THE INTEGRAL CLOSURE OF IDEALS AND WHITNEY EQUISINGULARITY OF GERMS OF HYPERSURFACES

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Abstract

In this article we use the Newton polyhedron of an ideal in the ring of complex holomorphic germs (or real analytic germs) to calculate the integral closure of such an ideal.
Applying these results, we follow the algebraic approach used by Teissier to characterize the Whitney equisingularity of hypersurfaces through the integral closure of ideals to show when the stratification of a real or complex hypersurface $X = X_G$ in $k^n \times k (k\ \mathbb{R}$ ou $\mathbb{C})$ along the parameter space $0 \times k$ at zero satisfies the Whitney conditions a and $b (X = X_G$ is defined by $G^{-1}(0)$, where $G : k^n \times k, 0 \to k, 0$ is a one parameter deformation of a germ g which has an algebraically isolated singularity at 0).
We consider the Newton polyhedron of the ideal $I = \left\langle x_i \frac{\partial g}{\partial x_j}; i, j = 1, ..., n \right\rangle$ to characterise the integral closure of this ideal.
We also give necessary and sufficient conditions for the Whitney equisingularity of the pair $\{X_G, 0 \times k\}$ in terms of integral closure of this ideal.

Resumo

Neste artigo nós consideramos o poliedro de Newton de um ideal com codimensão finita no anel dos germes de funções holomorfas (ou germes de funções analíticas reais) para calcular o fecho integral deste ideal.
Como aplicação, seguindo a abordagem algébrica de Teissier que caracteriza a equisingularidade de Whitney para hipersuperfícies em termos do fecho integral de ideais, mostramos quando a estratificação de uma hipersuperfície real ou complexa $X = X_G$ em $k^n \times k (k = \mathbb{R}$ ou $\mathbb{C})$ ao longo do espaço de parâmetros $0 \times k$ satisfaz as condições a e b de Whitney $(X = X_G$ é definida como $G^{-1}(0)$, onde $G : k^n, 0 \to k, 0$ é uma deformação a um parâmetro de um germe g com singularidade algebricamente isolada na origem.

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Considerando o poliedro de Newton do ideal $I = \left\langle x_i \frac{\partial g}{\partial x_j}; i, j = 1, ..., n \right\rangle$ caracterizamos o fecho integral deste ideal.
Além disto são dadas condições necessárias e suficientes para a equisingularidade de Whitney do par $\{X_G, 0 \times k\}$ em termos do fecho integral do ideal I.

The link between the integral dependence relations of ideals and some algebraic geometrical incidence relations is very interesting, as we can see in the works of Teissier, [T1], [T2] and Gaffney [G].

On the other side, it is well known that the Newton polyhedron of an arbitrary germ of function contains much useful information, for example, Kouchnirenko gives in [5] a formula for the Milnor number of Newton non-degenerate germs, Damon-Gaffney in [2] and Yoshinaga in [11] give conditions for the topological triviality of families of isolated singularity deformations, Varchenco in [10] shows a formula for the index of singularity of oscillatory integrals.

In this article we use the Newton polyhedron of an ideal in the ring of complex holomorphic germs (or real analytic germs) to calculate the integral closure of such an ideal. We showed in [7] that the integral closure of an ideal of holomorphic germs which has Newton non-degenerate polyhedrom is determined precisely by the set of elements that are in this Newton polyhedron.

Here, extending the results of [7], we show how to characterize the monomials x^m which are in the integral closure of any ideal of finite codimension in these rings.

In the second part of this article we are interested in the problem of to find a stratification of an hypersurface which satisfies the Whitney conditions **a** and **b** of equisingularity.

We shall consider a real or complex hypersurface $X = X_G$ in $k^n \times k$ ($k = \mathbb{R}$ or $\mathbb{C}$), defined by $G^{-1}(0)$, where $G : k^n \times k, 0 \to k, 0$ is a one parameter deformation of a germ g which has an algebraically isolated singularity at 0, we shall show when the stratification of X along the parameter space $0 \times k$ at zero satisfies the Whitney conditions.

Following the algebraic approach used by Teissier to characterize the Whitney equisingularity through the integral closure of ideals, we defined in [6] the polyhedron of equisingularity of a germ g, denoted by $\mathcal{E}(g)$ and characterized a convex subset of the Newton polyhedron of a commode germ g which is a subset of $\mathcal{E}(g)$.

