THE CHERN-EULER NUMBER OF CIRCLE BUNDLE VIA SINGULARITY THEORY

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Abstract

Some formulas representing the Chern-Euler class of circle bundle over a Riemannian surface in terms of global singularities of restrictions of a generic function to the fibers are given. These formulas provide new invariants of curves in Euclidean and projective two-spaces.

1. Main results

Let $c \subset \mathbb{R}^2$ be a smooth closed convex curve. We associate with this curve the family of functions on the curve $f_q(x) = ||q - x||^2$, $x \in c$, $q \in \mathbb{R}^2$, parameterized by points of the plane $\mathbb{R}^2$. The differential-geometric invariants of c can be expressed in terms of invariants of this family of functions. For example, the curvature circles touching the curve at points of local minimum of curvature and having no other points of intersection with c correspond to the points $q \in \mathbb{R}^2$ for which the function f_q gets its global minimum at a degenerate critical point of multiplicity 3. The circles lying inside c and having 3 points of tangency with c correspond to the points $q \in \mathbb{R}^2$ for which f_q gets its minimum at 3 different points. Denote by C and T the numbers of circles of the first and the second type respectively. According to Bose's formula (refined by Sedykh, see [2, 9])

$$C - T = 2\,. \tag{1}$$

In particular, the inequality $C = 2 + T \geq 2$ implies the four-vertices theorem that a convex plane curve has at least four points of extremum of curvature.

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Indeed, between these two points of local minimum of curvature there should exist another two points of local maximum of curvature.

In this paper we establish some formulas similar to (1) in terms of other singularities of the family f_q. The results are formulated in terms of the more general setting.

Throughout the paper we use the following notations. We consider a smooth locally trivial bundle $\pi\colon\ W \to M$. The base M of π is a closed oriented manifold of dimension 2 (a Riemannian surface). We assume that the fibers of π are diffeomorphic to the circle S^1 and oriented. Denote by $e(\pi)$ the Chern-Euler number of the bundle, the value of the first Chern class of π on the fundamental cycle of the base M.

Let $f\colon\ W \to \mathbb{R}$ be a generic smooth function. We consider f as the family of functions f_q, $q \in M$, where $f_q = f|_{\pi^{-1}(q)}$ is the restriction of f to the fiber over q. The *bifurcation diagram* $\Sigma \subset M$ is the set of the points $q \in M$ for which f_q is not a Morse function.

If $e(\pi) \neq 0$ then Σ is not empty. Indeed, otherwise, the bundle π has a global continuous section formed, for example, by the points of global minimum of f_q. Therefore, π is trivial and $e(\pi) = 0$.

In general, the bifurcation diagram consists of two components, $\Sigma = \Sigma^{(2)} \cup$

$\Sigma^{(11)}$. The set $\Sigma^{(2)}$ called the *discriminant* or the *caustic* is formed by the points $q \in M$ for which f_q has degenerate critical points and the *Maxwell stratum* $\Sigma^{(11)}$ corresponds to the functions f_q which have two critical points with the same critical value.

Both $\Sigma^{(2)}$ and $\Sigma^{(11)}$ are one-dimensional subvarieties smooth everywhere except some finite number of singular points. All possible singularities of Σ are those of the types $\Sigma^{(3)}$, $\Sigma^{(21)}$, $\Sigma^{(111)}$ shown in Fig. 1, and 3 possible types $\Sigma^{(2)(2)}$, $\Sigma^{(2)(11)}$, $\Sigma^{(11)(11)}$ of transversal selfintersections of Σ. The superscript $(a_1 \dots a_l)$ in the notations of strata of the bifurcation diagram stands for the functions with critical points of multiplicities $a_1, \dots, a_l$ on the same critical level. The diagrams shown in Fig. 1 correspond to all possible bifurcation diagrams of multigerms of codimension 2 of functions in 1 variable (cf. [3]).

The functions with the given multiplicities of critical points on the given critical level may have different number of other critical points. This gives the possibility to subdivide each of the classes $\Sigma^{(3)}, \dots, \Sigma^{(11)(11)}$.

Definition 1.1. The subset $\Sigma^{(11)(11)}_{\mathrm{extr}} \subset \Sigma^{(11)(11)}$ corresponds to the functions which have two points of global minimum, two points of global maximum, and such that the two points of global minimum and the two points of global maximum alternate.

Definition 1.2. The subsets $\Sigma^{(3)}_{\min} \subset \Sigma^{(3)}$, $\Sigma^{(111)}_{\min} \subset \Sigma^{(111)}$ (resp. $\Sigma^{(3)}_{\max} \subset \Sigma^{(3)}$, $\Sigma^{(111)}_{\max} \subset \Sigma^{(111)}$) correspond to the functions for which the multiple critical level is the level of global minimum (resp. maximum) of the function.

Definition 1.3. The subset $\Sigma^{(3)}_{m} \subset \Sigma^{(3)}$ ($m \geq 1$ is odd) corresponds to the functions which have m nondegenerate critical points and one degenerate critical point of multiplicity 3. The subset $\Sigma^{(2)(2)}_{l,m} \subset \Sigma^{(2)(2)}$ ($0 \leq l \leq m$, $l + m$ is even) corresponds to the functions which have l and m nondegenerate critical points respectively on the two arcs bounded by the two degenerate critical points of the function.

Theorem 1.4. *There is a natural way to define a sign of every singular point from Definitions 1.1–1.3. With the notation $\#\Sigma^{\alpha}_{\beta}$ for the algebraic number of points of the type Σ^{α}_{β} counted with their signs, the following relations hold*

$$
\begin{aligned}
e(\pi) &= \#\Sigma^{(11)(11)}_{\mathrm{extr}}, && (2)\\
e(\pi) &= \tfrac{1}{2}\#\Sigma^{(3)}_{\min} - \tfrac{1}{2}\#\Sigma^{(111)}_{\min} = \tfrac{1}{2}\#\Sigma^{(3)}_{\max} - \tfrac{1}{2}\#\Sigma^{(111)}_{\max}, && (3)\\
e(\pi) &= \sum \frac{2}{(m+1)(m+3)}\#\Sigma^{(3)}_{m} + \sum \frac{(-1)^{l+1}4(m-l)}{(l+m)(l+m+2)(l+m+4)}\#\Sigma^{(2)(2)}_{l,m} = \\
&= \tfrac{1}{4}\#\Sigma^{(3)}_{1} - \tfrac{1}{6}\#\Sigma^{(2)(2)}_{0,2} + \tfrac{1}{12}\#\Sigma^{(3)}_{3} + \tfrac{1}{24}\#\Sigma^{(2)(2)}_{1,3} - \tfrac{1}{12}\#\Sigma^{(2)(2)}_{0,4} \\
&\quad + \tfrac{1}{24}\#\Sigma^{(3)}_{5} + \dots, && (4)
\end{aligned}
$$

where $e(\pi)$ is the Chern-Euler number of the bundle π. In particular, the right hand side expressions of (2)–(4) *do not depend on the choice of (generic) function $f\colon W \to \mathbb{R}$.*

The equality (2) describes the Chern-Euler number $e(\pi)$ in terms of self-intersection points of the Maxwell stratum; the equality (3) describes $e(\pi)$ in terms of singularities of the minimum (maximum) function; and the equality (4) describes $e(\pi)$ in terms of singularities of the caustic. Note, that the coefficients entering into formula (4) depend on the behavior of the function outside the degenerate critical level. Therefore, these coefficients cannot be seen from the caustic itself.

Remark 1.5. Independent proofs of equalities (3) and (4) are given in [7]. In that paper we used notations '$\mathcal{M}$-singularities of the types (3), (1^3)' and '$\mathcal{C}$-singularities of types (20^m), (10^l10^m)' for functions on the circle corresponding to singular points of Σ of types $\Sigma^{(3)}_{\min}$, $\Sigma^{(111)}_{\min}$, $\Sigma^{(3)}_{m}$, $\Sigma^{(2)(2)}_{l,m}$ respectively.

The actual definition of the sign of a singular point $q_0 \in \mathrm{Sing}\Sigma$ is the following. We identify the fibers $\pi^{-1}(q)$ in a neighborhood of q_0 with the circle $\mathbb{R}/2\pi\mathbb{Z}$ using some trivialization of the bundle π.

