HOMOGENEOUS SPACES IN COINCIDENCE THEORY

Daciberg L. Gonçalves **Peter N.-S. Wong***

Abstract

In this paper, we study the Nielsen coincidence theory for maps $f_1, f_2 : M \to X$ between closed orientable n-manifolds where X is $\mathcal{C}$-nilpotent. As an application, we show, when $X = G/K$ is the homogeneous space of left cosets of a finite subgroup K in a compact connected Lie group G, that $L(f_1, f_2) = 0 \Rightarrow N(f_1, f_2) = 0$ and $L(f_1, f_2) \neq 0 \Rightarrow N(f_1, f_2) = R(f_1, f_2)$, where $L(f_1, f_2), N(f_1, f_2)$ and $R(f_1, f_2)$ denote the coincidence Lefschetz, Nielsen and Reidemeister numbers respectively.

Resumo

Neste trabalho, nós estudamos a teoria de Nielsen de coincidência para aplicações $f_1, f_2 : M \to X$ entre n-variedades fechadas orientavéis onde X é $\mathcal{C}$-nilpotente. Como aplicação, quando $X = G/K$ é o espaço homogeneo das classes laterais a esquerda de um grupo de Lie compacto e conexo por um subgrupo finito K, we mostramos que $L(f_1, f_2) = 0 \Rightarrow N(f_1, f_2) = 0$ e $L(f_1, f_2) \neq 0 \Rightarrow N(f_1, f_2) = R(f_1, f_2)$ onde $L(f_1, f_2), N(f_1, f_2)$ e $R(f_1, f_2)$, são os números de Lefschetz de coincidencia, de Nielsen and de Reidemeister respectivamente.

1. Introduction

In topological fixed point theory, the non-vanishing of the Lefschetz number $L(f)$ of a selfmap on a compact connected polyhedron M guarantees the existence of fixed points. However, the converse is not true in general. The Nielsen number $N(f)$ is a homotopy invariant which gives a lower bound for the number of fixed points of maps in the homotopy class of f. If M is a manifold of dimension $n \geq 3$, a classical result of Wecken shows that $N(f)$ is indeed a sharp lower bound so the converse of the Lefschetz theorem holds if $L(f) = 0 \Rightarrow N(f) = 0$.

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If, in addition, M satisfies the so-called Jiang condition (or M is a Jiang space), then all fixed point classes have the same fixed point index. In particular, either (C1) $L(f) = 0 \Rightarrow N(f) = 0$ or (C2) $L(f) \neq 0 \Rightarrow N(f) = R(f)$ where $R(f)$ denotes the Reidemeister number of f. Thus, under the Jiang condition, the Nielsen number $N(f)$ can be computed algebraically since $R(f)$ is defined at the fundamental group level. Nielsen fixed point theory has been generalized by H. Schirmer [19] to coincidences of maps $f, g : M_1 \to M_2$ between closed orientable n-manifolds. She defined an appropriate Nielsen coincidence number $N(f, g)$ and proved a Wecken type result for $n \geq 3$ so that $N(f, g)$ gives a sharp lower bound for the number of coincidences of maps f', g' in the homotopy classes of f, g respectively. If M_2 is a Jiang space then the converse of the Lefschetz coincidence theorem holds and $N(f, g) = R(f, g) = \#Coker(f_* - g_*)$ in the case $L(f, g) \neq 0$, where $R(f, g)$ is the Reidemeister coincidence number and f_*, g_* are the induced homomorphisms on the first integral homology groups.

One of the central issues in Nielsen fixed point or coincidence theory is to find algebraic means to compute the Nielsen number. While the Jiang condition is satisfied by a large class of spaces which include generalized lens spaces, H-spaces and homogeneous spaces G/G_0 of a compact topological group G by a connected subgroup G_0, it is restrictive in the sense that the fundamental group of a Jiang space must be abelian. In [5], the Reidemeister trace [1] was employed to show that under a weaker Jiang condition, we have either $L(f) = 0 \Rightarrow N(f) = 0$ or $L(f) \neq 0 \Rightarrow N(f) = R(f)$. This Jiang type result holds, for instance, for all selfmaps of the orbit space of an odd sphere under a free action of a finite group. In [1], D. Anosov proved (See E. Fadell and S. Husseini [4] for another proof) that $N(f) = |L(f)|$ for any selfmap f on a compact nilmanifold. This result was strengthened by B. Norton-Odenthal [17] so that $N(f) = R(f)$ in the case where $L(f) \neq 0$. More recently, the second author [24] showed that conditions (C1) and (C2) hold for selfmaps of orientable homogeneous spaces

[1] The term generalized Lefschetz number was used instead in [5]. Ross Geoghegan pointed out to us that the term *Reidemeister trace* was coined by F. Wecken in section 4 on p.226 of [23].

