BOUNDS ON THE DIMENSION OF MANIFOLDS WITH CERTAIN Z_2 FIXED SETS

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Abstract

In this paper we prove that if a smooth involution defined on a smooth closed $(2n+k)$-dimensional manifold fixes the disjoint union of a $2n$-dimensional manifold and a point with n odd, then k is less than or equal to $n+3$.

Resumo

Neste artigo provamos que, se uma involução C^∞ definida em uma variedade fechada e C^∞ de dimensão $2n+k$ fixa a união disjunta de uma variedade fechada de dimensão $2n$ com um ponto, onde n é ímpar, então k é menor que ou igual a $n+3$.

1. Introduction

Suppose M^m is a smooth closed m-dimensional manifold and $T : M^m \longrightarrow M^m$ is a smooth involution defined on M^m. The fixed set of T, F, is a disjoint union of closed manifolds, $F = \bigcup_{j=0}^{n} F^j$, where F^j denotes the union of those components of F having dimension j. It is well-known from equivariant bordism theory that if (T, M^m) is non-bounding then F cannot be too low dimensional. The best result in this direction is the famous 5/2-theorem of Boardman ([2] and [3]), which establishes that if M^m is non-bounding then $m \leq 5/2\dim(F)$; by $\dim(F)$ we mean the dimension of the highest dimensional non-empty component of

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F. A strengthened version of this fact was obtained by Kosniowski and Stong in [6], namely, that if (T, M^m) is a non-bounding involution then $m \leq 5/2n$. In particular, if $F = \bigcup_{j=0}^{n} F^j$ is non-bounding and if (T, M^m) is an involution fixing F, then $m \leq 5/2n$; this follows from the fact that the bordism class of (T, M^m) is determined by the bordism class of its fixed data. This result allows the possibility that fixed components of all dimensions j, $0 \leq j \leq n$, occur. In this way, it is a interesting question to ask whether there exists a better upper bound for m when we omit some components of F. For example, Kosniowski and Stong proved that if $F = F^n$ is of constant dimension n and non-bounding, then $m \leq 2n$ (see the first proposition of Section 6 of [6]). Certainly this is the best upper bound for this particular situation; to see this, it is sufficient to consider the involution $(S, RP(2r) \times RP(2r))$, where $RP(2r)$ is the $2r$-dimensional real projective space and S switches coordinates. So, by considering fixed sets of non-constant dimension, the next case of interest will be $F = F^n \cup \{p\}$, where p=point and $n > 0$ (which evidently does not bound). In this case, Royster proved that if n is odd then $m = n + 1$ (see Theorem 2.3 of [7]). This result is not true for n even; in fact, the involution defined on the complex projective plane $CP(2)$ by $[z_0, z_1, z_2] \to [-z_o, z_1, z_2]$ fixes the disjoint union of a 2-sphere and a point. Our interest is looking closer at this question; specifically, we prove in this note the following

Theorem. *If (T, M^{2n+k}) is an involution fixing $F^{2n} \cup \{p\}$ with n odd, then $k \leq n + 3$.*

We remark that this result is not valid for n even. In fact, consider the involution $(\tau, RP(3))$ given by $\tau[x_0, x_1, x_2] = [-x_0, x_1, x_2]$, which fixes $RP(2) \cup \{p\}$. Denoting by $\mathcal{N}_*^{Z_2}$ the unrestricted bordism group of smooth manifolds with involution, consider the standard endomorphism of degree one $\Gamma : \mathcal{N}_n^{Z_2} \to \mathcal{N}_{n+1}^{Z_2}$, introduced by J. Alexander in [1] and which is given by $\Gamma[T, M^m] = [S, \frac{M^m \times S^1}{\sim}]$, where $\sim$ identifies (m, r) with $(Tm, -r)$ and $S[m, r] = [m, \bar{r}]$, the bar denoting complex conjugation. By computing characteristic numbers and using Theorem 24.2 of [5], one can prove that $\Gamma^2[\tau, RP(3)]$ contains an involution (μ, V^5) fixing

$RP(2) \cup \{p\}$, and $(\mu \times \mu, V^5 \times V^5)$ is bordant to an involution defined on a 10-dimensional manifold whose fixed set is $(RP(2) \times RP(2)) \cup \{p\}$ (actually, the product involutions $(\mu^{2^r}, (V^5)^{2^r})$, $r \in N$, which are bordant to involutions fixing $(RP(2))^{2^r} \cup \{p\}$, provide examples where the Boardman's limit is attained).