Here we consider this problem for any germ g which has an algebraically isolated singularity at 0 and any deformation G of g. We consider the Newton polyhedron of the ideal $I = \left\langle x_i \dfrac{\partial g}{\partial x_j}; i, j = 1, \ldots, n \right\rangle$ instead of the Newton polyhedron of the germ g, considered in [6].

Applying the results of the first part of this article, we calculate the integral closure of the ideal I and show precisely how to characterise the polyhedron $\mathcal{E}(g)$.

We also give necessary and sufficient conditions for the Whitney equisingularity of the pair $\{X_G, 0 \times k\}$ in terms of the polyhedron $\mathcal{E}(g)$.

1. The Integral closure of ideals

We fix a system of local coordinates x of K^n and consider germs $g : k^n, 0 \to k, 0$ which are real analytic when $k = \mathbb{R}$ or holomorphic when $k = \mathbb{C}$.

We denote by K_n (in the appropriate category), the ring $K[[x]]$ of convergent power series.

Let I be an ideal in a ring A, an element $h \in A$ is said to be integral over I if it satisfies an integral dependence relation $h^n + a_1 h^{n-1} + \ldots + a_n = 0$ with $a_i \in I^i$.

The set of such elements form an ideal called the integral closure of I.

We shall denote this set by $\overline{I}$.

When $A = \mathcal{O}_{X,x_0}$, the local ring of a complex analytic set, Teissier gives in [8] various notions equivalent to the above concept.

Proposition 1.1. *Let I be an ideal in $\mathcal{O}_{X,x_0}$. The following statements are equivalent:*

i) $h \in \overline{I}$

ii) **Growth condition:** *For each choice of generators* $\{g_i\}$ *of* I *there exists a neighbourhood* U *of* x_0 *and a constant* $\mathcal{E} > 0$ *such that for all* $x \in U$*:*

$$| \; h(x) \; | \;\leq\; \mathcal{E} \cdot sup_i \; |g_i(x)|$$

iii) **Valuative criterion:** *For each analytic curve* $\varphi\colon (\mathbb{C}, 0) \to (X, x_0)$*,* $h \circ \varphi$ *lies in* $(\varphi^*(I))\, K_1$*.*

iv) *There exists a faithful* $\mathcal{O}_{X,x_0}$*-module* L *of finite type such that* $h \cdot L \subset I \cdot L$*.*

The real case: When X is real analytic, this algebraic definition of the integral closure is not adequate, Gaffney adopted in [3] the Valuative criterion as the definition for the real integral closure of an ideal, this definition coincides with Robson's notion of the complete hull of the ideal I, see [1].

We shall consider here Gaffney's definition for the real integral closure of an ideal. In this case, the equivalence between the Growth condition and the Valuative criterion remains true.

For each germ $g(x) = \sum a_k x^k$, we define $supp\ g = \{k \in \mathbb{Z}^n \;:\; a_k \neq 0\}$.

Definition 1.2. *Let* I *be an ideal in* K_n*, we define:* $supp\ I = \cup \{supp\ g \;:\; g \in I\}$.

Definition 1.3. *The Newton polyhedron of* I*, denoted by* $\Gamma_+(I)$*, is the convex hull in* $\mathbb{R}^n_+$ *of the set* $\cup \left\{k + v \;:\; k \in supp\ I,\ v \in \mathbb{R}^n_+\right\}$.

We shall denote by $\Gamma(I)$*, the union of all compact faces of* $\Gamma_+(I)$*.*

In the sequel we shall consider $I = \langle g_1, g_2, \ldots, g_s \rangle$ an ideal of finite codimension in K_n.

For any ideal I we describe, with respect to $\Gamma_+(I)$, a partition into convex cones of the positive octant in a space $\mathbb{R}^{n*}$ which is dual to $\mathbb{R}^n$.

Let $(a_1, \ldots, a_n)$ be dual coordinates in $\mathbb{R}^{n*}$. For each $a = (a_1, \ldots, a_n) \in \mathbb{R}^{n*}_+$ we define:

Definition 1.4.

(a) $\ell(a) = min\{\langle a, k\rangle \ : \ k \in \Gamma_+(I)\}$, *where* $\langle a, k\rangle = \sum_{i=1}^{n} a_i k_i$.

(b) $\Delta(a) = \{k \in \Gamma_+(I) : \ \langle a, k\rangle = \ell(a)\}$.

