# ON THE CUP PRODUCT ON SEIFERT MANIFOLDS

**Claude Hayat-Legrand** **Heiner Zieschang**

## 1. Introduction

For the computation of the cohomology ring of a manifold, there are several classical approaches, for instance: a) using a chain approximation of the diagonal, b) translating the problem into the intersection theory of chains utilizing the Poincaré duality, or c) applying the cohomological properties of fiber bundles. The explicitly calculation for given manifolds may be quite cumbersome, even if the cohomology groups are easily determined. In this paper we give a survey on the cohomology rings of orientable Seifert manifolds with orientable base space, refering to results in [3], [4], [1], and illustrate the first two methods of calculation.

The first and main motivation of this work is a question of Peter Zvengrowski and Jeff Williams. In relativity theory one is interested in the homotopy classification of Lorentzian metrics over the space-time manifold $M\times\mathbf{R}$ where $M$ is an orientable closed connected 3-manifold. Steenrod's classical results show that the set of homotopy classes of Lorentzian metrics is equivalent to $[M, SO(3)]$, which is isomorphic to the homotopy classes of pointed maps from $M$ to $\mathbf{RP}^3$. From the universal cover $S^3\to\mathbf{RP}^3$ and the inclusion $\mathbf{RP}^3\to K(\mathbf{Z}_2,1)$ one obtains a short exact sequence of groups:

$$(*)\qquad 0\to[M,S^3]\to[M,\mathbf{RP}^3]\to H^1(M,\mathbf{Z}_2)\to 0.$$

Shastri, Williams and Zvengrowski [9] proved that $[M,\mathbf{RP}^3]$ is an abelian group and remarked that the calculation of $[M,\mathbf{RP}^3]$ depends only on the homology of $M$ and on whether or not this exact sequence splits. Let $H^1(M;\mathbf{Z}_2)\cong\mathbf{Z}_2^m$.

Then $[M, \mathbf{RP}^3] = \mathbf{Z} \oplus \mathbf{Z}_2^{m-1}$ if the exact sequence $(*)$ does not split, otherwise $[M, \mathbf{RP}^3] = \mathbf{Z} \oplus \mathbf{Z}_2^m$. In the first case we say that $M$ has type 1: type$(M) = 1$, in the second type$(M) = 2$. By a theorem of Shastri, Williams, Zvengrowski [9] type$(M) = 1$ if and only if there is an $\alpha \in H^1(M; \mathbf{Z}_2)$ such that $\alpha \cup \alpha \cup \alpha \neq 0$. Hence, if the ring structure of $H^*(M; \mathbf{Z}_2)$ is known the set of homotopy classes of Lorentzian metrics on $M$ is determined. Collecting the results of [9], [11], [7] and others, we get the following theorem:

**Theorem 1.1.** *Let $M$ be a closed orientable 3-manifold. The following four assertions are equivalent:*

(*a*) *The exact sequence $(*)$ does not split.*

(*b*) *There exists a degree-one map $f\colon M \to \mathbf{RP}^3$.*

(*c*) *There exists an embedded closed surface $F$ in $M$ with odd Euler characteristic.*

(*d*) *There exists $\zeta \in H^1(M; \mathbf{Z}_2)$ with $\zeta \cup \zeta \cup \zeta \neq 0$.*

Shastri and Zvengrowski [9] have shown that for a connected sum $M_1 \sharp M_2$, one has

$$\mathrm{type}(\mathrm{M}_1 \sharp \mathrm{M}_2) = \min(\mathrm{type}(\mathrm{M}_1), \mathrm{type}(\mathrm{M}_2));$$

hence, it suffices to determine the type of irreducible 3-manifolds. Translated into terms of embedded surfaces this result is close to the result of Bredon and Wood [2, th. 5.1] for non-orientable embedded closed surfaces.

All our considerations will be based on cellular homology and cohomology. In section 2 we describe a classical cell decomposition of an orientable Seifert manifold $M$ with orientable basis. In section 3 we formulate the main results, namely we give the cohomology groups and the cup products. Using this we determine type$(M)$. In section 4 we explicitly calculate the cup products using Poincaré duality and the intersections of chains. We restrict ourselves to some special Seifert manifolds to avoid too many case considerations. In section 5 we write down a chain approximation to the diagonal and show how to calculate cup products. Finally, in section 6, we consider cup products of cohomology classes which "vanish on parts of $M$" and apply this to determine the cohomology ring for Seifert manifolds with finite fundamental group.