If q_0 is the point of the type $\Sigma^{(11)(11)}_{\mathrm{extr}}$ we denote by s_1, s_2 the two points of global minimum of f_{q_0} and by y_1, y_2 the two points of global maximum such that the four points of global extremum go in the order y_1, s_1, y_2, s_2 on the circle $\pi^{-1}(q_0)$. Put

$$\lambda_1(q) = f_q(y_2) - f_q(y_1)\,, \quad \lambda_2(q) = f_q(s_2) - f_q(s_1)\,.$$

If q_0 is any point of the type $\Sigma^{(3)}$ we denote by $s_0 \in \pi^{-1}(q_0)$ the degenerate critical point of f_{q_0} and put

$$\lambda_1(q) = f_q'(s_0)\,, \qquad \lambda_2(q) = f_q''(s_0)\,.$$

If q_0 is the point of the type $\Sigma^{(111)}_{\min}$ we denote by s_1, s_2, s_3 the three points of global minimum of f_{q_0} going in this order on $\pi^{-1}(q_0)$. Put

$$\lambda_1(q) = f_q(s_2) - f_q(s_1)\,, \quad \lambda_2(q) = f_q(s_3) - f_q(s_2)\,.$$

At last, assume that q_0 is of the type $\Sigma^{(2)(2)}_{l,m}$. Let s_1, s_2 be the two degenerate critical points of the function f_{q_0} such that the arc $\overset{\frown}{s_1 s_2}$ going from s_1 to s_2 in positive direction contains l nondegenerate critical points and the arc $\overset{\frown}{s_2 s_1}$ contains m critical points. Put

$$\lambda_1(q) = f_q'(s_1)\,, \qquad \lambda_2(q) = f_q'(s_2)\,.$$

If the function f is in general position then in the either case above the functions λ_1, λ_2 define a coordinate system in a neighborhood of the point $q_0 \in M$. In fact, the condition that the mapping $q \mapsto (\lambda_1(q), \lambda_2(q))$ is nondegenerate at q_0 is equivalent to the condition that the family of functions f_q forms a versal deformation of the corresponding multigerm.

Definition 1.6. We call a singular point q_0 *positive* (resp. *negative*) if the natural orientation of the plane of variables λ_1, λ_2 coincides with (resp. is opposite to) the orientation of the manifold M.

Proposition 1.7. *The signs of singular points used in Theorem* 1.4 *are those* *of Definition* 1.6.

The S^1-bundle of the family of functions associated with a convex curve $c \subset \mathbb{R}^2$ considered at the beginning of this Section is the trivial bundle $c \times \mathbb{R}^2 \to \mathbb{R}^2$. The base of this bundle is not compact. Hence, its Chern-Euler number is not defined. Nevertheless, if $||q||$ is large enough then f_q has a nondegenerate point of global minimum, a nondegenerate point of global maximum and no other critical points. Therefore, the bifurcation diagram is compact and the value of the right hand side expressions of (2)–(4) does not depend on the curve. We call this common value the *Chern-Euler number associated with a convex curve.*

Theorem 1.8. *The Chern-Euler number associated with a convex curve is equal* *to* 1*. Moreover, let* $\Sigma \subset \mathbb{R}^2$ *be the bifurcation diagram of the family of functions associated with a convex smooth curve. Then the sign of every singular point of the types* $\Sigma_*^{(3)}$, $\Sigma_*^{(111)}$, $\Sigma_{\mathrm{extr}}^{(11)(11)}$ *is always positive.*

An example of the bifurcation diagram associated with a convex curve is shown in Fig. 2. The three numbers in the brackets near the singular points of Σ are terms entering into the right hand side expressions of equalities (2), the first one of (3), and (4) respectively.

For the relation (3) of Theorem 1.4 we get as a consequence the relation (1). For the relation (2) of Theorem 1.4 we have.

Corollary 1.9. *For any smooth convex curve there is exactly one ring* R *containing* c *such that* c *touches both concentric circles of the boundary* ∂R *at two different points and the points of touching of* c *with the two components of* ∂R *alternate on* c *(see Fig.* 3)

1.10. A modification of formulas (2)–(4) leads to invariants of projective curves. Let $c \subset P^2$ be a generic smooth immersed closed curve. Denote by $[c] \in H_1(P^2) = \mathbb{Z}_2$ the homology class represented by c. The curve c may have any number of components. The only assumption we use is that the intersection of c with any projective line is not empty. For example, this is the case if $[c] \neq 0$.

A projective line $\lambda \subset P^2$ is called *special* if the number of intersection points of c and λ is two less than the total multiplicity of the intersection of c and λ. If c is in general position then a special line is one of the following.

a) λ is a tangent line to c at a point of inflection. Let m be the number of another transversal intersection points of c and λ. The set of such λ is denoted by I_m (note, that $m \equiv [c] \pmod 2$).

b) λ is a tangent line to c at a selfintersection point of c. Let m be the number of another transversal intersection points of c and λ ($m \equiv [c] \pmod 2$). The set of such λ is denoted by X_m. Note, that with each selfintersection point of c we associate two lines of this kind corresponding to different branches of c at this point.

c) λ touches c at two different points. Define numbers l and m as follows. Let a and b be the points of touching of c and λ. Let $n_a \in T_aP^2$ and $n_b \in T_bP^2$ be tangent vectors transversal to λ and such that they belong locally to the same half-plane bounded by λ that c. This vectors induce the same coorientation on one of the segments of λ bounded by a and b and opposite one on the other segment. We call these segments *positive* and *negative* ones respectively. Let l and m be the numbers of transversal intersection points of c with these segments respectively ($l{+}m \equiv [c]{+}1 \pmod 2$). The set of such λ with prescribed numbers l and m is denoted by $B_{l,m}$.

d) λ either passes through two selfintersection points of c or it passes through a selfintersection point of c and touches c at some other point. We use no notations for these special lines.

Denote by P^{2*} the space of all projective lines in P^2. For every $\lambda \in P^{2*}$ define its index as follows

$$\mathbf{ind}(\lambda) = \frac{2}{(m{+}1)(m{+}3)}\,, \qquad \text{if } \lambda \in I_m \text{ or } \lambda \in X_m,$$

$$\mathbf{ind}(\lambda) = \frac{4(l-m)}{(l{+}m)(l{+}m{+}2)(l{+}m{+}4)}\,, \qquad \text{if } \lambda \in B_{l,m},$$

and $\mathbf{ind}(\lambda) = 0$ for all other $\lambda \in P^{2*}$.

Note, that the index of any line of types I_m and X_m is positive while that of a line of type $B_{l,m}$ may have any sign.

Theorem 1.11. *If c is in general position and the intersection of c with any projective line is not empty then*

$$\sum_{\lambda \in P^{2*}} \mathbf{ind}(\lambda) \;=\; 2$$

For example, if $[c] \neq 0$ then we have

$$\begin{aligned} 2 \;=\; & \frac{2}{3}\#I_0 + \frac{2}{15}\#I_2 + \frac{2}{35}\#I_4 + \cdots + \frac{2}{3}\#X_0 + \frac{2}{15}\#X_2 + \frac{2}{35}\#X_4 + \cdots + \\ & \frac{4}{15}\#B_{1,0} - \frac{4}{15}\#B_{0,1} + \frac{4}{35}\#B_{3,0} - \frac{4}{35}\#B_{0,3} + \frac{4}{105}\#B_{2,1} - \frac{4}{105}\#B_{1,2} + \dots \end{aligned}$$

Here, # denotes the cardinality of corresponding sets. An example of indices associated with a projective noncontractible curve is given in Fig. 4.

According to Möbius' theorem *a smooth embedded noncontractible curve in* P^2 *has at least* 3 *points of inflection.* Let us show that Theorem 1.11 implies Möbius' theorem in case when the curve c is close enough to a projective line (together with derivatives, see also [1]). Indeed, if c is embedded then $\#X_m = 0$ for all m. Hence, if c has less than 3 points of inflection then $\#B_{l,m} > 0$ for some l, m that is c has a bitangent line λ. But if c is close to this bitangent line then it follows from the Roll theorem that it has at least 5 points of inflection. At least 2 of them are situated near the positive segment of λ bounded by points of touching with c and at least 3 of them are situated near the negative one, see Fig. 5.

2. Chern-Euler number

Let M be a closed oriented manifold of dimension 2 (a Riemannian surface). Consider an oriented locally trivial bundle $\pi\colon W \to M$ over M the fibers of which are diffeomorphic to the circle S^1 and oriented.

Consider a Riemannian metric on W such that the length of every fiber equals 2π. Then the arc-length parameters on the fibers are defined up to a shift of the origin. So, the structure group of the bundle is reduced to $S^1 = SO(2)$ and W can be represented as the bundle of unit circles in some 2-dimensional vector bundle $E \to M$.

Definition 2.1. The *Chern-Euler number* $e(\pi)$ of the bundle π is the selfintersection number of the zero section of E.

In other words, $e(\pi)$ is the value of the characteristic Chern-Euler class $c_1 = e \in H^2(M)$ of the bundle E on the fundamental cycle of M.

The Chern-Euler number can be defined purely in terms of the bundle π as the obstruction to existing of global section. Suppose that some finite set of points $X = \{x_1, \dots x_n\} \in M$ and a continuous section $s\colon M \setminus X \to W \setminus \pi^{-1}(X)$ of π are given. By a slightly abuse of notations, we call the points x_i the *singular points* of the section s. Let $D_i \subset M$ be a small disk centered at x_i. The bundle π can be trivialized over D_i. If some trivialization is chosen then the section s is given over D_i by a continuous function $s_i\colon D_i \setminus x_i \to S^1$.