(c) *Two vectors* $a, a' \in \mathbb{R}_+^{n*}$ *are said to be equivalent if* $\Delta(a) = \Delta(a')$.

The vector a is called a primitive integer vector if a is the vector with minimum length in $\mathrm{C}(a) \cap (\mathbb{Z}_+^n - \{0\})$, where $\mathrm{C}(a)$ is the half ray emanating from 0 passing through a.

It is easy to see that each $n-1$-dimensional face Δ in $\Gamma(I)$ is associated to a primitive integer $a \in \mathbb{R}_+^{n*}$ such that $\Delta = \Delta(a)$. Given an $\ell \in \mathbb{R}_+$, we call

$$\Delta_\ell(a) = \{m \in \Gamma_+(I)\colon \ < m, a > \leq \ell\}.$$

Given a finite subset $\Delta \subset \Gamma_+(I)$, for any germ $g(x) = \sum a_k x^k$ we call $g_\Delta = \sum_{k\in\Delta} a_k x^k$.

If Δ is a face of $\Gamma_+(I)$, $\mathrm{C}(\Delta)$ denotes the cone of half-rays emanating from 0 and passing through Δ. Since I is an ideal of finite codimension in K_n and $\Gamma_+(I)$ is a convex polyhedron in $\mathbb{R}_+^n$, the collection of all $\mathrm{C}(\Delta)$ gives a polyhedral decomposition for $\mathbb{R}_+^n$.

Definition 1.5. *A subset* $\Delta \subset \Gamma_+(I)$ *is Newton non-degenerate if the ideal generated by* $g_{1_\Delta}, g_{2_\Delta}, \ldots, g_{s_\Delta}$ *has finite codimension in* $C[[\Delta]]$.

Definition 1.6. *An ideal* I *is Newton non-degenerate if all compact faces* $\Delta \subset \Gamma(I)$ *are Newton non-degenerate.*

The above definition is equivalent to the following.

I is Newton non-degenerate if for each compact face $\Delta \subset \Gamma(I)$, the equations $g_{1_\Delta}(x) = g_{2_\Delta}(x) = \ldots = g_{s_\Delta}(x) = 0$ have no common solution in $(K-\{0\})^n$.

Definition 1.7. *For each* $n-1$*-dimensional compact face* $\Delta(a^j) \subset \Gamma(I)$ *we let*

$$Q_j = min.\{\ell\colon \Delta_\ell(a^j) \ \text{ is } \ \text{non-degenerate}\}.$$

Definition 1.8. *We denote by $C(\overline{I})$ the convex hull in $\mathbb{R}^n_+$ of the set $\cup\{m : x^m \in \overline{I}\}$.*

We showed in [7] the following results.

Lemma 1.9. $C(\overline{I}) \subset \Gamma_+(I)$.

Theorem 1.10. *Let I be an ideal of finite codimension in K_n.*

Then I is Newton non-degenerate if and only if $\Gamma_+(I) = C(\overline{I})$.

When the Newton polyhedron of an ideal I has compact faces which are degenerate, for each $n-1$-dimensional compact face $\Delta(a^j)$ of $\Gamma(I)$, the number Q_j is the key to compute $C(\overline{I})$

Theorem 1.11. *Let I be an ideal of finite codimension in K_n. For each $m \in \mathbb{Z}^n_+$ the following statements are equivalent:*

1. $m \in C(\overline{I})$.
2. *The inequality $Q_j \leq \langle m, a^j\rangle$ holds for each $(n-1)$-dimensional compact face $\Delta(a^j) \subset \Gamma(I)$.*

Examples:

1. Let $I_1 = \langle x^8 + y^8,\ xy^5 - x^5y\rangle$

The vertices of the polyhedron $\Gamma_+(I_1)$ are $\{(0,8),(1,5),(5,1),(8,0)\}$.

The primitive integers corresponding to the 1-dimensional faces of $\Gamma(I_1)$ are $a^1 = (1,0)$, $a^2 = (0,1)$, $a^3 = (3,1)$, $a^4 = (1,1)$ and $a^5 = (1,3)$, with $l(a^1) = l(a^2) = 0$, $l(a^3) = l(a^5) = 8$, and $l(a^4) = 6$.

This ideal I_1 is degenerate in the face $\Delta(a^4)$ since any point (x,x), with $x \neq 0$ is a solution for the equations $g_{1_{\Delta(a^4)}} = g_{2_{\Delta(a^4)}} = 0$. $\Delta_8(a^4)$ is the first non-degenerate set associated to $\Delta(a^4)$, hence $Q_4 = 8$.

