Using Example 6 and Prop. 9 repeatedly, we get that the dihedral Milnor representation on the space $\widetilde{H}_*(F(f_p)) = \widetilde{H}_n(F(f_p)) = \mathbb{C}$ is isomorphic to

$$\underbrace{\psi_3 \otimes \cdots \otimes \psi_3}_{(p+1)-times} \otimes \underbrace{\psi_4 \otimes \cdots \otimes \psi_4}_{(n-p)-times}$$

which is $\psi_1$ when $p+1$ and $n-p$ are both even, $\psi_3$ when $p+1$ odd and $n-p$ even, $\psi_4$ when $p+1$ even and $n-p$ odd and, finally, $\psi_2$ when $p+1$ and $n-p$ are both odd. (Use Lemma 4, (i)).

**Remark 11.** The real Milnor fibers behave in a similar way under the Thom-Sebastiani construction.

For $f$ and $g$ as above, we introduce the notations

$$F^+(f) = F_{\mathbb{R}}(f), F^-(f) = F_{\mathbb{R}}(-f)$$

(and similarly for $g$).

Then using similar arguments to [O] one can prove the following.

(i) If $F^+(f) \neq \emptyset$ and $F^+(g) \neq \emptyset$, then there is a homotopy equivalence

$$F^+(f) * F^+(g) \to F^+(f+g)$$

(ii) If $F^+(f) \neq \emptyset$ and $F^+(g) = \emptyset$, then there is a homotopy equivalence

$$F^+(f) \to F^+(f+g)$$

(iii) If $F^+(f) = F^+(g) = \emptyset$, then $F^+(f+g) = \emptyset$.

Note that there are similar properties for $F^-(f)$.

As an example, using (ii) for $f = x_0^2 + \cdots + x_p^2$ and $g = -x_{p+1}^2 - \cdots - x_n^2$ it follows that the real Milnor fiber $F_{\mathbb{R}}(f_p)$ has the homotopy type of $S^p$ as claimed in Example 1.

## 3. Equivariant Euler characteristics

To simplify the notation, let $G$ denote the dihedral group $D_d$. Then we can consider the equivariant Euler characteristic of the Milnor fiber $F$ given by

$$\chi_G(F) = \sum_{j=0}^{n} (-1)^j [H_j(F)] \tag{12}$$

the classes [ ] being evaluated in the complex representation ring $R(G)$, see Wall [W].

It is convenient to introduce also the reduced equivariant Euler characteristic $\widetilde{\chi}_G(F)$ defined by the equality

$$\chi_G(F) = \psi_1 + (-1)^n \widetilde{\chi}_G(F) \tag{13}$$

Using Prop. 9 it follows that

$$\widetilde{\chi}_G(F(f+g)) = \widetilde{\chi}_G(F(f)) \otimes \widetilde{\chi}_G(F(g)) \tag{14}$$

i.e. $\widetilde{\chi}_G$ has a nice multiplicative property with respect to the Thom-Sebastiani construction.

Consider now the decomposition of $\widetilde{\chi}_G(F)$ into irreducible G-representations

$$\widetilde{\chi}_G(F) = a_1\psi_1 + a_2\psi_2 + a_3\psi_3 + a_4\psi_4 + \sum_h b_h\rho^h \tag{15}$$

where $a_j, b_h$ are integers. The usual Lefschetz fixed point formula [Sp] applied to $c$ gives the following

$$\chi(F_{\mathbb{R}}) = 1 + (-1)^n(a_1 - a_2 + a_3 - a_4) \tag{16}$$

Let $X = \{x \in \mathbb{R}^{n+1}; f(x) = 0\}$, $S^n$ be the unit sphere in $\mathbb{R}^{n+1}$ and $K_{\mathbb{R}} = X \cap S^n$ be the real link of our polynomial $f$ as in the introduction.

**Proposition 17.**

$$\chi(K_{\mathbb{R}}) = 1 + (-1)^{n+1} + 2(a_2 - a_1)$$

**Proof.** Using the $\mathbb{R}^*_+$-action on $\mathbb{R}^{n+1}$ associated with the weights of $f$, we see that the spaces $\mathbb{R}^{n+1} \setminus X$ and $S^n \setminus K_{\mathbb{R}}$ have the same homotopy type. Hence, using Alexander duality, [Sp] p.296, we get

$$\begin{aligned}\chi(\mathbb{R}^{n+1} \setminus X) &= \chi(S^n \setminus K_{\mathbb{R}}) = (-1)^n\chi(S^n, K_{\mathbb{R}}) \\ &= (-1)^n(\chi(S^n) - \chi(K_{\mathbb{R}})) = (-1)^{n+1}\chi(K_{\mathbb{R}}) + 1 + (-1)^n.\end{aligned}$$