Proof of Theorem. We establish first some facts and notations. Denote by $\eta \longrightarrow F^{2n}$ the normal bundle of F^{2n} in M^{2n+k}, and by $W(\eta) = 1+v_1+v_2+...+v_k$ and $W(F^{2n}) = 1 + w_1 + w_2 + ... + w_{2n}$ the total Stiefel-Whitney classes of η and $\tau(F^{2n})$, respectively; here $\tau(F^{2n}) \longrightarrow F^{2n}$ is the tangent bundle of F^{2n}. Let $\lambda \to RP(\eta)$ be the usual line bundle over the projective space bundle associated to η, and let $c \in H^1(RP(\eta), Z_2)$ be the first characteristic class of λ. Write $W(RP(\eta)) = 1 + W_1 + W_2 + ... + W_{2n+k-1}$; from [4], one has that

$$W(RP(\eta)) = (1 + w_1 + w_2 + ... + w_{2n})(\sum_{j=0}^{k}(1 + c)^{k-j} v_j)$$

That is,

$$W_r = \sum_{t+p+q=r} \binom{k-p}{q} w_t v_p c^q,$$

$\binom{k-p}{q}$ denoting the binomial coefficient taken mod 2 (here we are suppressing all bundle maps, and shall continue to do so throughout of this paper).

Write $W(RP(2n+k-1)) = 1+\theta_1+\theta_2+...+\theta_{2n+k-1}$, and let $\alpha \in H^1(RP(2n+k-1), Z_2)$ be the generator; one knows that

$$W(RP(2n + k - 1)) = (1 + \alpha)^{2n+k} = (1 + \alpha)^{2n}(1 + \alpha)^k$$

That is,

$$\theta_r = \sum_{i+j=r} \binom{2n}{i}\binom{k}{j}\alpha^{i+j}$$

Denoting by $\xi \longrightarrow RP(2n + k - 1)$ the canonical line bundle, one has from [5;28.1] that the bordism classes of λ and ξ are equal as elements of $\mathcal{N}_{2n+k-1}(BO(1))$, the bordism group of 1-dimensional vector bundles over smooth closed $(2n + k - 1)$-dimensional manifolds; on the other hand, by [5;23.1], these classes are determined by their characteristic numbers. This is the key which will allow us to obtain our theorem.

Suppose for a moment that we can find a homogeneous polynomial over Z_2 of degree three involving c, W_1, W_2 and W_3, which we denote by $P_3 = P_3(c, W_1, W_2, W_3)$, such that:

i) P_3 can be expressed as $P_3 = p_3 + cp_2$, where p_2 and p_3 are homogeneous polynomials of degree two and three, respectively, with p_2 involving only v_1, v_2, w_1 and w_2, and p_3 involving only v_1, v_2, v_3, w_1, w_2 and w_3;

ii) $P_3(\alpha, \theta_1, \theta_2, \theta_3) = \alpha^3$.

Assuming this, suppose first that k is odd. Then $2n + k - 1$ is even and hence $\theta_1 = \alpha$. Now write $2n + k - 1 = 3x + t$, where $t = 0, 1$ or 2. Then

$$W_1^t P_3(c, W_1, W_2, W_3)^x [RP(\eta)]$$

is a characteristic number of λ, and the corresponding number of ξ is

$$\theta_1^t P_3(\alpha, \theta_1, \theta_2, \theta_3)^x [RP(2n + k - 1)],$$

which by ii) is

$$\alpha^t \alpha^{3x} [RP(2n + k - 1)] = \alpha^{2n+k-1} [RP(2n + k - 1)] = 1$$

Together with i), this yields the fact that $(p_3 + cp_2)^x$ is an element of $H^{3x}(RP(\eta), Z_2)$ different from zero. But

$$(p_3 + cp_2)^x = \sum_{i=0}^{x} \binom{x}{i} p_3^i (cp_2)^{x-i} = \sum_{i=0}^{x} \binom{x}{i} c^{x-i} p_3^i p_2^{x-i}$$

Therefore there is some $0 \leq i \leq x$ so that $p_3^i p_2^{x-i}$ is different from zero. Looking at the description of p_2 and p_3 and because the projection $p : RP(\eta) \to F^{2n}$ induces monomorphisms on the Z_2-cohomology groups, one has that the element $p_3^i p_2^{x-i}$ can be considered as belonging to $H^*(F^{2n}, Z_2)$. It follows that

$$3i + 2(x - i) = 2x + i \leq 2n$$

Since $i \geq 0$, this implies $x \leq n$, that is,

$$\frac{2n + k - 1 - t}{3} \leq n;$$

thus $n \geq k-1-t$, and finally

$$k \leq n+1+t \leq n+3$$

We assume now that k is even. Then $2n+k-1$ is odd and so $\theta_1+\alpha=\alpha$. Hence, by replacing the characteristic number $W_1^t P_3(c, W_1, W_2, W_3)^x[RP(\eta)]$ by

$$(W_1+c)^t P_3(c, W_1, W_2, W_3)^x[RP(\eta)]$$

in the above argument, we can prove in a completely similar fashion that also $k \leq n+3$ in this case.