The faces $\Delta(a^3)$ and $\Delta(a^5)$ are non-degenerate, then we have $Q_3 = \ell(a^3) = 8$ and $Q_5 = \ell(a^5) = 8$.

Applying the Theorem 1.11, we see that $x^m \in C(\overline{I}_1)$ if and only if $m_1 + m_2 \geq 8$.

2. Let $I_2 = \langle x^8 + x^5 y,\ y^8 + xy^5 \rangle$

We have $\Gamma_+(I_2) = \Gamma_+(I_1)$, but the ideal I_2 is degenerate in the faces $\Delta(a^3)$ and $\Delta(a^5)$, here $Q_3 = 16$, $Q_5 = 16$, and $Q_4 = 6$.

Applying the Theorem 1.11, we see that $x^m \in C(\overline{I}_2)$ if and only if $3m_1 + m_2 \geq 16$ and $m_1 + 3m_2 \geq 16$

3. Let $I_3 = \langle x^8 + xy^5,\ y^8 + x^5 y \rangle$

We have $\Gamma_+(I_3) = \Gamma_+(I_1)$, but I_3 is Newton non-degenerate, hence $C(\overline{I}_3) = \Gamma_+(I_3)$.

Toroidal Embeddings

The theory of toroidal embeddings was developed by Kempf et all in [4]. The procedure of constructing the toroidal embedding associated to a given Newton polyhedron is a local modification of Khovanskii's method of assigning a compact complex nonsingular toroidal manifold to an integer-valued compact convex polyhedron in K^n. This construction, that we summarise below, is due to Varchenko [10, pp.183–184] and form the essential tool used in proving Theorem 1.11.

Considering the equivalence defined in the item [c] of the definition 1.4, we see that any equivalence class is naturaly identified with a convex cone with its vertex at zero that is specified by finitely many linear equations and strictly linear inequalities with rational coefficients.

The closures of equivalence classes specify a partition Σ_0 of the positive cone $\mathbb{R}_+^{n*}$ into closed convex cones that have the properties:

1. *If σ_1 is a face of a cone $\sigma \in \Sigma_0$, then $\sigma_1 \in \Sigma_0$.*

2. *For any cones σ_1 and σ_2 in Σ_0, $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.*

Following the algorithm described in the proof of Theorem 11 of [3 p.32], we construct on the basis of Σ_0, a partition Σ of the cone $\mathbb{R}_+^{n*}$ into finitely many closed convex with their vertices at zero, satisfying the conditions 1. and 2. above and the following:

3. *Any cone belonging to Σ lies in one of the cones in Σ_0 and is specified by finitely many linear equalities and linear inequalities with rational coefficients.*

4. *Any cone q-dimensional σ in Σ is simplicial and unimodular, i.e., there exist a set of primitive integer vectors $a^1(\sigma), \ldots, a^q(\sigma)$ which are linearly independent over $\mathbb{R}$ and $n-q$ primitive integer vectors $a^{q+1}(\sigma), \ldots, a^n(\sigma)$ such that $\mathbb{Z}a^1(\sigma) + \ldots + \mathbb{Z}a^n(\sigma) = \mathbb{Z}^n$.*

Let σ be an n-dimensional cone in Σ and $a^1(\sigma), a^2(\sigma), \ldots, a^n(\sigma)$ the corresponding set of primitive integer vectors of σ that has been ordered once and for all. We associate to each such σ a copy of K^n denoted by $K^n(\sigma)$. Let us denote by $\pi_\sigma\colon K^n(\sigma) \to K^n$ the mapping given by the formulae

$$x_i = y_1^{a_i^1(\sigma)} \cdot \ldots \cdot y_n^{a_i^n(\sigma)}$$

where $x_1, x_2, \ldots, x_n$ are coordinates in K^n, $y_1, y_2, \ldots, y_n$ are coordinates in $K^n(\sigma)$ and $a_1^j(\sigma), \ldots, a_n^j(\sigma)$ denote the coordinates of the vector $a^j(\sigma)$.

We shall glue any two copies $K^n(\sigma)$ and $K^n(\tau)$ via the following equivalence relation. Let $y_\sigma \in K^n(\sigma)$ and $y_\tau \in K^n(\tau)$, then $y_\sigma \sim y_\tau$ if and only if $\pi_\sigma(y_\sigma) = \pi_\tau(y_\tau)$.