G/K where G is a compact connected Lie group and K a closed (not necessarily connected) subgroup. However, the techniques used in [24] cannot readily be generalized to coincidences.

The purpose of this paper is to show that conditions (C1) and (C2) hold for coincidences when the target manifold M_2 is a homogeneous space G/K where G is a compact connected Lie group and K is a finite subgroup. In fact, we prove a more general result for M_2 a $\mathcal{C}$-nilpotent space whose fundamental group has center of finite index (see Theorem 2). Our approach uses the notion of $\mathcal{C}$-nilpotent actions and follows the approach in [7].

This paper is organized as follows. Section 2 reviews some background in Nielsen coincidence theory and $\mathcal{C}$-nilpotent actions. We prove our main results in section 3 for $\mathcal{C}$-nilpotent spaces. We show in section 4 that the homogeneous space G/K when K is finite is $\mathcal{C}$-nilpotent and hence the Jiang type results for coincidences may be obtained in this case(see Theorem 4). Finally, in section 5, we discuss the coincidence theory for homogeneous spaces of non-compact Lie groups, such as compact nilmanifolds.

The basic references in Nielsen fixed point theory are [3], [12] and [13]. For $\mathcal{C}$-nilpotent spaces, see [6] and [10].

2. Preliminares

In this section, we review some basic elements of Nielsen coincidence theory and the notion of a $\mathcal{C}$-nilpotent space.

Let $f, g : X \to Y$ be two maps between two closed connected orientable n-manifolds. The coincidence set $C(f,g) = \{x \in X | f(x) = g(x)\}$ is compact in X. If $x_1, x_2 \in C(f,g)$, we say that x_1 and x_2 are **Nielsen equivalent** if there is a path $\alpha : [0,1] \to X$ with $\alpha(0) = x_1, \alpha(1) = x_2$ such that $f \circ \alpha$ is homotopic to $g \circ \alpha$ ($f \circ \alpha \sim g \circ \alpha$) relative to the endpoints. Given an isolated subset $\gamma \subset C(f,g)$, the **coincidence index** of γ with respect to f, g is defined to be the integer (see Chap. 5 of [22])

$$I(f,g;\gamma) = I(f,g;U_\gamma) = < (f,g)^*\mu_2, o_\gamma >$$

where U_γ is an open neighborhood of γ in X which does not contain any other coincidences of f and g; $o_\gamma \in H_n(X, X-\gamma)$ is the fundamental homology class around γ; $\mu_2 \in H^n(Y \times Y, Y \times Y - \Delta_Y)$ is the Thom class and $(f,g) : X \to Y \times Y$ is given by $(f,g)(x) = (f(x), g(x))$. Here H_n and H^n denote the singular homology and cohomology with integer coefficients, respectively. Analogous to the Lefschetz coincidence number $L(f,g)$ ([22]), we have the following

Definition 1. *A coincidence class γ is* **essential** *if $I(f,g;\gamma) \neq 0$. Then the* **Nielsen coincidence number** *of f and g, denoted by $N(f,g)$ is defined to be the (finite) number of essential coincidence classes of f and g.*

Using covering spaces, we may treat the coincidence classes algebraically as follows. Following [5], by choosing base points and lifts to the universal covers, the maps f, g induce homomorphisms φ_1, φ_2 on the fundamental groups. Then $\pi_1(X)$ acts on $\pi_1(Y)$ via

$$\sigma \bullet \alpha = \varphi_2(\sigma)\alpha\varphi_1(\sigma)^{-1}.$$

Denote by $\mathcal{R}(f,g)$, called the set of Reidemeister coincidence classes, the set of orbits of this action and by $R(f,g)$, called the Reidemeister number, the cardinality of $\mathcal{R}(f,g)$. It is well-known that $R(f,g)$ is independent of the choice of base points and lifts and it is invariant under homotopy. Furthermore, every coincidence class (non-empty) corresponds to a unique Reidemeister coincidence class in $\mathcal{R}(f,g)$. Throughout, a coincidence class means a non-empty Reidemeister coincidence class.