With great pleasure we thank Kerstin Aaslepp, John Bryden, Derek Hacon, and Peter Zvengrowski for common work. The results mentioned in this survey are mostly from [1], [3], [4], new proofs are given in sections 4 and 6.

## 2. A cell complex of a Seifert manifold.

We use the following description of a Seifert manifold

$$M = SF(g; e; m : (a_1, b_1), \dots, (a_m, b_m)).$$

The manifold consists of $m+1$ solid tori $V_0, V_1, \dots, V_m$, and a *central part* $M' = F'_g \times S^1$ where $F'_g = \overline{F_g \setminus (D_0^2 \cup \dots D_m^2)}$ is the closure of the orientable surface $F_g$ of genus $g$ minus $m+1$ disks, the boundaries of which are denoted by $\rho_0^1, \dots, \rho_m^1$. We call $F_g$ the *base-surface.* The images of the $S^1$-factor are called *(normal) fibers.* The fibers of the extension from the boundaries to the solid tori $V_j$ are also called *fibers.* The solid tori $V_j$, for $1 \le j \le m$, are regular neighborhoods of the $m$ *exceptional fibers* with characteristic numbers $(a_j, b_j)$ and $V_0$ is a regular neighborhood of a normal fiber with characteristic numbers $(1, e)$. On each boundary torus we take a fiber $\eta_j^1$ intersecting $\rho_j^1$ exactly once. In addition we take pairwise disjoint simple arcs $\sigma_1^1, \dots, \sigma_j^1$ in $F'_g \subset M'$ where $\sigma_j^1$ goes from the $j$-th torus to the 0-th one. Together we obtain the following cell complex:

| | | | |
|---|---|---|---|
| $(G_0)$ | $\dim 0:$ | $\sigma_0^0, \dots, \sigma_m^0$ | vertices in $F'_g$; |
| $(G_{1,1})$ | $\dim 1:$ | $\sigma_1^1, \dots, \sigma_m^1$ | arcs in $F'_g$; |
| $(G_{1,2})$ | | $\rho_0^1, \dots, \rho_m^1$ | boundary curves of $F'_g$; |
| $(G_{1,3})$ | | $\eta_0^1, \dots, \eta_m^1$ | fibers; |
| $(G_{1,4})$ | | $\nu_1^1, \omega_1^1, \dots, \nu_g^1, \omega_g^1$ | closed curves in $F'_g$; |
| $(G_{2,1})$ | $\dim 2:$ | $\sigma_1^2, \dots, \sigma_m^2$ | from the annuli over $\sigma_j^1$ |
| $(G_{2,2})$ | | $\rho_0^2, \dots, \rho_m^2$ | from the boundary tori of $F'_g \times S^1$; |
| $(G_{2,3})$ | | $\mu_0^2, \dots, \mu_m^2$ | from the meridian disks in $V_j$; |
| $(G_{2,4})$ | | $\nu_1^2, \omega_1^2, \dots, \nu_g^2, \omega_g^2$ | from the annuli in $F'_g \times S^1$; ; |
| $(G_{2,5})$ | | $\delta^2$ | from $F'_g$; |
| $(G_{3,1})$ | $\dim 3:$ | $\sigma_0^3, \dots, \sigma_m^3$ | from the solid tori; |
| $(G_{3,2})$ | | $\delta^3$ | from $F'_g \times S^1$. |

The boundary maps are given by

$(R_{1,1})$ $\quad \partial\sigma_j^1 = \sigma_j^0 - \sigma_0^0, \quad 1 \le j \le m;$

$(R_{1,2})$ $\quad \partial\rho_j^1 = 0, \quad 0 \le j \le m;$

$(R_{1,3})$ $\quad \partial\eta_j^1 = 0, \quad 0 \le j \le m;$

$(R_{1,4})$ $\quad \partial\nu_j^1 = \partial\omega_j^1 = 0, \quad 1 \le j \le g;$

$(R_{2,1})$ $\quad \partial\sigma_j^2 = \eta_0^1 - \eta_j^1, \quad 1 \le j \le m;$