To finalize the proof, all that remains is then to exhibit the polynomials P_3. To do this, we will handle the previously presented formulae for θ_r and W_r using extensively the known fact that $\binom{a}{b} \equiv 1 \pmod 2$ if and only if every power of 2 occuring in the dyadic expression of b also occurs in the dyadic expression of a. To understand our considerations we remark that the dyadic expression of $2n$ does contain 2, since n is odd by hypothesis. We divide the construction into several cases:

Case 1) k is odd and its dyadic expression contains 2: it is an easy computation to show in this case that $\theta_1=\alpha$ and $\theta_3=0$. On the other hand, a tedious computation shows that $W_1=c+v_1+w_1$ and $W_3=c^3+c^2v_1+cv_2+v_3+c^2w_1+cw_2+v_2w_1+v_1w_2+w_3$. Hence

$$\begin{aligned} P_3(c, W_1, W_2, W_3) = W_3 + cW_1^2 + W_1^3 + c^3 = \\ = v_3+v_2w_1+v_1w_2+w_3+v_1^3+w_1^2v_1+v_1^2w_1+w_1^3+c(v_2+w_2) \end{aligned}$$

satisfies the required conditions.

Case 2) k is odd and its dyadic expression does not contain 2: in this case, $\theta_3=\alpha^3$ and $W_3=v_3+v_2w_1+v_1w_2+w_3+c(v_2+w_2)$, so

$$P_3(c, W_1, W_2.W_3) = W_3$$

is the desired polynomial.

Case 3) k is even and its dyadic expression contains 2: in this case, one has

$\theta_1 = 0$, $\theta_2 = 0$ and $\theta_3 = 0$; on the other hand, $W_1 = v_1 + w_1$, $W_2 = c^2 + cv_1 + v_2 + v_1w_1 + w_2$ and $W_3 = v_3 + c^2w_1 + cv_1w_1 + v_2w_1 + v_1w_2 + w_3$. Therefore

$$\begin{aligned} P_3(c, W_1, W_2, W_3) &= W_3 + cW_2 + c^2W_1 + c^3 = \\ &= v_3 + v_2w_1 + v_1w_2 + w_3 + c(v_2 + w_2) \end{aligned}$$

is as required.

Case 4) k is even and its dyadic expression does not contain 2: in this case $\theta_2 = \alpha^2$, $\theta_3 = 0$, $W_2 = cv_1+v_2+v_1w_1+w_2$ and $W_3 = c^2v_1+v_3+cv_1w_1+v_2w_1+v_1w_2+w_3$. Hence we can take

$$\begin{aligned} P_3(c, W_1, W_2, W_3) = W_3 + cW_2 = \\ = v_3 + v_2w_1 + v_1w_2 + w_3 + c(v_2 + w_2) \end{aligned}$$

□

By introducing a slight modification which is suggested by the above method,

we can shorten the proof presented by Royster of the result mentioned in the introduction, namely, that if (T, M^{n+k}) fixes $F^n \cup \{p\}$ with n odd, then $k = 1$. We present this proof: by [5;27.2] one has $\chi(M^{n+k}) \equiv \chi(F^n \cup \{p\}) \equiv 1 \pmod 2$, where χ denotes the Euler characteristic. Thus, $n + k$ must be even and so k must be odd, which implies that $W_1 = v_1 + w_1 + c$ and $\theta_1 = 0$. Therefore

$$(W_1+c)^{n+k-1}[RP(\eta)] = (v_1+w_1)^{n+k-1}[RP(\eta)] = (\theta_1+\alpha)^{n+k-1}[RP(n+k-1)] = 1$$

So $(v_1 + w_1)^{n+k-1} \in H^{n+k-1}(F^n, Z_2)$ is different from zero, which implies that $n + k - 1 \leq n$, that is, $k \leq 1$.

Despite this simplification, we emphasize that the argument used in this paper was inspired by the method developed by Royster.

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