We now turn to $\mathcal{C}$-nilpotent spaces. The notion of *class* is a generalization of the definition of a class of abelian groups introduced by J.-P. Serre [20] and that of a Serre class of nilpotent groups given by P. Hilton and J. Roitberg [10]. The following was introduced in [6].

Definition 2. *A family $\mathcal{C}$ of groups is called a* **class** *of groups if it satisfies the following property:*

Given a short exact sequence of groups

$$1 \to A \to B \to C \to 1$$

we have $A, C \in \mathcal{C}$ if and only if $B \in \mathcal{C}$.

Example 1.

(i) The family of all finite groups is a class.

(ii) All torsion groups form a class.

Let $\mathcal{C}$ be a class of groups.

Definition 3. *A group π is said to be $\mathcal{C}$-**nilpotent** if $\Gamma^n(\pi) \in \mathcal{C}$ for some n where $\Gamma^{n-1}(\pi)$ denotes the n-th term in the lower central series of π:*

$$\pi = \Gamma^0(\pi) \supset [\pi, \pi] = \Gamma^1(\pi) \supset [\pi, \Gamma^1(\pi)] = \Gamma^2(\pi) \supset ...$$

*An action $\theta : \pi \to Aut(G)$ of a group π on a group G is said to be $\mathcal{C}$-**nilpotent** if $\Gamma^n_\pi(G) \in \mathcal{C}$ for some positive integer n, where $\Gamma^n_\pi(G)$ is the smallest normal π-subgroup that contains $[G, \Gamma^{n-1}_\pi(G)]$ and the set $\{(\alpha \cdot g)g^{-1} \mid \alpha \in \pi,\ g \in \Gamma^{n-1}_\pi(G)\}$. Moreover, a space X is $\mathcal{C}$-**nilpotent** if (i) $\pi_1(X, x_0)$ is a $\mathcal{C}$-nilpotent group and (ii) the action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ is $\mathcal{C}$-nilpotent, for all $x_0 \in X$ and for all $n \geq 1$. **Nilpotent** spaces are $\mathcal{C}$-nilpotent where $\mathcal{C} = \{1\}$ is the trivial class.*

Proposition 1. *Let G be a torsion free group. If a finite group π acts on G nilpotently, then π acts trivially on G.*

Proof. Let $\alpha \in \pi$ and $< \alpha >$ be the cyclic subgroup generated by α. Since π acts nilpotently on G, there exists an n such that $\Gamma^n_{<\alpha>}(G) = 1$ and $\Gamma^{n-1}_{<\alpha>}(G) \neq 1$. Let $x \in \Gamma^{n-2}_{<\alpha>}(G)$.

Then

$$\begin{aligned}
&(x^{-1}\alpha \cdot x)^{-1}\alpha \cdot (x^{-1}\alpha \cdot x)) = (\alpha \cdot x)^{-1}x(\alpha \cdot x^{-1})(\alpha^2 \cdot x) = 1 \\
&\Rightarrow x(\alpha \cdot x^{-1})(\alpha^2 \cdot x) = \alpha \cdot x \\
&\Rightarrow x = (\alpha \cdot x)(\alpha^2 \cdot x)^{-1}(\alpha \cdot x^{-1})^{-1}.
\end{aligned}$$

Let $p = o(\alpha)$, the order of α. Then $\alpha^p \cdot x = x$ and thus

$$\alpha^p \cdot x = (\alpha \cdot x)(\alpha^2 \cdot x)^{-1}(\alpha \cdot x)$$

or

$$\alpha \cdot (\alpha^{p-1} \cdot x) = \alpha \cdot (x(\alpha \cdot x)^{-1} x).$$