We shall denote this thus-obtained set by $X = X(\Gamma_+(I)) = \cup K^n(\sigma)/\sim$, where $\cup K^n(\sigma)$ denotes the disjoint union of the $K^n(\sigma)$.

It follows from the properties 1–4 of the partition Σ and Theorems 6,7 and 8 of [4 pp.24–26] that X is a nonsingular n-dimensional algebraic complex manifold and that $\pi\colon X \to K^n$ defined by $\pi(y) = \pi_\sigma(y_\sigma)$ is a proper analytic mapping onto K^n (y_σ denotes a representative in $K^n(\sigma)$ of the equivalence class y in X).

Therefore, for any monomial $x^m \in K^n$ and any n-dimensional cone $\sigma \in \Sigma$ we have the equivalence between the conditions:

(*) $|x^m| \leq \mathcal{E}.sup_i \, \{|g_i(x)|\}$ *for all* x *in a neighbourhood* U *of* 0.

(**) $|x^m| \circ \pi_\sigma(y_\sigma) \leq \mathcal{E}.sup_i \, \{|g_i| \circ \pi_\sigma(y_\sigma)\}$ *for all* $y_\sigma \in \pi^{-1}(U)$.

We shall use the condition $(**)$ to prove the Theorem 1.11.

Proof of the Theorem 1.11.
Given an n-dimensional cone σ, with a set of generators $a^1, \dots, a^n$, since I has finite codimension in K_n, for each point $y_\sigma \in \pi_\sigma^{-1}(0)$ there exist an n-tuple of numbers

$$(Q_1(y_\sigma), \dots, Q_n(y_\sigma))$$

with $\ell(a^j) \leq Q_j(y_\sigma) \leq Q_j$ such that

$$sup_i \, \{|g_i| \circ \pi_\sigma(y_\sigma)\} = \; |y_1^{Q_1(y_\sigma)} \dots y_n^{Q_n(y_\sigma)}| \; sup_i \, \{|h_i(y_\sigma)|\}$$

and for all $y_\sigma \in \pi_\sigma^{-1}(0)$ we have $sup_i|h_i(y_\sigma)| > 0$.

Then there exists a neighbourhood V of $\pi_\sigma^{-1}(0)$ such that $sup._i|h_i(y_\sigma)| > 0$ for all $y_\sigma \in V$

On the other side, for all y_σ, we have

$$x^m \circ \pi_\sigma(y_\sigma) \;=\; y_1^{M_1} y_2^{M_2} \dots y_n^{M_n}$$

where $M_j = \langle m, a^j \rangle$, hence the inequality 2 of the Theorem 1.11 holds if and only if the inequality $(**)$ holds. $\square$

Remark. Since the statement of the Theorem 1.11 shows us when a monomial is in the integral closure of an ideal, the remaining question is to show when a germ of function $h \in K_n$ is in the integral closure of an ideal.

If we consider the ideal I_1 given in the above example, we can see that all polynomials h which have only terms of degree higher than 8 are in the integral closure of I_1, but there are some special polynomials which have some terms with degree less than 8 and belong to the integral closure of I_1, endeed let $h = x^9 + x^5y - xy^5$, following the proof of the Theorem 1.11. it is easy to show that $h \in \overline{I}_1$.

2. Whitney equisingularity of germs of hypersurfaces

2.1. Whitney equisingularity and the integral closure of ideals

Let $X_G = G^{-1}(0)$ be an hypersurface in $k^n \times k$, where $G : k^n \times k, 0 \to k, 0$ is a one parameter deformation of a germ g. The pair $\{X_G, 0 \times k\}$ is Whitney equisingular at 0 if $0 \times k$ is a stratum of a Whitney stratification of X_G in a neighbourhood of 0 (see [9]).

For complex germs, Teissier shows the importance of the integral dependence relation in the determinacy of the Whitney conditions.

Theorem 2.1. The (c) condition of Teissier. *The pair $\{X_G, 0 \times \mathbb{C}\}$ is Whitney equisingular if and only if $\dfrac{\partial G}{\partial t}$ belongs to the integral closure of the ideal in $\mathbb{C}_{n+1}$ generated by $\left\{x_i \dfrac{\partial G}{\partial x_j}\right\}$ for all $i, j = 1, \ldots, n$.*

When g has an algebraically isolated singularity at 0, Gaffney shows in [3, p.319] that the (c) condition of Teissier is also valid in the real analytic category. Hence, we shall consider this condition to show the Whitney equisingularity.