Definition 2.2. The *index* of singular point x_i of the section s is the degree of the restriction of s_i to the circle $\partial D_i \cong S^1$.

Note, that both circles, the boundary ∂D_i and the fiber S^1 of π have natural orientations. Therefore, the index is a well defined integer.

Proposition 2.3. *The Chern-Euler number $e(\pi)$ is equal to the sum of indices of all singular points of the section s.*

Proof. Let W be the bundle of unite circles in the linear bundle E. Let

$\rho\colon M \to \mathbb{R}$ be a continuous function satisfying the following properties

- $\rho(x_i) = 0$, $i = 1, \dots, n$;
- $\rho(x) > 0$ for $x \in M \setminus X$.

Then the section $\rho\, s\colon M \setminus X \to E$ can be continued to the global section $\bar{s}\colon M \to E$. Zeros of $\bar{s}$ correspond one-to-one to the points of X. The index of each point x_i is equal to the local intersection number of $\bar{s}$ with the zero section. Now the Proposition follows from Definition 2.1.

3. Proof of equalities (2) and (3) of Theorem 1.4

Equalities (2) and (3) of Theorem 1.4 are the simplest applications of Proposition 2.3. Using a given function in the total space of the bundle we try to build a global section. The obstructions are expressed in terms of singularities of the restriction of the function to the fibers. Let $\pi\colon W \to M$ be an S^1-bundle over a Riemannian surface M. Let $\Sigma \subset M$ be a close subvariety of dimension one smooth outside of a finite set $\mathrm{Sing}\Sigma \subset \Sigma$. Consider a continuous section $\tilde{s}\colon M \setminus \Sigma \to \pi^{-1}(M \setminus \Sigma)$ given over the domain $M \setminus \Sigma$. Let Γ be a connected component of $\Sigma \setminus \mathrm{Sing}\Sigma$. Choose any coorientation of Γ. Denote by U^+ and U^- the two components of $M \setminus \Sigma$ bounded by Γ and situated from its positive and negative side respectively. We assume that for every $y \in \Gamma$ the following limits exist

$$\tilde{s}^+(y) = \lim_{x \in U^+, x \to y} \tilde{s}(x)\,, \qquad \tilde{s}^-(y) = \lim_{x \in U^-, x \to y} \tilde{s}(x)\,, \qquad y \in \Gamma\,.$$

Moreover, we assume that these limits satisfy the following condition: either $\tilde{s}^+(y) = \tilde{s}^-(y)$ for all $y \in \Gamma$ or $\tilde{s}^+(y) \neq \tilde{s}^-(y)$ for all $y \in \Gamma$.

If Σ is the bifurcation diagram of the function $f\colon W \to \mathbb{R}$ then the point of global minimum of f_q on the fiber over q gives a section satisfying the properties above.

Denote by $\Sigma^=$ and $\Sigma^{\neq}$ the union of those components of $\Sigma \setminus \mathrm{Sing}\Sigma$ for which $\tilde{s}^+ = \tilde{s}^-$ and $\tilde{s}^+ \neq \tilde{s}^-$ respectively. The section $\tilde{s}$ has a natural continuous extension over $\Sigma^{=}$. When the point of the base crosses through $\Sigma^{\neq}$ the section $\tilde{s}$ makes a jump. This jump can be done continuous after a change of $\tilde{s}$ in a neighborhood of $\Sigma^{\neq}$. More precisely, there exists a continuous section $s\colon W \setminus \mathrm{Sing}\Sigma \to \pi^{-1}(M \setminus \mathrm{Sing}\Sigma)$ coinciding with $\tilde{s}$ over the complement of some tubular neighborhood of Σ.

The section s is not defined uniquely up to homotopy. Its homotopy type is defined by the homotopy types of paths on the circles $\pi^{-1}(q)$, $q \in \Sigma^{\neq}$, connecting the points $\tilde{s}^{-}(q)$ and $\tilde{s}^{+}(q)$.

Suppose that every component of $\Sigma^{\neq}$ is cooriented. This allows to say which of the two domains $U^{\pm}$ bounded by $\Gamma \subset \Sigma \setminus \mathrm{Sing}\Sigma$ is considered as positive and negative one respectively.

Definition 3.1. The section s over $M \setminus \mathrm{Sing}\Sigma$ is called the *natural continuation of the section* $\tilde{s}$ if it makes half turn rotation in positive (resp. negative) direction when the point of the base crosses through $\Sigma^{\neq}$ in positive (resp. negative) direction.

Denote by $i(q)$, $q \in \mathrm{Sing}\Sigma$, the index of the point q with respect to the section s defined in 2.2. According to Proposition 2.3 we have the relation

$$e(\pi) = \sum_{q \in \mathrm{Sing}\Sigma} i(q)\,. \tag{5}$$

Let $S^1_\varepsilon(q)$, $q \in \mathrm{Sing}\Sigma$, be a small circle centered at q and oriented counterclockwise. The intersection number $(S^1_\varepsilon(q), \Sigma^{\neq})$ is well defined because $\Sigma^{\neq}$ is cooriented. Denote

$$\mathrm{ind}(q) = i(q) - \tfrac{1}{2}(S^1_\varepsilon(q), \Sigma^{\neq})\,.$$

Proposition 3.2. *a) The index* $\mathrm{ind}(q)$ *of a singular point* q *does not depend on the coorientation of* $\Sigma^{\neq}$.

b) For the Chern-Euler number we have

$$e(\pi) = \sum_{q \in \mathrm{Sing}\Sigma} \mathrm{ind}(q)\,. \tag{6}$$

Proof. a) Let q be one of the end points of the component $\Gamma \subset \Sigma$. The change of the coorientation of Γ leads to the change of both $i(q)$ and $\frac{1}{2}(S^1_\varepsilon(q), \Sigma^{\neq})$ by 1 with the same sign. Hence, their difference does not change.

b) We have

$$\begin{aligned} \textstyle\sum_{q\in \mathrm{Sing}\Sigma} \mathrm{ind}(q) &= \textstyle\sum_{q\in \mathrm{Sing}\Sigma} i(q) - \frac{1}{2}\sum_{q\in \mathrm{Sing}\Sigma}(S^1_\varepsilon(q), \Sigma^{\neq}) = e(\pi) \\ &\quad -\tfrac{1}{2}\left(\textstyle\bigcup_{q\in \mathrm{Sing}\Sigma} S^1_\varepsilon(q), \Sigma^{\neq}\right) . \end{aligned}$$

The last intersection number vanishes because $\bigcup_{q\in \mathrm{Sing}\Sigma} S^1_\varepsilon(q)$ represents trivial element in the homology group of the complement $M \setminus \mathrm{Sing}\Sigma$: it is the boundary of the complement to the union of small disks centered at points $q \in \mathrm{Sing}\Sigma$.

For a function $f\colon W \to \mathbb{R}$, we denote by f_q, $q \in M$, the restriction of f to the fiber of π over q. Let $\Sigma \in M$ be the bifurcation diagram. If $q \in M \setminus \Sigma$ then f_g is a Morse function. Denote by $\widetilde{s}(q) \in \pi^{-1}(q)$ the point where f_q gets its global minimum. This defines a section $\widetilde{s}\colon M\setminus\Sigma \to W\setminus\pi^{-1}(\Sigma)$ over $M\setminus\Sigma$.

The set $\Sigma^{\neq}$ for this section $\widetilde{s}$ consists of those $q \in M$ for which f_q has two nondegenerate points of global minimum and all other critical points of f_q are also nondegenerate and have different critical values. Let $q \in \Sigma^{\neq}$. According to our notations introduced in the beginning of this Section $\widetilde{s}^{\pm}(q)$ are the two points of global minimum of f_q. The choice of the coorientation of $\Sigma^{\pm}$ permits to say which of the two points of minimum is considered as $\widetilde{s}^{+}(q)$ and $\widetilde{s}^{-}(q)$ respectively. Denote $y(q) \in \pi^{-1}(q)$ the point of global maximum of f_q.

Definition 3.3. The coorientation of $\Sigma^{\neq}$ is called *natural* if for every $q \in \Sigma^{\neq}$ the points $y(q)$, $\widetilde{s}^{-}(q)$, $\widetilde{s}^{+}(q)$ go in this order on the fiber $\pi^{-1}(q)$ with its orientation.

Lemma 3.4. *For the natural continuation of the section defined by points of global minimum, the index $i(q)$ is equal to ± 1 for $q \in \Sigma^{(11)(11)}_{\mathrm{extr}}$ and to 0 for any other singular point of Σ.*

Lemma 3.5. *For the natural continuation of the section defined by points of* *global minimum, the index* $\operatorname{ind}(q)$ *is equal to* $\pm\frac{1}{2}$ *for* $q \in \Sigma^{(3)}_{\min}$ *or* $q \in \Sigma^{(111)}_{\min}$ *and to* 0 *for any other singular point of* Σ.