This implies that

$$\begin{aligned}
&\alpha^{p-1} \cdot x = x(\alpha \cdot x)^{-1} x \\
&= (\alpha \cdot x)(\alpha^2 \cdot x)^{-1}(\alpha \cdot x)(\alpha \cdot x)^{-1}(\alpha \cdot x)(\alpha^2 \cdot x)^{-1}(\alpha \cdot x) \\
&= \alpha \cdot (x(\alpha \cdot x)^{-1} x(\alpha \cdot x)^{-1} x). \\
&\Rightarrow \alpha^{p-2} \cdot x = x(\alpha \cdot x)^{-1} x(\alpha \cdot x)^{-1} x.
\end{aligned}$$

Similarly, we obtain

$$\alpha^{p-k} \cdot x = (\prod_{i=1}^{k} x(\alpha \cdot x)^{-1})x.$$

In particular, when $k = p$, we have

$$x = (\prod_{i=1}^{p} x(\alpha \cdot x)^{-1})x$$

which implies that

$$\prod_{i=1}^{p} x(\alpha \cdot x)^{-1} = 1.$$

Since G is torsion free, we must have $\alpha \cdot x = x$ and so every $\alpha \in \pi$ acts trivially on G.

$\square$

For the rest of the paper, $\mathcal{C}$ will denote the class of finite groups unless otherwise stated and the fundamental group of a space X is finitely generated.

Proposition 2. *Let $Cov\eta$ be the group of covering transformations of a regular cover $\eta : \hat{X} \to X$. If X is a $\mathcal{C}$-nilpotent space then $Cov\eta$ acts nilpotently on $H_*(\hat{X}; \mathbf{Q})$.*

Proof. It follows from Prop. 2.10 of [6] that $Cov\eta$ acts $\mathcal{C}$-nilpotently on $H_*(\hat{X}; \mathbf{Z})$. It is easy to see that the action of $Cov\eta$ on $H_*(\hat{X}; \mathbf{Q})$ is nilpotent (see also Prop. 2.3 of [7]).

□

The following is a useful characterization of $\mathcal{C}$-nilpotency.

Proposition 3. [[6]] *A space X is $\mathcal{C}$-nilpotent if and only if $\pi_1(X)$ is $\mathcal{C}$-nilpotent and the action of $\pi_1(X)$ on $H_*(\tilde{X};\mathbf{Z})$ is $\mathcal{C}$-nilpotent, where $\tilde{X}$ is the universal cover of X.*

Example 2. (Jiang spaces) The Jiang subgroup $J(X)$ (or the Gottlieb subgroup $G(X)$) of $\pi_1(X)$ is the set of elements (when identified with deck transformations) that are $\pi_1(X)$-equivariantly homotopic to the identity $1_{\tilde{X}}$, where $\tilde{X}$ denotes the universal cover of X. It follows from Theorem I.4 of [9] that $G(X)$ is a subgroup of the group of elements in $\pi_1(X)$ which act trivially on $\pi_*(X)$. Recall that a space X is Jiang if $G(X) = \pi_1(X)$. Since $G(X)$ is central in $\pi_1(X)$, it follows that a Jiang space must be $\mathcal{C}$-nilpotent (in fact, nilpotent).

Example 3. Let G be a finite group acting freely on an odd sphere S^{2n-1}. The orbit space $X = S^{2n-1}/G$ is $\mathcal{C}$-nilpotent. Note that if X is a Jiang space then G must be abelian. The converse is also true, i.e., if G is abelian then X is Jiang. This follows from the fact that the Gottlieb group $G(X)$ is the center of G [18].

3. Coincidence theory for $\mathcal{C}$-nilpotent spaces

We now compute the Nielsen coincidence number when the target manifold is $\mathcal{C}$-nilpotent.

Theorem 1. *Let M and X be closed connected orientable n-manifolds such that X is $\mathcal{C}$-nilpotent and $\pi_1(X)$ is abelian. For any two maps $f_1, f_2 : M \to X$, the Reidemeister coincidence classes of f_1, f_2 have the same coincidence index.*