$(R_{2,2})$ $\quad \partial\rho_j^2 = 0, \quad 0 \le j \le m;$

$(R_{2,3})$ $\quad \partial\delta^2 = \sum_{j=0}^{m} \rho_j^1;$

$(R_{2,4})$ $\quad \partial\mu_j^2 = a_j\rho_j^1 + b_j\eta_j^1, \quad 0 \le j \le m;$

$(R_{2,5})$ $\quad \partial\nu_j^2 = \partial\omega_j^2, \quad 0 \le j \le g;$

$(R_{3,1})$ $\quad \partial\sigma_j^3 = \rho_j^2, \quad 0 \le j \le m;$

$(R_{3,2})$ $\quad \partial\delta^3 = -\sum_{j=0}^{m} \rho_j^2.$

## 3. The cohomology ring of a Seifert manifold.

In this section we will give the main result on the cohomology ring of a Seifert manifold and determine the type. For the cochains we take the bases dual to those for the chains and denote the corresponding elements by adding a hat, for example, $\hat{\rho}_j^1$ is defined by

$$\langle \hat{\rho}_j^1, \rho_k^1 \rangle = \delta_{jk}, \ \langle \hat{\rho}_j^1, \sigma_k^1 \rangle = \langle \hat{\rho}_j^1, \eta_j^1 \rangle = \langle \hat{\rho}_j^1, \nu_j^1 \rangle = \langle \hat{\rho}_j^1, \omega_j^1 \rangle = 0.$$

Here and in the following $\delta_{jk}$ denotes the Kronecker symbol. The (co)homology class of a (co)cycle $\alpha$ is denoted by $[\alpha]$.

**Theorem 3.1.** *Let $M = SF(g; e; m : (a_1, b_1), \dots, (a_m, b_m))$ be an orientable Seifert fibered 3-manifold with orientable base-surface.*

**Case (A):** *Let $a_1, \dots, a_n \equiv 0 \bmod 2$, $1 \le n \le m$, and $a_{n+1}, \dots, a_m \equiv 1 \bmod 2$.*

*Then*

$$H^i(M;\mathbf{Z}_2) \cong \begin{cases} \mathbf{Z}_2, & i=0,3 \\ \mathbf{Z}_2^{2g+n-1}, & i=1,2 \\ 0, & i \geq 3. \end{cases}$$

*Moreover, generators are*

$$\begin{cases} 1 = \left[\sum_{j=0}^{m} \hat{\sigma}_j^0\right], & \textit{dimension } 0; \\ \alpha_j := \left[\hat{\rho}_j^1 + \hat{\rho}_1^1\right],\ \theta_k := [\hat{\nu}_k^1],\ \theta'_k := [\hat{\omega}_k^1], & \\ \qquad 2 \leq j \leq n,\ 1 \leq k \leq g, & \textit{dimension } 1; \\ \beta_j := \left[\hat{\mu}_j^2\right] = \left[\hat{\sigma}_j^2\right],\ \varphi_k := [\hat{\nu}_k^2],\ \varphi'_k := [\hat{\omega}_k^2], & \\ \qquad 2 \leq j \leq n,\ 1 \leq k \leq g, & \textit{dimension } 2; \\ \gamma := \left[\hat{\delta}^3\right] = [\hat{\sigma}_0^3] = \ldots = [\hat{\sigma}_m^3], & \textit{dimension } 3. \end{cases}$$

**<u>Case (B):</u>** *Assume that* $n=0$ *and* $b_i \equiv 0 \bmod 2$ *for* $1 \leq i \leq r$, $b_i \equiv 1 \bmod 2$ *for* $r+1 \leq i \leq m$. *Then*

$$H^i(M;\mathbf{Z}_2) \cong \begin{cases} \mathbf{Z}_2 & \textit{for} \quad i=0,3; \\ \mathbf{Z}_2^{2g+1} & \textit{for} \quad i=1,2 \quad \textit{if} \quad m-r+e \equiv 0 \bmod 2, \\ \mathbf{Z}_2^{2g} & \textit{for} \quad i=1,2 \quad \textit{if} \quad m-r+e \equiv 1 \bmod 2; \\ 0 & \textit{for} \quad i>3. \end{cases}$$