In the sequel we shall denote the ideal $\left\langle x_i \dfrac{\partial g}{\partial x_j},\ i, j = 1, \ldots, n \right\rangle$ by I.

Definition 2.2. [6] *The polyhedron of equisingularity of a germ g, denoted by $\mathcal{E}(g)$ is the convex hull of the set*

$$\cup \left\{ m + \mathbb{R}^n_+ \ :\ x^m \in \overline{I} \right\}.$$

A natural question is to identify all directions θ such that $\Gamma_+(\theta) \subset \mathcal{E}(g)$ and give necessary and sufficient conditions in terms of the polyhedron $\mathcal{E}(g)$ for the

Whitney equisingularity of the pair $\{G^{-1}(0),\ 0 \times k\}$, where $G = g + t\theta$.

In the Proposition 2.6., applying the Theorem 1.11. to the ideal I we characterize the polyhedron $\mathcal{E}(g)$.

In the next lemma we show that $m \in \mathcal{E}(g)$ is a necessary condition for a monomial $x^m \in \overline{\left\langle x_i \dfrac{\partial G}{\partial x_j} \right\rangle}$, hence it is a necessary condition for the Whitney equisingularity of the pair $\{X_G, 0 \times k\}$.

Lema 2.3. *Let $G(x,t) = g(x) + tx^m$ be a deformation of a germ g which has isolated singularity at 0.*

A necessary condition for $x^m \in \overline{\left\langle x_i \dfrac{\partial G}{\partial x_j} \right\rangle}$ in K_{n+1} is $m \in \mathcal{E}(g)$ in K_n.

Proof: If we suppose that $m \notin \mathcal{E}(g)$ in K_n it follows from the valuative criterion of the Proposition 1.1 that there exists a curve $\phi : (K,0) \rightarrow (K^n,0)$ such that

$$fil_{K_1}(x^m \circ \phi) < fil_{K_1}\left(x_i \frac{\partial g}{\partial x_j} \circ \phi\right) \text{ for all } i,j = 1,\dots,n;$$

where $fil_{K_1}(\sum a_\ell t^\ell) = min\ \{\ \ell : a_\ell \neq 0\}$ for any Taylor series $\sum a_\ell t^\ell$ in K_1.

Let ψ be the curve in K_{n+1} defined as $\psi(\lambda) = (\phi(\lambda), 0)$.

Since $x_i \dfrac{\partial G}{\partial x_j} = x_i \dfrac{\partial g}{\partial x_j} + t x_i \dfrac{\partial x^m}{\partial x_j}$, for all $i,j = 1,\dots,n;$, we have

$$fil_{K_1}\left(x_i \frac{\partial g}{\partial x_j} \circ \phi\right) = fil_{K_1}\left(x_i \frac{\partial G}{\partial x_j} \circ \psi\right)$$

and the result also follows from the Valuative Criterion. □

In the next lemma we show that the polyhedron of equisingularity $\mathcal{E}(g)$ also gives a sufficient condition for the Whitney equisingularity of the pair $\{G^{-1}(0), 0 \times k\}$.

Lemma 2.4. *Let $G(x,t) = g(x) + tx^m$, $|t| \leq 1$, be a deformation of a germ g.*

If $x_i \dfrac{\partial x^m}{\partial x_j} \in \overline{I}$ for all $i,j = 1,\dots,n$, then $x^m \in \overline{\left\langle x_i \dfrac{\partial G}{\partial x_j} \right\rangle}$.

Proof: It follows from the hypothesis that there exists a constant $0 < \epsilon < 1$

such that

$$|t|\left|x_i\frac{\partial x^m}{\partial x_j}\right| \leq \epsilon\ sup_{i,j}\left|x_i\frac{\partial g}{\partial x_j}(x)\right|$$

for all (x,t) in a neighbourhood U of 0×0 in $K^n\times K$.