Proof. Let $\varphi_1, \varphi_2 : \pi_1(M) \to \pi_1(X)$ be the homomorphisms induced by f_1, f_2, respectively and let $H = \{\varphi_2(\alpha)\varphi_1(\alpha)^{-1} | \alpha \in \pi_1(M)\}$. Since $\pi_1(X)$ is abelian, H is a normal subgroup of $\pi_1(X)$. Let $R_1, ..., R_r$ be the Reidemeister classes in $\pi_1(X)$ corresponding to the essential coincidence classes of f_1, f_2. Note that a Reidemeister class R takes the form $\{\varphi_2(\alpha)x\varphi_1(\alpha)^{-1}|\alpha \in \pi_1(M)\}$ so that $R = Hx$, i.e., cosets of H are precisely the Reidemeister classes of φ_1 and φ_2. Thus, there exist $x_i, i = 1, ..., r$ such that $R_i = Hx_i$. The set $\{[x_1], ..., [x_r]\}$ in $\pi_1(X)/H$ is a finite set and $[x_i] \neq [x_j]$ for $i \neq j$.

Now let $W = \{[x_i x_j^{-1}]|1 \leq i, j \leq r, i \neq j\}$. There exists a normal subgroup N of finite index in $\pi_1(X)/H$ such that $N \cap W = \{1\}$. If $p : \pi_1(X) \to \pi_1(X)/H$ is the canonical projection, then $H_1 = p^{-1}(N)$ is an abelian subgroup of finite index in $\pi_1(X)$ such that H_1 intersects at most one of the R_i's.

Let $\eta : \bar{X} \to X$ be the finite cover corresponding to H_1 and $q : \bar{M} \to M$ be the finite cover corresponding to $C = \varphi_1^{-1}(H_1) \cap \varphi_2^{-1}(H_1)$. Then f_1, f_2 can be lifted to $F_1, F_2 : \bar{M} \to \bar{X}$. Since X is $\mathcal{C}$-nilpotent, it follows from Proposition 1 and Proposition 2 that $\pi_1(X)/H_1$ acts trivially on $H_*(\bar{X}; \mathbf{Q})$. Thus, for any $\alpha \in Cov\eta$, we have $L(\alpha F_1, F_2) = L(F_1, F_2)$. By the choice of H_1, the set $q(Coin(\alpha F_1, F_2))$ consists of a single coincidence class $\mathbf{C}$ of f_1, f_2 and the coincidence index of $\mathbf{C}$ is given by $L(\alpha F_1, F_2)/|\pi_{\mathbf{C}}|$ where $\pi_{\mathbf{C}}$ depends upon $\mathbf{C}$. By assuming (without loss of generality) that $C(\alpha F_1, F_2)$ is finite, $|\pi_{\mathbf{C}}|$ is equal to the number of coincidence classes of $\alpha F_1, F_2$ times the number of coincidences in each class. Since $\pi_1(X)$ is abelian, the number of coincidences in each class is independent of the class $\mathbf{C}$ (see section 5 of [16]) and the number of coincidence classes of $\alpha F_1, F_2$ is independent of α and hence of $\mathbf{C}$, so $|\pi_{\mathbf{C}}|$ is constant. Hence, every Reidemeister coincidence class of f_1, f_2 has the same index.

□

As an immediate corollary of Theorem 1, we recover the following well-known fact.

Corollary 1. *Let M and X be closed orientable n-manifolds and X be a Jiang space. Then for any two maps $f_1, f_2 : M \to X$, the Reidemeister coincidence classes have the same coincidence index. In the fixed point situation, i.e., when $M = X, f_2 = 1_X$ and X is a compact polyhedron satisfying the Jiang condition, all fixed point classes of f_1 have the same fixed point index.*

Theorem 2. *Let M and X be closed connected orientable n-manifolds such that X is $\mathcal{C}$-nilpotent and $[\pi_1(X) : Z(\pi_1(X))] < \infty$. For any two maps $f_1, f_2 : M \to X$, the Reidemeister coincidence classes of f_1, f_2 have the coincidence index of the same sign. In particular, either $L(f_1, f_2) = 0 \Rightarrow N(f_1, f_2) = 0$ or $L(f_1, f_2) \neq 0 \Rightarrow N(f_1, f_2) = R(f_1, f_2)$.*