Proof of equalities (2) and (3) of Theorem 1.4. By Lemmas 3.4, 3.5 the formulas (5), (6) take form of formulas (2), (3) respectively.

Proof of Lemma 3.4. When the point q of the base crosses through $\Sigma^{\neq}$ the section s makes a jump along the arc connecting the two points of minimum. By Definition 3.3, *this arc does not contain the point of global maximum of* f_q. It follows, that the index $i(q)$ of every singular point of Σ is equal to 0 provided the function f_q has either the only point of global minimum or the only point of global maximum. In the same way the index is equal to 0 if f_q has two points of global minimum and two points of global maximum but the two points of global minimum and the two points of global maximum *do not* alternate. Indeed, in this case the section s nowhere crosses that arc connecting the points of maximum which does not contain the points of minimum.

It remains to compute the index of a singular point of the type $\Sigma^{(11)(11)}_{\text{extr}}$. Let q_0 be such point. The change of orientation of M changes the signs of all singular points. Therefore, it is enough to show that *the index of positive (in the sense of Definition 1.6) point of the type* $\Sigma^{(11)(11)}_{\rm extr}$ *is equal to* $+1$.

The proof of that is seen from Fig. 6. Let $s_1(q)$, $s_2(q)$ be the two points of local minimum of the function f_q close to the two points of global minimum of f_{q_0}. The closure of the set $\Sigma^{\neq}$ is smooth at q_0. A small circle $S^1_\varepsilon(q_0)$ around q_0 intersects $\Sigma^{\neq}$ at two points. When the point q goes along this circle, the section s makes two jumps, first from $s_1(q)$ to $s_2(q)$, and then from $s_2(q)$ to $s_1(q)$, both in positive direction. Hence, the section $s(q)$ makes one turn rotation in positive direction when the point q goes along $S^1_\varepsilon(q_0)$ which means that $i(q_0) = 1$.

Proof of Lemma 3.5. Let $\Sigma^{\neq}$ be, as above, the set of points $q \in M$ for which the function f_q gets its global minimum at two different points. Singular points of $\Sigma^{\neq}$ are endpoints of type $\Sigma^{(3)}_{\rm min}$ and triple points of type $\Sigma^{(111)}_{\rm min}$ (see Fig. 7 and 8). Therefore, the index $\mathrm{ind}(q)$ may not be trivial for these points only.

Let q_0 be a point of the type $\Sigma^{(3)}_{\rm min}$. Without loss of generality we can assume that q_0 is positive in the sense of Definition 1.6. For computation of the index we can take any coorientation of $\Sigma^{\neq}$, for example, that which coorients $\Sigma^{\neq}$ counterclockwise as shown in Fig. 7. Let the point q goes along a small circle $S^1_\varepsilon(q_0)$ around q_0. By Definition 3.1, the section $s(q)$ remains close to the point of global minimum of f_{q_0} everywhere except a small neighborhood of the intersection point of $S^1_\varepsilon(q_0)$ with $\Sigma^{\neq}$, where the section $s(q)$ makes a jump close to 2π. Therefore, for this choice of the coorientation of $\Sigma^{\neq}$ we have $i(q) = 1$. Thus,

$$\mathrm{ind}(q_0) = i(q_0) - \tfrac{1}{2}(S^1_\varepsilon(q_0), \Sigma^{\neq}) = 1 - \tfrac{1}{2} = \tfrac{1}{2}\,.$$

Let q_0 be now a positive point of the type $\Sigma^{(111)}_{\rm min}$. Choose counterclockwise coorientation of $\Sigma^{\neq}$ as shown in Fig. 8. The set $\Sigma^{\neq}$ divides the neighborhood of q_0 into three domains. In every of these domains the section $s(q)$ is close to the corresponding point of global minimum of f_{q_0}. Every time when the point q of the base goes along the circle $S^1_\varepsilon(q_0)$ and crosses through $\Sigma^{\neq}$ the section $s(q)$ makes a jump in positive direction. It follows, that $i(q_0)$, the total sum of

these jumps, is equal to 2π and we have

$$\operatorname{ind}(q_0) = i(q_0) - \frac{1}{2}(S^1_\varepsilon(q_0), \Sigma^{\neq}) = 1 - \frac{3}{2} = -\frac{1}{2}\,.$$

Lemma 3.5 is proved.

4. Multivalued sections and their singularities

Let $\pi\colon W \to M$ be an oriented S^1-bundle over a Riemannian surface M. Let $\Sigma \subset M$ be a closed one-dimensional variety smooth outside a finite set $\mathrm{Sing}\Sigma \subset \Sigma$. Suppose for every connected component $U_\alpha \subset M \setminus \Sigma$ some finite number of sections are chosen $s_{\alpha i}\colon U_\alpha \to \pi^{-1}(U_\alpha)$, $i = 1, \ldots, n_\alpha$, such that $s_{\alpha i}(x) \neq s_{\alpha j}(x)$ for any $x \in U_\alpha$, $i \neq j$. We assume that these sections satisfy the following boundary condition. Let $\Gamma \subset \Sigma \setminus \mathrm{Sing}\Sigma$ be any connected component. Let U^+, U^- be two components of $M \setminus \Sigma$ bounded by Γ. Let s^+, s^- be some sections chosen over U^+ and U^- respectively. We assume that for every $y \in \Gamma$ the following limits exist

$$s^+(y) = \lim_{x \in U^+,\, x \to y} s^+(x)\,, \qquad s^-(y) = \lim_{x \in U^-,\, x \to y} s^-(x)\,, \qquad y \in \Gamma\,.$$

Moreover, we assume that these limits satisfy the following condition: either $s^+(y) = s^-(y)$ for all $y \in \Gamma$ or $s^+(y) \neq s^-(y)$ for all $y \in \Gamma$. We show in this Section how these data can be used to express the Chern-Euler number of the bundle.

Remark 4.1. Consider the cohomology spectral sequence of the bundle π. Its second term is $E_2^{p,q} \cong H^p(M) \otimes H^q(S^1)$. One can prove that the Chern-Euler class of π is the image of the class dual to the fundamental class of the fiber under the homomorphism $\delta_2\colon E_2^{0,1} \cong H^1(S^1) \to E_2^{2,0} \cong H^2(M)$. The spectral sequence of the bundle can be defined using some cellular partition of W. The system of sections above produces a natural partition of W. In fact, our calculations below are the calculations of the homomorphism δ_2 in terms of this partition.

Consider the collection of sections $\{s_{\alpha i}\}$, $U_\alpha \subset M \setminus \Sigma$, as a multivalued section $\mathbf{s}$ over $M \setminus \Sigma$. Let q_0 be some point. Let $U \subset M$ be a small disk centered at q_0. Consider a continuous section over $U \setminus (\Sigma \cap U)$ which coincides over each component of $U \setminus (\Sigma \cap U)$ with some branch of $\mathbf{s}$. The construction of Sect. 3 gives rise to the index $\operatorname{ind}(q_0)$ for the one-valued section obtained. The value of this index depends on particular choice of the branches of $\mathbf{s}$ over each component of $U \setminus (\Sigma \cap U)$.

Definition 4.2. The index $\mathbf{ind}(q_0)$ of the point q_0 is the arithmetical mean of indices $\operatorname{ind}(q_0)$ over all arbitrariness used in its definition.

Corollary 4.3. (of Definition 4.2) *If the number n_α of chosen sections is equal to* 1 *for each component $U_\alpha \subset M \setminus \Sigma$ then the index $\mathbf{ind}(q_0)$, $q_0 \in Sing\Sigma$, is equal to the index $\operatorname{ind}(q_0)$ introduced in Sect. 3.*

Theorem 4.4. For the Chern-Euler number we have

$$e(\pi) = \sum_{q \in \operatorname{Sing}\Sigma} \mathbf{ind}(q)\,. \tag{7}$$

Proof. To shorter arguments we shall use the language of probability theory.

We call the variety $\Sigma \subset M$ *simple* if the intersection of any connected component of $M \setminus \Sigma$ with a small disc around any point of Σ is also connected if not empty.

Assume for a moment that Σ is simple. Let us choose in a random way some branches of $\mathbf{s}$ over each component of $M \setminus \Sigma$. The index $\mathrm{ind}(q)$, $q \in \mathrm{Sing}\Sigma$, for the one-valued section obtained depends on the particular choice of branches of $\mathbf{s}$. Thus, we can consider $\mathrm{ind}(q)$ as a random variable. The condition of simplicity of Σ implies that the branches of $\mathbf{s}$ over every of $U \setminus (\Sigma \cap U)$ are chosen independently, where U is a small disc centered at q. Therefore, by Definition 4.2 the index $\mathbf{ind}(q)$ is the mathematical expectation of the random index $\mathrm{ind}(q)$.