*In this case generators are:*

$$\begin{cases} 1 = \left[\sum_{j=0}^{m} \hat{\sigma}_j^0\right] & \textit{in dimension } 0; \\ \alpha := \left[\sum_{j=0}^{m} \hat{\eta}_j^1 + \sum_{j=r+1}^{m} = \hat{\rho}_j^1\right],\ \theta_k := [\hat{\nu}_k^1], & \textit{in dimension } 1 \textit{ and} \\ \theta'_k := [\hat{\omega}_k^1], 1 \leq k \leq g, & \textit{if } m-r+e \equiv 0 \bmod 2, \\ \theta_k := [\hat{\nu}_k^1],\ \theta'_k := [\hat{\omega}_k^1],\ 1 \leq k \leq g, & \textit{if } m-r+e \equiv 1 \bmod 2; \\ \xi := \left[\hat{\delta}^2\right],\ = \varphi_k := [\hat{\nu}_k^2],\ \varphi'_k := [\hat{\omega}_k^2], & \\ \qquad 1 \leq k \leq g, & \textit{if } m-r+e \equiv 0 \bmod 2 \textit{ in dim } 2, \\ \varphi_k := [\hat{\nu}_k^2],\ \varphi'_k := [\hat{\omega}_k^2],\ 1 \leq k \leq g, & \textit{if } m-r+e \equiv 1 \bmod 2 \textit{ in dim } 2; \\ \gamma := \left[\hat{\delta}^3\right] & \textit{in dimension } 3. \end{cases}$$

Let us now give the cup products.

**Theorem 3.2.** *Let $M = SF(g; e; m : (a_1, b_1), \dots, (a_m, b_m))$ be an orientable Seifert manifold with orientable base-surface $F_g$ and let $\delta_{jk}$ denote the Kronecker delta.*

**Case (A):** *For $2 \le j, k \le n$, the cup products in $H^*(M; \mathbf{Z}_2)$ are given by:*

$$\alpha_j \cup \alpha_k = \binom{a_1}{2} \beta_1 + \delta_{jk} \binom{a_j}{2} \beta_j \quad \textit{and} \quad \alpha_j \cup \beta_k = \delta_{jk}\gamma,$$

*where $\beta_1 = \beta_2 + \ldots + \beta_n$. Furthermore, for $1 \le l, m \le g$,*

$$\theta_l \cup \varphi_m = \theta'_l \cup \varphi'_m = \delta_{lk}\gamma.$$

*Moreover, for $2 \le i, j, k \le n$*

$$\alpha_i \cup \alpha_j \cup \alpha_k = \binom{a_1}{2} \gamma, \text{ if } i \ne j \text{ or } j \ne k, \qquad \alpha_i \cup \alpha_i \cup \alpha_i = \left[\binom{a_1}{2} + \binom{a_i}{2}\right] \gamma.$$

*The remaining cup products in $H^*(M; \mathbf{Z}_2)$ are zero.*

**Case (B):** *If $n = 0$, then*

(1) *when $m - r + e \equiv 1 \bmod 2$*

$$\theta_j \cup \varphi'_k = \theta_j \cup \varphi_k = \gamma;$$

(2) *when $m - r + e \equiv 0 \bmod 2$*

$$\begin{aligned} \alpha \cup \theta_k &= \varphi_k,\ \alpha \cup \theta'_k = \varphi'_k,\ \theta_j \cup \theta'_j = \xi; \\ \alpha \cup \xi &= \gamma,\ \theta_j \cup \varphi'_j = \theta'_j \cup \varphi_j = \gamma. \end{aligned}$$

*The remaining cup products are zero.*

Different proofs of this theorem or parts of it will be considered in the following sections. From theorem 3.2 we directly obtain the calculation of the type of Seifert manifolds.