Proof. Since $[\pi_1(X) : Z(\pi_1(X))] < \infty$, there exists a finite covering $\eta : \bar{X} \to X$ corresponding to $Z(\pi_1(X))$. Then $\bar{X}$ is $\mathcal{C}$-nilpotent and $\pi_1(\bar{X})$ is abelian. Lift f_1, f_2 to $F_1, F_2 : \bar{M} \to \bar{X}$ where $q : \bar{M} \to \bar{X}$ is the finite cover corresponding to $\varphi_1^{-1}(Z(\pi_1(X))) \cap \varphi_2^{-1}(Z(\pi_1(X)))$ and φ_1, φ_2 are the respective homomorphisms induced by f_1, f_2. By Theorem 1, for any $\alpha \in Cov\eta$, all coincidence classes of $\alpha F_1, F_2$ have the same index. Furthermore, since $1 \to \pi_1(\bar{X}) \to \pi_1(X) \to Cov\eta \to 1$ is a central extension, α acts trivially on $\pi_1(\bar{X})$ and hence the Reidemeister numbers $R(\alpha F_1, F_2)$ and $R(F_1, F_2)$ coincide. It follows that the coincidence classes of f_1, f_2 are essential if and only if $L(\alpha F_1, F_2) \neq 0$. Since the set of Reidemeister classes of $\alpha F_1, F_2$ (with α ranging over $Cov\eta$) surjects onto those of f_1, f_2, we have $N(f_1, f_2) = R(f_1, f_2)$ when $L(f_1, f_2) \neq 0$.

$\square$

Corollary 2. *Let X be a compact connected polyhedron. Suppose that X is $\mathcal{C}$-nilpotent and $[\pi_1(X) : Z(\pi_1(X))] < \infty$. Then for any selfmap $f : X \to X$, either $L(f) = 0 \Rightarrow N(f) = 0$ or $L(f) \neq 0 \Rightarrow N(f) = R(f)$.*

Remark 1. In the case where $\pi_1(X)$ is finite, Corollary 2 follows from Theorem 5.6 of [12] since $\pi_1(X)$ acts nilpotently and hence trivially on $H_*(\tilde{X}; \mathbf{Q})$, the rational homology of the universal cover $\tilde{X}$.

Corollary 3. *Let G be a finite group acting freely on an odd sphere S^{2n-1}. For any two maps $f_1, f_2 : M \to X = S^{2n-1}/G$, where M is a connected orientable $(2n-1)$-dimensional closed manifold, either $L(f_1, f_2) = 0 \Rightarrow N(f_1, f_2) = 0$ or $L(f_1, f_2) \neq 0 \Rightarrow N(f_1, f_2) = R(f_1, f_2)$.*

Remark 2. Corollary 3 in the fixed point case was obtained by E. Fadell and S. Husseini in Corollary 6.38 of [5].

4. Coincidence theory for homogeneous spaces

In this section we study the coincidence theory of two maps $f_1, f_2 : M_1 \to M_2$ where $M_2 = G/K$ is the homogeneous space of a compact connected Lie group G by a finite subgroup K.

Theorem 3. *Let G be a compact connected Lie group and K a finite subgroup. The homogeneous space of left cosets $M = G/K$ is $\mathcal{C}$-nilpotent.*

Proof. Write $A = \pi_1(G, e), B = \pi_1(M, eK), e =$ identity in G. Since the canonical projection $G \to M$ is a finite regular cover, by homotopy exact sequence, we have the following short exact sequence of groups

$$0 \to A \to B \to K \to 1. \qquad (*)$$

Note that K acts on G via $k * g = gk^{-1}$. Given a loop g_t (based at e) in G, $\beta_t = k * g_t = g_t k^{-1}$ is a loop based at k^{-1}. Let α_s be a path in G from e to k^{-1}. Then $\alpha_s \cdot \beta_t \cdot \alpha_s^{-1}$ is a loop based at e. Let $\beta_{s,t} = g_t \alpha_s^{-1}$. Clearly, $\beta_{0,t} = g_t$ and $\beta_{1,t} = g_t k^{-1} = \beta_t$. Therefore, $\alpha_s \cdot \beta_t \cdot \alpha_s^{-1}$ yields a homotopy from g_t to $\beta_t = k * g_t$. In other words, K acts trivially on A. It follows that the extension $(*)$ is central. A similar argument shows that K acts trivially on $\pi_n(G) \cong \pi_n(M)$ for $n \geq 2$. Hence $B = \pi_1(M)$ acts trivially on $\pi_n(M)$. It remains to show that B is $\mathcal{C}$-nilpotent.