On the other hand, the space $\mathbb{R}^{n+1} \setminus X$ has the same homotopy type as the disjoint union $F^+(f) \cup F^-(f)$, using the notation from Remark 11. Therefore

$$\chi(K_{\mathbb{R}}) = 1 + (-1)^n + (-1)^{n+1}\Big(\chi\Big(F^+(f)\Big) + \chi\Big(F^-(f)\Big)\Big).$$

To prove the result, it is enough to show that

$$\widetilde{\chi}_G\left(F(f)\right) = a_1\phi_1 + a_2\phi_2 + a_4\phi_3 + a_3\phi_4 + \sum b'_h\rho^h$$

where $a_1, \cdots, a_4$ are the same integers from (15).

When $d$ is odd, this is very simple. Indeed the homeomorphism

$\varphi : F(f) \to F(-f)$ given by
$\varphi(x) = (-1) \cdot x = ((-1)^{w_0}x_0, \cdots, (-1)^{w_n}x_n)$

induces an isomorphism of $G$-modules $H_j(F(f)) \simeq H_j(F(-f))$ for all $j$. Moreover, in this case $a_3 = a_4 = 0$ since the representations $\phi_3$ and $\phi_4$ do not occur!

When $d$ is even, one can again define a homeomorphism $\varphi$ as above by the formula $\varphi(x) = \lambda \cdot x$, where $\lambda = \exp(\pi i/d)$.

Then it is easy to check that $\varphi h = h\varphi$, i.e. $\varphi$ commutes with the corresponding monodromy homeomorphisms. On the other hand, $hc\varphi = \varphi c$. When $h_* = 1$, this gives $c_*\varphi_* = \varphi_* c_*$ i.e. $\varphi_*$ preserves the representations of type $\psi_1$ and $\psi_2$. When $h_* = -1$, we have $-c_*\varphi_* = \varphi_* c_*$ i.e. $\varphi_*$ interchanges the representations of type $\psi_3$ and $\psi_4$ (a special case of this phenomenon is seen in Example 6). This ends the proof of Prop. 17.

It is usual to consider the zeta function $\zeta(h)(t)$ of the monodromy homeomorphism $h : F \to F$ defined by

$$\zeta(h)(t) = \prod_{j=0}^{n} \det\left(tI - h_*|H_j(F)\right)^{(-1)^j} \tag{18}$$

This zeta function is known for large classes of weighted homogeneous singularities $f$, in particular for all isolated singularities [MOr] and for large classes of homogeneous singularities, see [D2].

It is clear that the knowledge of this zeta function $\zeta(h)$ determines the following integers $a_1 + a_2$, $a_3 + a_4$ and $b_k$, where $a_j$ and $b_k$ are as in (15). Indeed, one has to consider the multiplicity of 1, $-1$ and respectively, $\exp(2\pi ik)/d)$ as a root (or a pole) in $\zeta(h)$. This fact, together with (16), (17), gives

**Corollary 19.** *The monodromy zeta function $\zeta(h)$ and the Euler numbers $\chi(F_{\mathbb{R}})$ and $\chi(K_{\mathbb{R}})$ determine the equivariant Euler characteristic $\chi_G(F)$ and conversely.*

Let us consider now the case when 0 is an isolated singularity for $f$. Then $\widetilde{H}_j(F) = 0$ except for $j = n$ and $K_{\mathbb{R}}$ is a smooth manifold of dimension $n-1$.

**Proposition 20.** *When $O$ is an isolated singularity for $f$, the equivariant Euler characteristic $\widetilde{\chi}_G(F)$ satisfies the conditions*

(i) *$a_1 = a_2$ when $n$ is even, and*

(ii) $a_3 = a_4$ *when* $n$ *is odd.*

**Proof.** Assume that $n$ is even. Then $K_{\mathbb{R}}$ is an odd-dimensional oriented closed manifold and hence $\chi(K_{\mathbb{R}}) = 0$. This implies $a_1 = a_2$ by (17). Assume now that $n$ is odd. If $d$ is odd as well, $a_3 = a_4 = 0$ and there is nothing to prove. When $d = 2d_1$ is even, consider the polynomial

$$\tilde{f}(x_0, \cdots, x_{n+1}) = f(x_0, \cdots, x_n) + x_{n+1}^2$$

which is weighted homogeneous if we set $wt(x_{n+1}) = d_1$. Using (14) and Example 6, if follows that

$$\begin{aligned} \widetilde{\chi}_G\left(F(\tilde{f})\right) &= (a_1\psi_1 + a_2\psi_2 + a_3\psi_3 + a_4\psi_4 + \cdots) \otimes \psi_3 \\ &= a_1\psi_3 + a_2\psi_4 + a_3\psi_1 + a_4\psi_2 + \cdots \end{aligned}$$

where $\cdots$ denotes a sum of 2-dimensional irreducible representations. Applying (i) to the polynomial $\tilde{f}$ we get $a_3 = a_4$.