The equality (6) for the random indices $\mathrm{ind}(q)$ implies similar equality for their mathematical expectations. This proves Theorem 4.4 for the case when Σ is simple.

To prove Theorem 4.4 in the general case note that there exists some simple $\Sigma' \subset M$ such that $\Sigma \subset \Sigma'$. In other words, Σ can be done simple by adding some number of lines, see Fig. 9.

Let $\mathbf{s}'$ be the restriction of $\mathbf{s}$ to $M \setminus \Sigma'$ considered as a multivalued section of π over $M \setminus \Sigma'$. The following Lemma completes the proof of Theorem 4.4.

Lemma 4.5. *The index* $\mathbf{ind}(q)$*,* $q \in \mathit{Sing}\Sigma'$*, for the multivalued section* $\mathbf{s}'$ *is equal to* 0 *if* $q \notin \mathit{Sing}\Sigma$ *and coincides with that for* $\mathbf{s}$ *if* $q \in \mathit{Sing}\Sigma$*.*

One of the possible proofs of this Lemma is given below.

It is very hard to compute the index directly from Definition 4.2. The computations in the next Sections rely on the following formula for the index.

Let $q_0 \in \mathrm{Sing}\Sigma$ be some singular point of Σ. The intersection of Σ with a small neighborhood U of q_0 form some finite number of curves $\Gamma_1, \dots, \Gamma_n$ going out of q_0. Denote by U_k^+, U_k^- the components of $U \setminus (\Sigma \cap U)$ such that Γ_k enters with coefficients $+1$ and -1 into the expressions of ∂U_k^+ and ∂U_k^- respectively (we consider the orientation of $U_k^\pm$ induced by the orientation of M and the orientation Γ_k directed out of q_0). Let $s_{k1}^+, \dots, s_{kn^+}^+$ and $s_{k1}^-, \dots, s_{kn^-}^-$ be the sections chosen over U_k^+ and U_k^- respectively. If the curves Γ_k are numerated counterclockwise then $U_k^+ = U_{k+1}^-$, $n_k^+ = n_{k+1}^-$ and so on, see Fig. 10. Consider some trivialization of the bundle π over U, $\pi^{-1}(U) \cong U \times (\mathbb{R}/2\pi\mathbb{Z})$, such that the section $U \times \{0\}$ does not intersect any chosen section over any component of $U \setminus (\Sigma \cap U)$.

Proposition 4.6. *The index of the point q_0 is given by*

$$\mathbf{ind}(q_0) = \sum_{k=1}^{n} \frac{1}{n_k^+ n_k^-} \sum_{\substack{1 \le i \le n_k^+ \\ 1 \le j \le n_k^-}} \sigma_k(i,j)\,, \tag{8}$$

$$\textit{where} \quad \sigma_k(i,j) = \begin{cases} 0\,, & \textit{if} \quad s_{ki}^- = s_{kj}^+ \quad \textit{on } \Gamma_k \\ -\frac{1}{2}\,, & \textit{if} \quad 0 < s_{ki}^- < s_{kj}^+ < 2\pi \quad \textit{on } \Gamma_k \\ \frac{1}{2}\,, & \textit{if} \quad 0 < s_{ki}^+ < s_{kj}^- < 2\pi \quad \textit{on } \Gamma_k \end{cases}$$

Proof. Let s be a section over $U \setminus (\Sigma \cap U)$ the restriction of which to every component of $U \setminus (\Sigma \cap U)$ coincides with some branch of $\mathbf{s}$. Denote by $s_k^{\pm}$ the restriction of this section to $U_k^{\pm}$. Denote

$$\sigma_k = \begin{cases} 0\,, & \text{if} \quad s_k^- = s_k^+ \quad \text{on } \Gamma_k \\ -\frac{1}{2}\,, & \text{if} \quad 0 < s_k^- < s_k^+ < 2\pi \quad \text{on } \Gamma_k \\ \frac{1}{2}\,, & \text{if} \quad 0 < s_k^+ < s_k^- < 2\pi \quad \text{on } \Gamma_k \end{cases}$$

Lemma 4.7. *For the section s we have*

$$\operatorname{ind}(q_0) = \sum_{k=1}^{n} \sigma_k\,.$$

Proof. The set $\Sigma^{\neq}$ for the section s consists of those components Γ_k for which $s_k^- \neq s_k^+$ on Γ_k. Let us change the coorientation of $\Gamma_k \subset \Sigma^{\neq}$ so that the local intersection number of $S^1_\varepsilon(q_0)$ with Γ_k is positive if $0 < s_k^- < s_k^+$ on Γ_k and negative if $0 < s_k^+ < s_k^-$ on Γ_k. For this choice of the coorientation of $\Sigma^{\neq}$ the natural continuation of the section s over $S^1_\varepsilon(q_0)$ nowhere intersects the section $U \times \{0\}$. So, we have $i(q_0) = 0$ and

$$\mathrm{ind}(q_0) = i(q_0) - \frac{1}{2}(\Sigma^{\neq}, S^1_\varepsilon(q_0)) = \sum_{\Gamma_k \subset \Sigma^{\neq}} -\frac{1}{2}(\Gamma_k, S^1_\varepsilon(q_0)) = \sum \sigma_k \,.$$

It follows from Lemma 4.7 that if σ_k are considered as random variables then $\mathbf{ind}(q_0)$ is the sum of their mathematical expectations. As the branches of $\mathbf{s}$ over U_k^+ and U_k^- are chosen independently, we have that the mathematical expectation of σ_k is given by

$$\overline{\sigma}_k = \frac{1}{n_k^- n_k^+} \sum_{\substack{1 \le i \le n_k^+ \\ 1 \le j \le n_k^-}} \sigma_k(i,j)\,.$$

Proposition 4.6 follows.

We call the section s_{ki}^- (resp. s_{kj}^+) *continuable* over Γ_k if there exists a number i' (resp j') such that $s_{ki}^- = s_{ki'}^+$ (resp. $s_{kj'}^- = s_{kj}^+$) on Γ_k and the number with this property is unique.

Suppose that s_{ki}^- and s_{kj}^+ are two continuable sections and $s_{ki}^- = s_{ki'}^+$, $s_{kj'}^- = s_{kj}^+$ on Γ_k. Then $\sigma_k(i,j) = -\sigma_k(j',i')$. Therefore, all terms $\sigma_k(i,j)$ for which both s_{ki}^- and s_{kj}^+ are continuable cancel in the expression (8) for $\mathbf{ind}(q_0)$.

Corollary 4.8. *The formula* (8) *remains valid if the second summation is taken over those pairs* (i,j), $1 \le i \le n_k^-$, $1 \le j \le n_k^+$, *for which at least one of the sections* s_{ki}^- *and* s_{kj}^+ *is not continuable.*

Proof. of Lemma 4.5. Suppose that the set Σ' is obtained by adding one new line Γ to the given set $\Sigma \subset M$. We need to prove that this does not change the indices of singular points for a given multivalued section $\mathbf{s}$ over $M \setminus \Sigma$. It is clear that adding Γ does not affect on the indices of the points which are not endpoints of Γ.

Suppose that q_0 is one of the two points of $\partial\Gamma$ that is $\Gamma \cup U$ coincides with Γ_k for some k, where U is as in the proof of Proposition 4.6. For this k we have that $n_k^- = n_k^+$ and all the sections $s_{ki}^{\pm}$ are continuable over Γ_k. Therefore, the kth summand in the expression (8) for $\mathbf{ind}(q_0)$ calculated with respect to the restriction of $\mathbf{s}$ to $M \setminus \Sigma'$ vanishes and all the other summands coincide with the corresponding summands for $\mathbf{ind}(q_0)$ calculated with respect to $\mathbf{s}$ itself. This completes the proofs of Lemma 4.5 and Theorem 4.4.

5. Indices of multivalued sections associated with a generic smooth function

Let $f\colon W \to \mathbb{R}$ be a generic smooth function on the total space of S^1-bundle $\pi\colon W \to M$ over a Riemannian surface M. Denote by $\Sigma \subset M$ the bifurcation diagram of f. There are several possibilities to define a multivalued section over $M \setminus \Sigma$.

Theorem 5.1. Let the set $\mathbf{s}(q)$, $q \in M \setminus \Sigma$ be the point of global minimum of f_q. Then the index $\mathbf{ind}(q_0)$ of every singular point $q_0 \in$ SingΣ coincides with that ind(q_0) of Sect. 3. In particular,

$$\begin{array}{lll} \mathbf{ind}(q_0) = \pm\frac{1}{2} \quad , & \textit{if} & q_0 = \Sigma^{(3)}_{\min}\,, \\ \mathbf{ind}(q_0) = \pm(-\frac{1}{2})\,, & \textit{if} & q_0 = \Sigma^{(111)}_{\min}\,, \end{array}$$

and $\mathbf{ind}(q_0) = 0$ for other $q_0 \in$ SingΣ. The sign $\pm$ above is that of Definition 1.6 and the formula (7) in this case is equivalent to the formula (3) of Theorem 1.4.