The 5-term low dimensional exact homology sequence [21] of the extension yields

$$H_2(K) \xrightarrow{I} A/[B, A] \to H_1(B) \to H_1(K) \to 0. \qquad (**)$$

Note that $H_2(K)$ is a finite group since K is finite. The subgroup A (identified with its image in B) is central in B so that $A/[B, A] = A$. Consider the following short exact sequence of finitely generated abelian groups

$$0 \to A/I(H_2(K)) \to B/[B, B] \to K/[K, K] \to 0.$$

Since K is finite, the groups $A/I(H_2(K))$ and $B/[B,B]$ have the same rank. Furthermore, $I(H_2(K))$ is finite and A is of finite index in B so that $[B,B]$ must also be finite, i.e., $[B,B] \in \mathcal{C}$. Thus B is $\mathcal{C}$-nilpotent.

□

Let B_2 be the fundamental group of M_2. Denote by $Z(B_2)$ the center of B_2. Since $(*)$ is a central extension and $A = \pi_1(G)$ is of finite index in B_2, $[B_2 : Z(B_2)] < \infty$. Let $\eta : \hat{M}_2 \to M_2$ be the finite cover corresponding to $Z(B_2)$.

Let φ_1, φ_2 be the homomorphisms induced by f_1, f_2, respectively, on fundamental groups. Then the subgroup $C = \varphi_1^{-1}(Z(B_2)) \cap \varphi_2^{-1}(Z(B_2))$ is normal in $\pi_1(M_1) = B_1$ and denote by $p : \bar{M}_1 \to M_1$ the cover corresponding to C. Note that $[B_1 : C] < \infty$ so p is a finite cover.

Theorem 4. *Let M_1 and M_2 be closed connected orientable n-manifolds such that $M_2 = G/K$ is the homogeneous space of a compact connected Lie group G by a finite subgroup K. For any two maps $f_1, f_2 : M_1 \to M_2$, the Reidemeister coincidence classes of f_1, f_2 have the coincidence index of the same sign. In particular, either $L(f_1, f_2) = 0 \Rightarrow N(f_1, f_2) = 0$ or $L(f_1, f_2) \neq 0 \Rightarrow N(f_1, f_2) = R(f_1, f_2)$.*

Proof. This follows from Theorem 2 and Theorem 3, which states that M_2 is $\mathcal{C}$-nilpotent, the remark following Theorem 3 and the center of $\pi_1(M_2)$ has finite index in $\pi_1(M_2)$.

□

Remark 3. Theorem 4 generalizes to coincidences a special case of a result for fixed points in [24] in which K need not be discrete. In the fixed point case, i.e., when $f_2 = 1_X$, our result here gives an alternate proof.

5. Coincidence theory for nilmanifolds

Consider a pair of maps $f_1, f_2 : M_1 \to M_2$ where M_1 and M_2 are compact orientable manifolds of the same dimension. We will consider the following question: if M_2 is a homogeneous space, when do the Reidemeister coincidence classes of f_1, f_2 have coincidence index of the same sign? Of course we know that a positive answer implies that $L(f_1, f_2) = 0 \Rightarrow N(f_1, f_2) = 0$ and $L(f_1, f_2) \neq 0 \Rightarrow N(f_1, f_2) = R(f_1, f_2)$. We have seen in section 4 that the homogeneous space G/K, which is the coset space of a compact connected Lie group with a finite subgroup K, provides an affirmative answer. The situation with G non-compact and K a co-compact closed subgroup is quite different.

If $f : T \to T$ is a map, where T is the torus, our question has a positive answer, from [3] chapter VIII Theorem 4. More generally, it follows from Proposition 5 of [14] and Corollary 7.3 of [13] that for a pair of maps $f_1, f_2 : M \to T$ the Reidemeister coincidence classes have the same coincidence index. So in both cases we have a positive answer. Of course, when we consider only one map and the domain is the same as the target space, then we have the fixed point situation. Despite the fact that T is a homogeneous space, the proofs of the abovementioned results are based on the fact that T is a Jiang space.