Hence $sup_{i,j}\left|x_i\dfrac{\partial G}{\partial x_j}(x)\right| = sup_{i,j}\left|x_i\dfrac{\partial g}{\partial x_j}(x)+tx_i\dfrac{\partial x^m}{\partial x_j}(x)\right| \geq$

$$\geq sup_{i,j}\left|x_i\frac{\partial g}{\partial x_j}(x)\right| - |t|sup_{i,j}\ \left|x_i\frac{\partial x^m}{\partial x_j}(x)\right| \geq (1-\epsilon)sup_{i,j}\left|x_i\frac{\partial g}{\partial x_j}(x)\right|.$$

From the growth condition, there exists a constant $\varepsilon>0$ such that

$$|x^m|\ \leq\ \varepsilon\ sup_{i,j}\left|x_i\frac{\partial g}{\partial x_j}(x)\right| \text{for all } x$$

in a neighbourhood V of 0 in K^n. Then

$$|x^m|\ \leq\ \varepsilon\ sup_{i,j}\left|x_i\frac{\partial g}{\partial x_j}(x)\right| \leq \frac{\varepsilon}{1-\epsilon}sup_{i,j}\left|x_i\frac{\partial G}{\partial x_j}(x)\right|$$

for all x in the neighbourhood $(V\times K)\cap U$ of 0×0 in $K^n\times K$. □

Our next step is a characteriztion of the polyhedron of equisingularity $\mathcal{E}(g)$ in terms of the Newton polyhedron $\Gamma_+(I)$ of the ideal I.

When $\Gamma_+(I)$ is non-degenerate, the following result is an easy consequence of the Lemma 1.9. and of the Theorem 1.10.

Proposition 2.5.

i) $\mathcal{E}(g)\subset\Gamma_+(I)$

ii) $\mathcal{E}(g)=\Gamma_+(I)$ *if and only only if* $\Gamma_+(I)$ *is non-degenerate.*

Applying the Theorem 1.11 we show how to obtain the polyhedron $\mathcal{E}(g)$ when the polyhedron $\Gamma_+(I)$, has degenerate compact faces.

Proposition 2.6. *For each* $m=(m_1,\dots,m_n)\in\mathbb{Z}_+^n$ *the following statements are equivalent:*

a) $m\in\mathcal{E}(g)$.

b) The inequality $Q_j \leq \langle m, a^j \rangle$ holds for each $(n-1)$-dimensional compact face $\Delta(a^j) \subset \Gamma(I)$.

Since g has isolated singularity at 0 the ideal I has finite codimension in K_n, therefore the result follows from the Theorem 1.11.

Examples

1. Let $g(x,y) = x^9 + y(x^3 - y^2)^2 = x^9 + x^6 y - 2x^3 y^3 + y^5$.

The ideal I is generated by

$$x\frac{\partial g}{\partial x} = 9x^9 + 6x^6 y - 6x^3 y^3,\ y\frac{\partial g}{\partial x} = 9x^8 y + 6x^5 y^2 - 6x^2 y^4,$$

$$x\frac{\partial g}{\partial y} = x^7 - 6x^4 y^2 + 5y^4 x \text{ and } y\frac{\partial g}{\partial y} = x^6 y - 6x^3 y^3 + 5y^5.$$

The vertices of the Newton polyhedron $\Gamma_+(I)$ are $(7,0), (1,4)$ and $(0,5)$. The primitive integer vectors in $\mathbb{R}_+^{2^*}$ corresponding to the 1-dimensional faces of $\Gamma_+(I)$ are $a^1 = (1,0)$, $a^2 = (1,1)$, $a^3 = (2,3)$ and $a^4 = (0,1)$.

The face $\Delta(a^3)$ with vertices $(7,0)$ and $(1,4)$ is degenerate since

$$x\frac{\partial g}{\partial y}_{\Delta(a^3)} = x^7 - 6x^4 y^2 + 5y^4 x, \text{ and}$$

$$x\frac{\partial g}{\partial x}_{\Delta(a^3)} = y\frac{\partial g}{\partial x}_{\Delta(a^3)} = y\frac{\partial g}{\partial y}_{\Delta(a^3)} = 0.$$

The sets $\Delta_{15}(a^3)$ and $\Delta_{16}(a^3)$ are degenerate because the points of type $(t^2, t^3) \in \mathbb{C}^2$ are solutions for all the equations $x_i \frac{\partial g}{\partial x_j}$ restricted to these sets.

The first non-degenerate set of $\Delta(a^3)$ is $\Delta_{18}(a^3)$, hence $Q_3 = 18$.

The face $\Delta(a^2)$ with vertices $(1,4)$ and $(0,5)$ is non-degenerate, hence $Q_2 = 5$.