**Example 21.** When $O$ is not an isolated singularity, the equalities in Prop. 20 do not hold in general. For example, take $f = x_0^2 x_1^2$. Then it is easy to see that $\widetilde{\chi}_G(F(f)) = \psi_2 - \psi_3 + \psi_4$.

Assume now that the weights $w_j$ of the polynomial $f$ have been numbered in such a way that $w_0, \cdots, w_k$ are odd for $0 \leq k \leq n$ and the rest $w_{k+1}, \cdots, w_n$ are even (if any). Then the group $\mathbb{Z}_2 = \{\pm 1\}$ acts on the sphere $S^n$ by the formula $\pm 1 \cdot x = (\pm 1) \cdot x$ and the fixed points are the $n-k-1$ sphere $S^{n-k-1}$ given by the linear section $x_0 = \cdots = x_k = 0$. The quotient $(S^n \setminus S^{n-k-1})/\mathbb{Z}_2$ can be identified with the product $\mathbb{R}P^k \times D^{n-k}$ where

$$D^{n-k} = \{y \in \mathbb{R}^{n-k}; y_1^2 + \cdots + y_{n-k}^2 < 1\}.$$

If the hypersurface $K_{\mathbb{R}}$ is disjoint from the fixed point set $S^{n-k-1}$, we can define a hypersurface in $\mathbb{R}P^k \times D^{n-k}$ by taking the quotient

$$\overline{K_{\mathbb{R}}} = K_{\mathbb{R}}/\mathbb{Z}_2 \tag{22}$$

When 0 is an isolated singularity for $f$, we get $\chi(\overline{K}_{\mathbb{R}}) = 0$ for $n$ even, while for $n$ odd we get from (16), (17) and (20) :

$$\chi(\overline{K}_{\mathbb{R}}) = 1 + a_2 - a_1 = \chi(F_{\mathbb{R}}) \tag{23}$$

In particular, for homogeneous polynomials $\overline{K}_{\mathbb{R}}$ is just the projective hypersurface $V_{\mathbb{R}}$ discussed in the introduction.

**Example 24.** Let $f_k(x_0, x_1)$ be a homogeneous polynomial of degree $d$ with $d - k$ an even non negative integer such that 0 is an isolated singularity for $f_k$ (over $\mathbb{C}$) and the irreducible factorization of $f_k$ in $\mathbb{R}[x_0, x_1]$ has $k$ distinct linear factors. Since the zeta function $\zeta(h)$ is known [M Or] and $K_{\mathbb{R}}$ consists of $2k$ points it follows easily that

$$a_1 = (d-k)/2, \ \ a_2 = (d+k-2)/2,$$
$$a_3 = a_4 = \begin{array}{cl} 0 & \text{for } d \ odd \\ (d-2)/2 & \text{for } d \text{ even} \end{array}$$

and $b_\ell = d - 2$ for $1 \le \ell \le (d-1)/2$.

Using the identification of elements in $R[G]$ to class functions on $G$ (by taking characters), we get $\widetilde{\chi}_G\left(F(f_k)\right)(s) = a_1 - a_2 + a_3 - a_4 = -(k-1)$. Using now the formula (14) we get $\widetilde{\chi}_G\left(F(f_{k_1} + f_{k_2})\right)(s) = (k_1 - 1)(k_2 - 1)$ or

$$\begin{aligned} \chi\left(V_{\mathbb{R}}(f_{k_1} + f_{k_2})\right) &= \chi\left(F_{\mathbb{R}}(f_{k_1} + f_{k_2})\right) \\ &= 1 - (k_1 - 1)(k_2 - 1) \end{aligned} \tag{24}$$

Hence we get these possibilities for $d = 4$

| $(k_1, k_2)$ | (0,2) | (0,4) | (2,2) | (2,4) | (4,4) |
|---|---|---|---|---|---|
| $\chi(V_{\mathbb{R}})$ | 2 | 4 | 0 | -2 | -8 |

Take now $g = x_0^2 x_1^2$. Using Example 21 it follows that

$$\widetilde{\chi}_G\left(F(g)\right)(s) = -3.$$

As above we get

$$\chi\left(F_{\mathbb{R}}(g + f_k)\right) = 1 - 3(k-1)$$

for $k = 0, 2$ or $4$.