Around 1985, D. Anosov [1] and E. Fadell and S. Husseini [4], considered maps $f : N \to N$ where N is a nilmanifold. They proved that $N(f) = |L(f)|$. Also implict in their work is the fact that all essential fixed point classes have index the same value, which is either +1 or -1. This is not sufficient to get a positive answer to our question because we need to know if one Reidemeister class corresponding to one essential fixed point class implies the same correspondence for all the other Reidemeister classes. Finally, relative to the two works cited above, we should point out that the fact that N is a homogeneous space plays a very important role in formulating our question.

J. Jezierski [11] in 1989 and R. Brooks and P. Wong [2] in 1992, considered the coincidence case where the domain and the target are the same compact nilmanifold. They showed that $N(f_1, f_2) = |L(f_1, f_2)|$. Also it is implicit in their work that the essential coincidence classes have coincidence index of the same sign. (In fact these coincidence indices are equal).

In 1995, the first author in [8] showed that our question has a positive answer for a pair of selfmaps $f_1, f_2 : N \to N$ where N is a nilmanifold. This is explicitly stated in Theorem 2.3 of [8]. Furthermore in Lemma 3.7 [8] this result is extended to a manifold M, which fibers over a nilmanifold and the fibration is nilpotent. Note that M is no longer necessarily a nilmanifold.

While positive results have been obtained, we cannot expect a positive answer for an *arbitrary* homogeneous space M_2. If one looks at the example given in [8] (first remark after Theorem 2.3), we have that $L(id, g) \neq 0$ but $N(id, g) \neq R(id, g)$, because $R(id, g) = \infty$. In fact, we have the following example from [8] in which $L(id, h) = 0$ but $N(id, h) \neq 0$.

Example 4. Consider $\mathbf{R}^3$ as a solvable Lie group with the group operation given by

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + x_2, y_1 + (-1)^{x_1} y_2, z_1 + (-1)^{x_1} z_2).$$

Let S be the solvmanifold which is the quotient space of $\mathbf{R}^3$ by the relation $(x, y, z) \sim (x + a, (-1)^a y + b, (-1)^a z + c)$, where a, b, c are integers. The fundamental group of S is the semi-direct product of $\mathbf{Z} \oplus \mathbf{Z}$ by $\mathbf{Z}$ and the action $\omega : \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z}$ is given by $\omega(1) = -id$. Now take $f = id$ and $h[x, y, z] = [-x, y + z, y]$. These two maps are fiber preserving maps relative to the fibration $T^2 \to S \to S^1$, where the map $S \to S^1$ is the projection on the first coordinate.

The induced maps $\bar{id}$, $\bar{h} : S^1 \to S^1$ are respectively the identity and $\bar{h}[x] = [-x]$ and thus $Fix(\bar{h}) = \{[0], [\frac{1}{2}]\}$. Let $T^2_{[0]}$, $T^2_{[\frac{1}{2}]}$ be the fibers over the points $[0], [\frac{1}{2}]$ respectively. Then $h[0, y, z] = [0, y + z, y]$. It follows that $h|_{T^2_{[0]}}$ has Lefschetz number -1. Now

$$h\left[\frac{1}{2}, y, z\right] = \left[-\frac{1}{2}, y + z, y\right] = \left[-\frac{1}{2} + 1, (-1)^1(y + z), (-1)^1 y\right] = \left[\frac{1}{2}, -y - z, -y\right]$$

so that $h|_{T^2_{[\frac{1}{2}]}}$ has Lefschetz number 1. It was shown in [8] that $L(h) = 0$ but $N(h) = 2$.

Finally, if we consider $f_1, f_2 : N_1 \to N_2$ where the domain and the target are compact nilmanifolds, possibly different, C. K. McCord [15] in 1995, showed that $N(f_1, f_2) = |L(f_1, f_2)|$ and that all essential coincidence classes have coincidence index the same value which is either +1 or -1. Our question still remains open for this case and is equivalent to determining whether $N(f_1, f_2) = R(f_1, f_2)$ when $L(f_1, f_2) \neq 0$. We believe that is the case.

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Departamento de Matemática
IME - USP
Caixa Postal 66.281
05315-970 São Paulo - SP
E-mail: dlgoncal@ime.usp.br

Department of Mathematics
Bates College
Lewiston, ME 04240, U.S.A.
E-mail: pwong@abacus.bates.edu

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