Proof. This Theorem follows from Corollary 4.3 and the proof of Lemma 3.5.

Theorem 5.2. *Let the set* $\mathbf{s}(q)$*,* $q \in M \setminus \Sigma$ *be the set of all critical points of* f_q*. Then the points* $q_0 \in$ *Sing*Σ *for which* $\mathbf{ind}(q_0) \neq 0$ *are singular points of the* *discriminant. Furthermore,*

$$\mathbf{ind}(q_0) = \pm\frac{2}{(m+1)(m+3)}\,, \qquad \textit{if} \quad q_0 = \Sigma_m^{(3)}\,,$$
$$\mathbf{ind}(q_0) = \pm\frac{(-1)^{l+1}4(m-l)}{(l+m)(l+m+2)(l+m+4)}\,, \qquad \textit{if} \quad q_0 = \Sigma_{l,m}^{(2)(2)}\,.$$

The sign $\pm$ *above is that of Definition* 1.6 *and the formula* (7) *in this case is equivalent to the formula* (4) *of Theorem* 1.4.

The equality (2) of Theorem 1.4 can also be interpreted in terms of the index **ind** for certain multivalued sections over $M \setminus \Sigma$.

Theorem 5.3. *Let* $\mathbf{ind}_{\min}$, $\mathbf{ind}_{\max}$, *and* $\mathbf{ind}_{\mathrm{extr}}$ *be the indices* **ind** *corresponding to the cases when the set* $\mathbf{s}(q)$, $q \in M\setminus\Sigma$ *consists of the point of global minimum, global maximum, and the two points of global extremum of* f_q *respectively. Put* $\mathbf{ind}(q_0) = 2\,\mathbf{ind}_{\mathrm{extr}}(q_0) - \frac{1}{2}\,\mathbf{ind}_{\min}(q_0) - \frac{1}{2}\,\mathbf{ind}_{\max}(q_0)$. *Then*

$$\mathbf{ind}(q_0) = \pm 1\,, \qquad \textit{if} \quad q_0 = \Sigma_{\mathrm{extr}}^{(11)(11)}\,,$$

and $\mathbf{ind}(q_0) = 0$ *for other* $q_0 \in \mathit{Sing}\Sigma$. *The sign* $\pm$ *above is that of Definition* 1.6. *This index satisfies* $\sum \mathbf{ind}(q) = e(\pi)$ *and this formula is equivalent to the formula* (2) *of Theorem* 1.4.

Proof of Theorem 5.2. Let $q_0 \in \mathrm{Sing}\Sigma$ be some point. The family of functions f_q, $q \in M$ formsGGG a deformation of the multigerm corresponding to the critical points of f_{q_0}. Let $(\mathbb{R}^2, 0)$ be the base of the versal deformation of this multigerm. The deformation given by the family f_q can be induced from the versal deformation by a smooth map germ $\varphi\colon (M, q_0) \to (\mathbb{R}^2, 0)$. If the function f is in general position then the mapping φ is nondegenerate for every $q_0 \in \mathrm{Sing}\Sigma$. Therefore, the index of the point q_0 coincides up to a sign with the index of the origin for the corresponding versal deformation. This sign is positive or negative depending on coincides or not the orientation of M with that induced by the mapping φ. Hence, to prove Theorem 5.2 it is enough to compute the index of the origin in the versal deformation of every singular multigerm of codimension 2. We fix the orientation of the base of the versal deformation so that the origin is a positive point in the sense of Definition 1.6.

The set of critical points $\mathbf{s}(q)$, $q \in M \setminus \Sigma$, does not depend on the critical values of the function f_q. Therefore, all possible points for which $\mathbf{ind}(q_0) \neq 0$ are singular points of the discriminant, that are cusp points of the discriminant of the type $\Sigma^{(3)}_m$ ($m > 0$ is odd), and its selfintersection points of the type $\Sigma^{(2)(2)}_{l,m}$ ($0 \le l \le m$, $l + m$ is even).

For the point of type $\Sigma^{(3)}_m$ the bifurcation diagram is shown in Fig. 11. The complement to the discriminant consists of the two domains $U_1^+ = U_2^-$ and $U_2^+ = U_1^-$. The function f_q has $m+3$ and $m+1$ critical points over the first and the second domain respectively. The sections which are not continuable correspond to the couples of critical points which cancel over the discriminant. Therefore, by Corollary 4.8

$$\sum \sigma_1(i,j) = 2(m+1)\frac{1}{2} = m+1\,, \qquad \sum \sigma_2(i,j) = 2\frac{1}{2} + 2m(-\frac{1}{2}) = 1-m\,,$$

and by Proposition 4.6 we get

$$\mathbf{ind}(q_0) = \frac{m+1}{(m+1)(m+3)} + \frac{1-m}{(m+1)(m+3)} = \frac{2}{(m+1)(m+3)}\,.$$

For the selfintersection point of the type $\Sigma_{l,m}^{(2)(2)}$ the complement to the discriminant consists of four components as shown in Fig. 12. The number of critical points of f_q over these domains is equal to $l+m$, $l+m+2$, $l+m+4$, and $l+m+2$ respectively. The signs of the indices $\sigma_k(i,j)$ are seen from Figure. We get

$$\sum \sigma_1(i,j) = 2l\frac{1}{2} + 2m(-\frac{1}{2}) = l - m\,,$$
$$\sum \sigma_2(i,j) = 2(l+m)\frac{1}{2} = l + m\,,$$
$$\sum \sigma_3(i,j) = 2(l+2)(-\frac{1}{2}) + 2m\frac{1}{2} = m - l - 2\,,$$
$$\sum \sigma_4(i,j) = 2(l+m+2)(-\frac{1}{2}) = -l - m - 2\,.$$

This gives by Proposition 4.6

$$\begin{aligned}\mathbf{ind}(q_0) \quad = \quad & \frac{\sum \sigma_1(i,j)}{(l+m)(l+m+2)} + \frac{\sum \sigma_2(i,j)}{(l+m)(l+m+2)} + \frac{\sum \sigma_3(i,j)}{(l+m+2)(l+m+4)} + \\ & + \frac{\sum \sigma_4(i,j)}{(l+m+2)(l+m+4)} = \frac{4(l-m)}{(l+m)(l+m+2)(l+m+4)}.\end{aligned}$$

It remains to observe that the counterclockwise orientation of the plane of Fig. 12 is positive in the sense of Definition 1.6 if l and m are even and it is negative if these numbers are odd. Note that if $l = m$ then there is no natural orientation in the space of the versal deformation. But for the either choice of this orientation the index **ind** vanishes.

Proof of Theorem 5.3. First, observe that

$$\begin{aligned}\sum \mathbf{ind}(q) &= 2\sum \mathbf{ind}_{\rm extr}(q) - \frac{1}{2}\sum \mathbf{ind}_{\min}(q) - \frac{1}{2}\sum \mathbf{ind}_{\max}(q)\\ &= (2-\tfrac{1}{2}-\tfrac{1}{2})e(\pi) = e(\pi)\end{aligned}$$

by Theorem 4.4. The indices $\mathbf{ind}_{\min}(q)$ and $\mathbf{ind}_{\max}(q)$ are given by Theorem 5.1. Calculations of the index $\mathbf{ind}_{\rm extr}(q)$ for different singular points $q \in \mathrm{Sing}\Sigma$ are similar to those in the proof of Theorem 5.2. It is left to the reader to verify that this index is equal to $\frac{1}{8}$ for a positive point of type $\Sigma^{(3)}_{\min}$ or $\Sigma^{(3)}_{\max}$; it equals $-\frac{1}{8}$ for a point of type $\Sigma^{(111)}_{\min}$ or $\Sigma^{(111)}_{\max}$; and it equals $\frac{1}{2}$ for a point of type $\Sigma^{(11)(11)}_{\rm extr}$ respectively. Theorem 5.3 follows.

6. Convex plane curves and positive coorientations of singularities

In this Section we prove Theorem 1.5. Let $c = (c_1, c_2)\colon S^1 \to \mathbb{R}^2$ be a convex plane curve parameterized counterclockwise. With this curve we associate the following family of functions on the circle

$$f_q(t) = ||c(t) - q||^2 \quad t \in S^1 = \mathbb{R}/2\pi\mathbb{Z}, \quad q = (q_1, q_2) \in \mathbb{R}^2.$$

Let $q^* \in \mathbb{R}^2$ be a singular point of the family f. The linearization of f at q^*

$$df(f, \Delta q_1, \Delta q_2) = \frac{\partial f}{\partial q_1}\Big|_{q^*}\Delta q_1 + \frac{\partial f}{\partial q_2}\Big|_{q^*}\Delta q_2 = 2(q_1^* - c_1(t))\Delta q_1 + 2(q_2^* - c_1(t))\Delta q_2$$

coincides, up to a linear transformation with the restriction to c of a linear function on $\mathbb{R}^2$.