Therefore we conclude that a monomial x^m lies in $\mathcal{E}(g)$ if and only if $m_1 + m_2 \geq 5$ and $2m_1 + 3m_2 \geq 18$. The vertices of polyhedron $\mathcal{E}(g)$ in $\mathbb{R}_+^n$ are $(0,6)$ and $(9,0)$.

It is easy to see that for all the monomials x^m such that m is in the compact face of $\mathcal{E}(g)$, the condition $x_i \dfrac{\partial x^m}{\partial x_j} \in \mathcal{E}(g)$ is satisfied only for the monomial x^9, hence the pair $\{X_G, 0\times k\}$ where $G(x,y,t)=g(x)+tx^9$ satisfies the Whitney conditions **a** and **b** .

A straightforward calculation shows us that, in this example, all monomials x^m such that $m\in\mathcal{E}(g)$ are in the integral closure of the ideal $\left\langle x_i\dfrac{\partial G}{\partial x_j}{}_{i,j=1,2}\right\rangle$, hence we have the Whitney equisingularity of the pair $\{X_G, 0\times k\}$ for any G of type $G(x,t)=g(x)+tx^m$, with $m\in\mathcal{E}_g$.

We can see from this example that the condition of the Lemma 2.4. is sufficient but not necessary for the Whitney equisingularity of the pair $\{X_G, 0\times k\}$.

2. Let $g(x,y)=y^9+x(x^3-y^2)^2=y^9+xy^4-2x^4y^2+x^7$.

The ideal I is generated by

$$x\frac{\partial g}{\partial x}=7x^7-8x^4y^2+xy^4,\ y\frac{\partial g}{\partial x}=7x^6y-8x^3y^3+y^5,$$

$$x\frac{\partial g}{\partial y}=9y^8x-4x^5y+4x^2y^3\ \text{ and }\ y\frac{\partial g}{\partial y}=9y^9-4x^4y^2+4xy^4.$$

The vertices of the Newton polyhedron $\Gamma_+(I)$ are $(7,0)$, $(5,1)$, $(2,3)$ and $(0,5)$.

The primitive integer vectors in $\mathbb{R}_+^{2^*}$ corresponding to the 1-dimensional faces of $\Gamma_+(I)$ are $a^1=(1,0)$, $a^2=(1,1)$, $a^3=(2,3)$, $a^4=(1,2)$ and $a^5=(0,1)$.

The faces $\Delta(a^2)$ and $\Delta(a^4)$ are non-degenerate, hence $Q_2=5$ and $Q_4=7$.

The face $\Delta(a^3)$ with vertices $(5,1)$ and $(2,3)$ is degenerate.

The first non-degenerate set of $\Delta(a^3)$ is $\Delta_{26}(a^2)$, hence $Q_3=26$.

We conclude that a monomial x^m lies in $\mathcal{E}(g)$ if and only if $m_1+m_2\geq 5$, $m_1+2m_2\geq 7$ and $2m_1+3m_2\geq 26$.

Therefore the vertices of polyhedron $\mathcal{E}(g)$ in $\mathbb{R}_+^n$ are $(0,9)$, $(1,8)$ and $(13,0)$.

Here, the condition $x_i\dfrac{\partial x^m}{\partial x_j}\in\mathcal{E}(g)$ is satisfied for the monomials y^9 and x^{13}, hence the pairs $\{X_G, 0\times k\}$ where $G(x,y,t)=g(x)+tx^{13}$ or $G(x,y,t)=g(x)+ty^9$ satisfy the Whitney conditions **a** and **b** .

In this example, we can show that each monomial $x^ay^b\neq x^{13}$ such that (a,b) is in the compact face with vertices $(1,8)$ and $(13,0)$ of $\mathcal{E}(g)$ does not lie in the integral closure of the ideal $\left\langle x_i \dfrac{\partial G}{\partial x_j}_{i,j=1,2} \right\rangle$ by constructing an analytic curve $\varphi\colon (\mathbb{C},0) \to (\mathbb{C}^3,0)$ such that the valuative criterion does not hold for these curves.

For example, let $G(x,y,t) = g(x,y) + txy^8$, the curve $\varphi\colon (\mathbb{C},0) \to (\mathbb{C}^3,0)$ is $\varphi(\lambda) = (\lambda^2, \lambda^3, -\frac{9}{8}\lambda)$ and the corresponding pair $\{X_G, 0 \times k\}$ is not Whitney equisingular.

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