On the other hand, to determine $\chi(V_{\mathbb{R}}(g+f_k))$ we have to compute first

$$\begin{aligned}\tilde{\chi}_G(F(g+f_k)) &= (\psi_2 - \psi_3 + \psi_4)(a_1\psi_1 + a_2\psi_2 + a_3\psi_3 + a_4\psi_4 + \cdots) \\ &= (a_2 - a_3 + a_4)\psi_1 + (a_1 + a_3 - a_4)\psi_2 \\ &+(-a_1 + a_2 + a_4)\psi_3 + (a_1 - a_2 + a_3)\psi_4 + \cdots \\ &= \tilde{a}_1\psi_1 + \tilde{a}_2\psi_3 + \tilde{a}_3\psi_3 + \tilde{a}_4\psi_4 + \cdots\end{aligned}$$

where $\cdots$ denotes a sum of 2-dimensional irreducible $G$-representations. Using (17) and (23) we get

$$\begin{aligned}\chi(V_{\mathbb{R}}(g+f_k)) &= 1 + \tilde{a}_2 - \tilde{a}_1 \\ &= 1 + (a_1 + a_3 - a_4) - (a_2 - a_3 + a_4) \\ &= 1 + a_1 - a_2 + 2(a_3 - a_4) = k.\end{aligned}$$

Note that for the *nonisolated singularities* of the form $f+g$ the computation of $\chi(V_{\mathbb{R}})$ or $\chi(K_{\mathbb{R}})$ cannot be done just by using the topological information contained in the Thom-Sebastiani formula. For such computations the consideration of dihedral representations is essential.

Besides (22), there is another *generalization of a real projective hypersurface.*

Let $\mathbb{P} = \mathbb{P}(w_0, \cdots, w_n)$ denote the complex weighted projective space associated to the weights $w_0, \cdots, w_n$, see [D2]. Let $V = V(f)$ be the hypersurface in $\mathbb{P}$ corresponding to the equation $f = 0$.

Let $V_{\mathbb{R}}$ be the set of real points of the variety $V$. In other words, $V_{\mathbb{R}}$ is the fixed point set of the involution $c : V \to V$ induced by the complex conjugation. When $f$ is homogeneous, this $V_{\mathbb{R}}$ is just the hypersurface in $\mathbb{R}P^n$ given by $f = 0$, as in the Introduction.

Let $K = f^{-1}(0) \cap S^{2n+1}$ be the complex link of the polynomial $f$ at the origin. Note that the projection $p : K \to V$, $x \mapsto [x]$ induces a surjective map $p_{\mathbb{R}} : K_{\mathbb{R}} \to V_{\mathbb{R}}$, for which each fiber consists of exacty two points. We warn the reader that this map $p_{\mathbb{R}}$ does not come from a $\mathbb{Z}_2$-action on $K_{\mathbb{R}}$. Indeed, the involution of $K_{\mathbb{R}}$ obtained by interchanging the two points in each fiber of $p_{\mathbb{R}}$ is not continuous! (see the example discussed below).

Nevertheless, we can stratify this map $p_{\mathbb{R}}$ in such a way that its restriction over every stratum in $V_{\mathbb{R}}$ is a $\mathbb{Z}_2$-bundle. Since Euler numbers behave additively with respect to stratifications (use Lefschetz duality theorem for relative manifolds, see [Sp], p.297 with $\mathbb{Z}_2$ coefficients), we get the following.

**Corollary 25.**

$$\chi(V_{\mathbb{R}}) = a_2 - a_1 + (1 + (-1)^{n+1})/2.$$

*In particular, when $f$ has an isolated singularity at the origin and $n-1 = \dim_{\mathbb{C}} V$ is odd, then $\chi(V_{\mathbb{R}}) = 0$.*

Finally, we note that in general $V_{\mathbb{R}}$ is not a $\mathbb{Q}$-manifold (as is $V$ itself being a quasismooth weighted hypersurface, see for instance [D2]). A very simple example is when $w_0 = \cdots = w_{n-1} = 1,\ \ w_n = 2$ and $f(x) = x_0$. Then the link $K_{\mathbb{R}}$ is just the sphere $S^{n-1}$. The projection $P_{\mathbb{R}}$ identifies the points $(0, x_1, \cdots, x_{n-1}, x_n)$ and $(0, -x_1, \cdots, -x_{n-1}, x_n)$ for all $(0, x_1, \cdots, x_n) \in S^{n-1}$ as well as the points $(0, \cdots, 0, 1)$ and $(0, \cdots, 0, -1)$. The latter identification comes from the multiplication by $i$.

Hence the space $V_{\mathbb{R}}$ is obtained from the product $\mathbb{R}P^{n-2} \times [-1, 1]$ by collapsing the boundary $\mathbb{R}P^{n-2} \times \{\pm 1\}$ to a point $y$. It is clear that $V_{\mathbb{R}}$ is not a $\mathbb{Q}$-manifold at this point $y$.

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