By Definition 1.6, the assertion on positiveness of the point q^* is equivalent to the following. Consider the following mappings of the space $\mathbb{R}^{2*}$ of linear functions to the coordinate space $\mathbb{R}^2$

$$\begin{aligned}&\text{a) } l \mapsto \Big(l(c(t_3)) - l(c(t_1)),\, l(c(t_4)) - l(c(t_2))\Big);\\ &\text{b) } l \mapsto \Big(l(c'(t_1)),\, l(c''(t_1))\Big);\\ &\text{c) } l \mapsto \Big(l(c(t_2)) - l(c(t_1)),\, l(c(t_3)) - l(c(t_2))\Big);\end{aligned}$$

where $t_1 < t_2 < t_3 < t_4 < t_1 + 2\pi$ are some fixed points. *These mappings are orientation preserving isomorphisms.*

This assertion is equivalent, in turn, to the assertion that *the following vectors*

$$\begin{aligned} &a)\ e_1 = c(t_3) - c(t_1)\,,\ \ e_2 = c(t_4) - c(t_2)\,;\\ &b)\ e_1 = c'(t_1)\,,\ \ e_2 = c''(t_1)\,;\\ &c)\ e_1 = c(t_2) - c(t_1)\,,\ \ e_2 = c(t_3) - c(t_2) \end{aligned}$$

form a positive basis on the plane, which is evident.

This essentially completes the proof of Theorem 1.5. To compute the Chern-Euler number associated with a convex plane curve it is enough to compute it for any particular curve, for example, for that shown in Fig. 2 for which it equals 1.

The reason that this number is 1 is the following. Let $D(R) \subset \mathbb{R}^2$, $R \gg 0$, be a disk of great radius R and $S(R) = \partial D(R)$. Then for $q \in S(R)$ the function f_q has a nondegenerate point of global minimum, a nondegenerate one of global maximum and no other critical points. When the point q goes along the circle $S(q)$ the two critical points of f_q make one turn rotation on the fiber S^1. Therefore, we can modify the function f over a neighborhood of $S(R)$ without changing the bifurcation diagram and identify the fibers over $S(R)$ so that the restrictions of f to the fibers over $S(R)$ are represented by the same function on the circle. Thus, we get a circle bundle over $D(R)/S(R) \cong S^2$ and a function in its total space. This bundle is not trivial. It is isomorphic to the Hopf bundle $S^3 \to S^2$. Therefore, the number 1 in Theorem 1.5 is the Chern-Euler number of the Hopf bundle.

7. Singularities of odd functions

A function $f\colon\ S^1 \to \mathbb{R}$ is called *odd* if it satisfies

$$f(t+\pi) = -f(t)\,, \quad t \in S^1 = \mathbb{R}/2\pi\mathbb{Z}\,.$$

A function on the total space of an S^1-bundle $\pi\colon\ W \to M$ is called *odd* if its restriction to each fiber is odd. Let $\overline{W} = W/\{\pm 1\}$, where we consider -1

as an element of the group $S^1 = U(1)$ with its natural action on W. Denote $\overline{\pi}\colon \overline{W} \to M$ the natural projection induced by π. Consider the one-dimensional vector bundle over $\overline{W}$ which changes its orientation along the fibers of $\overline{\pi}$. Sections of this bundle are called *Möbius functions.*

The function f on W defines a Möbius function $\overline{f}$ on $\overline{W}$. Critical points of restrictions of $\overline{f}$ to the fibers one-to-one correspond to the pairs of opposite critical points of the restrictions of f. Hence, the set of critical points of restrictions of $\overline{f}$ defines a multivalued section of the bundle $\overline{\pi}$, and we may make use of the results of Sect. 4 to find expressions for the Chern-Euler numbers of the bundles π and $\overline{\pi}$.

Our notations for singularities of odd functions are similar to those we used for usual functions. Let $\Sigma^{(3)}_{\mathrm{extr}}$ and $\Sigma^{(111)}_{\mathrm{extr}}$ be the sets of such points $q \in M$ that the Möbius function $\overline{f}_q$ gets its global extremum at a degenerate point and at three different points respectively. We subdivide the set $\Sigma^{(111)}_{\mathrm{extr}}$ into $\Sigma^{(111)}_{\mathrm{extr!}}$ and $\Sigma^{(111)}_{\mathrm{extr?}}$ according to alternate or not the three points of global minimum and those of global maximum of the odd function f_q. Let $\Sigma^{(3)}_m \subset M$ be the set of such points that the restriction of $\overline{f}$ to the corresponding fiber has a degenerate critical point of multiplicity 3 and m other nondegenerate critical points ($m \geq 0$ is even). Let $\Sigma^{(2)(2)}_{l,m} \subset M$ be the set of such points that the restriction of $\overline{f}$ to the corresponding fiber has 2 degenerate critical points and

l and m nondegenerate ones respectively on the two arcs with the ends at the degenerate critical points. In this case $l+m$ is odd, and to distinguish between $\Sigma^{(2)(2)}_{l,m} \subset M$ and $\Sigma^{(2)(2)}_{m,l} \subset M$ we assume that l is even and m is odd.

Theorem 7.1. *There is a natural way to define a sign of every singular point such that with the notation $\#\Sigma^\alpha_\beta$ for the algebraic number of points of the type Σ^α_β counted with their signs, the following relation hold*

$$
\begin{aligned}
e(\overline{\pi}) &= 2e(\pi) = \#\Sigma^{(111)}_{\mathrm{extr!}}\,, && (9)\\
e(\overline{\pi}) &= 2e(\pi) = \tfrac{1}{2}\#\Sigma^{(3)}_{\mathrm{extr}} + \tfrac{1}{2}\#\Sigma^{(111)}_{\mathrm{extr!}} - \tfrac{1}{2}\#\Sigma^{(111)}_{\mathrm{extr?}}\,, && (10)\\
e(\overline{\pi}) &= 2e(\pi) = \sum \frac{2}{(m+1)(m+3)}\#\Sigma^{(3)}_m + \\
&\quad + \sum \frac{4(l-m)}{(l+m)(l+m+2)(l+m+4)}\#\Sigma^{(2)(2)}_{l,m} = \\
&= \frac{2}{3}\#\Sigma^{(3)}_0 - \frac{4}{15}\#\Sigma^{(2)(2)}_{0,1} + \frac{2}{15}\#\Sigma^{(3)}_2 - \frac{4}{35}\#\Sigma^{(2)(2)}_{0,3} + \frac{4}{105}\#\Sigma^{(2)(2)}_{2,1} + \\
&\quad + \frac{2}{35}\#\Sigma^{(3)}_4 + \dots\,. && (11)
\end{aligned}
$$

Note, that the equalities (9) and (10) give

$$
e(\overline{\pi}) = 2e(\pi) \;=\; \tfrac{2}{3}\#\Sigma^{(3)}_{\mathrm{extr}} - \tfrac{2}{3}\#\Sigma^{(111)}_{\mathrm{extr?}}\,. \qquad (12)
$$

Example 7.2. Consider the unite sphere in $\mathbb{C}^2$ as the total space of the Hopf bundle $S^3 \to \mathbb{C}P^1 = S^2$. Consider the function $f\colon\ S^3 \to \mathbb{R}$ given by

$$
f(z_1, z_2) = \mathbb{R}e(z_1 + z_3^3)\,, \quad (z_1, z_2) \in S^3 \subset \mathbb{C}^2\,.
$$

This function is odd. The bifurcation diagram of this function is shown in Fig. 13. The numbers in the brackets near the singular points of Σ are terms entering into the right hand side expressions of equalities (9) and (11) (or (12)).

Proof. First, observe that if W is the bundle of unit circles in the complex linear bundle U over M then $\overline{W}$ is the bundle of unit circles in the bundle $U \otimes_{\mathbb{C}} U$ and we get

$$
e(\overline{\pi}) = c_1(U \otimes U) = 2c_1(U) = 2e(\pi)\,.
$$

The computation of indices of singular points of Σ is similar to that in the proof of Theorem 1.4. Formulas (9) and (10) correspond to the indices $i(q)$ and $\operatorname{ind}(q)$ defined in Sect. 3 for the section $s\colon M\setminus\Sigma\to\overline{W}$ given by points of global extremum of $\overline{f}_q$. Formula (11) corresponds to the index $\mathbf{ind}(q)$ for the multivalued section given by all critical points of $\overline{f}_q$.

An independent proof of (11) is given in [7].

Similar formula to (11) describes the Chern-Euler number of π in terms of degenerations of *zero level* of f_q (or $\overline{f}_q$). Denote by $Z^{(3)}_m$ ($m\geq 0$ is even) the set of points $q\in M$ such that the zero level of $\overline{f}_q$ has $m+1$ points one of which is a degenerate critical point. Denote by $Z^{(2)(2)}_{l,m}$ ($l\geq 0$ is even, $m>0$ is odd) the set of points $q\in M$ such that the zero level of $\overline{f}_q$ has two critical points and l and m nondegenerate ones on the two arcs connecting degenerate ones.

Corollary 7.3. *There is a natural way to define a sign of every point of type Z^α_β such that the formula* (11) *of Theorem* 7.1 *remains valid after exchanging $\#\Sigma^\alpha_\beta$ by $\#Z^\alpha_\beta$*

$$\begin{aligned}e(\overline{\pi}) &= 2e(\pi)=\tfrac{2}{3}\#Z^{(3)}_0-\tfrac{4}{15}\#Z^{(2)(2)}_{0,1}+\tfrac{2}{15}\#Z^{(3)}_2-\tfrac{4}{35}\#Z^{(2)(2)}_{0,3}+\\ &+\tfrac{4}{105}\#Z^{(2)(2)}_{2,1}+\tfrac{2}{35}\#Z^{(3)}_4+\dots.\end{aligned}$$

Proof. Critical points of a function are zero points of its derivative. And vice versa, every odd function has unique odd primitive the critical points of which are zeros of original function.

Example 7.4. Let $\lambda_0\subset P^2$ be a projective line. Consider a smooth curve $c\subset P^2$ close to λ_0 (together with derivatives). Fix an orientation of λ_0 so that c is also oriented. Let S^{2*} be the space of *oriented* projective lines. Let $U\subset S^{2*}$ be a small neighborhood of λ_0 such that any line $\lambda\in\partial U$ has unique transversal intersection point with c.

Realize P^2 as the quotient space of the unite sphere $S^2\subset\mathbb{R}^3$ over the antipodal involution. Let $\tilde{c}\subset S^2$ be the inverse image of c under this covering. The space S^{2*} can be realized as a unite sphere in the space $\mathbb{R}^{3*}$ of linear functions on $\mathbb{R}^3$. Denote by $f_\lambda\colon\ \tilde{c}\to\mathbb{R}$, $\lambda\in S^{2*}\subset\mathbb{R}^{3*}$ the restriction of the corresponding linear function to $\tilde{c}\subset S^2\subset\mathbb{R}^3$.

The family of functions f_λ can be considered as a function on the space of trivial bundle $\tilde{c}\times U\to U$. This function is odd. The points of type $Z^{(3)}_m$ correspond to tangent lines of type I_m of the curve c at inflection points and those of type $Z^{(2)(2)}_{l,m}$ correspond to bitangent lines of type $B_{l,m}$ (see 1.10a). The formula of Corollary 7.2 is equivalent in this case to the formula of Theorem 1.10. To prove that the 'Chern-Euler number' of this family is equal to 2 we use the same arguments as at the end of Sect. 6.

The same construction gives an index of any oriented projective line with respect to any noncontractible immersed projective curve. By Corollary 7.2, the sum of these indices is equal to the Chern-Euler number of the trivial bundle $\tilde{c}\times S^{2*}\to S^{2*}$, i.e. to zero. But this is evident, the change of orientation of a projective line changes also the sign of its index. Therefore, this is *not* the way we define the index in Theorem 1.10 in general case, see Sect. 8.

8. Global invariants of projective plane curves

Let $c\subset P^2$ be a generic smooth immersed closed curve. Denote by $[c]\in H_1(P^2)=\mathbb{Z}_2$ the homology class represented by c. The curve c may have any number of components. Assume the intersection of c with any projective line is not empty. For example, this is the case if $[c]\neq 0$. Denote by P^{2*} the space of all projective lines in P^2 and by S^{2*} its two-sheeted covering, the space of all *oriented* projective lines. Consider the tautological S^1-bundle

$$\pi\colon\ F\to S^{2*}\,.$$

The space F of this bundle is the set of pairs of kind (an oriented projective line, a point of this line). The projection π is the projection onto the first factor. For a generic line $\lambda\in S^{2*}$ this line intersects c transversally at some finite number of different points. The set $\lambda\cap c$ considered as a subset of λ defines a singular multivalued section $\mathbf{s}$ of the bundle π. We would like to apply the results of Sect.4 to this section to obtain global invariants of the curve c.

Denote by $\mathbf{ind}(\lambda)$ the index of $\lambda \in S^{2*}$ with respect to the section $\mathbf{s}$ as it was defined in Sect. 4. The multivalued section $\mathbf{s}$ is not related to any Möbius (or usual) function. So, we compute indices of its singular points directly from its definition using Corollary 4.8.

Lemma 8.1. *The index* $\mathbf{ind}(\lambda)$ *depends neither on the orientation of* λ *nor on the orientation of* c.

Proof. This index does not depend on the orientation of c by definition. Note, that the involution inverting orientations of projective lines changes orientations of both base and fibers of the bundle π. This proves that the index $\mathbf{ind}$ does not depend on the orientation of λ as well.

Lemma 8.2. *The Chern-Euler number of the bundle* π *is equal to* 4.

Proof. Let $\pi'\colon SU(2) \to S^2$ and $\pi''\colon SO(3) \to S^2$ be the natural bundles. Then there are natural two-sheeted coverings $SU(2) \to SO(3)$, $SO(3) \to F$. Therefore, as in the proof of Theorem 7.1, we have

$$e(\pi) = 2e(\pi'') = 4e(\pi') = 4\,,$$

because π' is the Hopf bundle and $e(\pi') = 1$.

Theorem 1.10 is the direct corollary of Lemmas 8.5, 8.6, and the following one.

Lemma 8.3. *The index* $\mathbf{ind}(\lambda)$ *of a projective line* λ *coincides with that defined in* 1.10.

Proof. Codimension 2 singularities for the section $\mathbf{s}$ are projective lines which are called special in 1.10. The bifurcation diagram in a neighborhood of points of types I_m, $B_{l,m}$ are diffeomorphic to those shown in Fig. 11 and 12. Calculations of indices are similar to those in the proof of Theorem 5.2. Thus, we get

$$\mathbf{ind}(\lambda) = \frac{2}{(m+1)(m+3)}\,, \qquad \lambda \in I_m\,,$$

$$\mathbf{ind}(\lambda) = \frac{4(l-m)}{(l+m)(l+m+2)(l+m+4)}\,, \qquad \lambda \in B_{l,m}\,.$$

For $\lambda \in X_m$ the bifurcation diagram is shown in Fig. 14. The complement to the bifurcation diagram consists of 4 components. The section $\mathbf{s}$ has $m+1$ branches over one of these domains and $m+3$ branches over the others. The signs of the indices $\sigma_k(i,j)$ are seen from Figure. By Corollary 4.8, we get

$$\sum \sigma_1(i,j) = 2(m+1)\frac{1}{2} = m+1\,, \qquad \sum \sigma_2(i,j) = 0\,,$$

$$\sum \sigma_3(i,j) = 0\,, \qquad \sum \sigma_4(i,j) = 2\frac{1}{2} + 2m(-\frac{1}{2}) = 1-m\,,$$

and by Proposition 4.6, we get

$$\mathbf{ind}(\lambda) = \frac{m+1}{(m+1)(m+3)} + \frac{1-m}{(m+1)(m+3)} = \frac{2}{(m+1)(m+3)}\,.$$

It remains to show that if λ passes through a selfintersection point a of the curve c and either passes through another selfintersection point of c or touches c at some point different from a then the index $\mathbf{ind}$ vanishes at such λ. Let $c' \subset P^2$ be a smooth closed immersed curve which coincides with c everywhere except a small neighborhood U of the point a where the curve c' consists of two nonintersecting arcs transversal to λ, as in Fig. 15. (the curve c' may consist of two components).

The assertion that $\mathbf{ind}(\lambda) = 0$ is equivalent to the following two.

1) *The index* $\mathbf{ind}(\lambda)$ *calculated with respect to the curve* c' *is equal to* 0.

2) *The indices* $\mathbf{ind}(\lambda)$ *calculated with respect to the curves* c *and* c' *coincide.*

The first assertion is evident, because λ is the point of singularity of codimension 1 of the multivalued section defined with respect to the curve c'.

Let us prove the second one. In a neighborhood of λ, the bifurcation diagram for the curve c is the union of the bifurcation diagram for the curve c' and the curve Γ of projective lines passing through the point a. For all strata of the bifurcation diagram for c', the indices $\sigma_k(i,j)$ coincide with the corresponding indices calculated for the curve c. It remains to show that $\sum_k \sigma_k(i,j) = 0$ for the numbers k corresponding to the two branches of the curve Γ and calculated with respect to the curve c. This assertion follows from the symmetry, as in the proof of Corollary 4.8.

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