**Theorem 3.3.** *With the above notation, if $n \ge 2$, then* type$(M) = 1$ *exactly when $\binom{a_i}{2} + \binom{a_j}{2} \equiv 1 \bmod 2$ for some $i$, $j$, $1 \le i, j \le n$. If $n = 1$, then* type$(M) = 2$. *Finally, if $n = 0$, then* type$(M) = 1$ *if and only if $m - r + e \equiv 0 \bmod 2$ and $e - 2p + \sum_{j=r+1}^{m} a_j b_j \equiv 2 \bmod 4$.*

## 4. Computing cup products using Poincaré duality and intersections of chains.

Following a suggestion of Derek Hacon, we compute explicitly the cup products using Poincaré duality and the intersection theory of chains. Merely to avoid repetitions, we will restrict ourselves to the case of an orientable Seifert manifold with base-surface $S^2$ where all $a_j$, $1 \leq j \leq m$, are even and hence all $b_j$ are odd. We recall (see notations of the paragraph 2) that $a_0 = 1$ and $b_0 = e$. We will calculate $H_1(M; \mathbf{Z}_2)$ and $H_2(M; \mathbf{Z}_2)$ and realize bases of these $\mathbf{Z}_2$-vector spaces by curves and surfaces in $M$. In the following we calculate mod 2.

**4.1. Basis of the homology groups.** By $(R_{1,3})$ from section 2 we know that $\eta_j^1$ is a cycle for $0 \leq j \leq m$, by $(R_{2,4})$ we obtain (with the hypothesis on the $a_j$) that $\eta_j^1$ is a boundary for $j \neq 0$. By $(R_{2,1})$ and $(R_{2,4})$, $\eta_0^1$ and $\rho_0^1$ are boundaries. By $(R_{1,2})$, $\rho_j^1$ is a cycle. Therefore it suffices to determine the solutions of

$$0 = \partial\Big(\sum_{j=1}^{m} k_j \sigma_j^1\Big) = \sum_{j=1}^{m} k_j \sigma_j^0 + \sum_{j=1}^{m} k_j \sigma_0^0$$

with variables $k_j \in \mathbf{Z}_2$. Hence $k_j = 0$, $1 \leq j \leq m$. From $(R_{2,3})$ we obtain $\sum_{j=0}^{m} \rho_j^1 \sim \sum_{j=1}^{m} \rho_j^1 \sim 0$; hence:

**Proposition 4.1.1.** *$H_1(M; \mathbf{Z}_2) \cong \mathbf{Z}_2^{m-1}$. A basis is $\{[\rho_2^1], \ldots, [\rho_m^1]\}$.*

To calculate $H_2(M; \mathbf{Z}_2)$ observe that $\rho_j^2$ is a cycle representing the torus $\partial V_j^3$, but at the same time also a boundary, namely that of the solid torus $V_j^3$. By dropping the $\rho_j^2$ we get rid of all boundaries and we have only to decide which expressions $\sum_{j=1}^{m} k_j \sigma_j^2 + d\delta^2 + \sum_{j=0}^{m} \ell_j \mu_j^2$ are cycles. But

$$0 = \partial\left(\sum_{j=1}^{m} k_j \sigma_j^2 + d\delta^2 + \sum_{j=0}^{m} \ell_j \mu_j^2\right) = \sum_{j=1}^{m} k_j(\eta_0^1 - \eta_j^1) + d\sum_{j=0}^{m} \rho_j^1 + \sum_{j=1} \ell_j \eta_j^1 + \ell_0 \rho_0^1 + \ell_0 e \eta_0^1$$

implies

$$d = \ell_0 = 0,\ k_j = \ell_j \text{ for } 1 \leq j \leq m, \sum_{j=2}^{m} k_j = \ell_1.$$

It is enough to choose $k_1 = k_i = 1$ and $k_j = 0$ for $2 \leq i \leq m$ and $i \neq j$.

**Proposition 4.1.2.** *$H_2(M; \mathbf{Z}_2) \cong \mathbf{Z}_2^{m-1}$. A basis is $\{[\sigma_1^2+\sigma_2^2+\mu_1^2+\mu_2^2], \ldots, [\sigma_1^2+\sigma_m^2 + \mu_1^2 + \mu_m^2]\}$.*

**4.2. Dual bases.** Since for $j \in \{1, \ldots, m\}$

$$\langle \hat\rho_j^1, \sum_{k=2}^m \ell_k \sigma_k^2 + \sum_{k=1}^m n_k \rho_k^2 + \sum_{k=1}^m m_k \mu_k^2 + \ell \delta^2 \rangle =$$

$$\langle \hat\rho_j^1, \sum_{k=2}^m \ell_k (\eta_1^1 - \eta_k^1) + \sum_{k=1}^m m_k (a_k \rho_k^1 + b_k \eta_k^1) + \ell \sum_{k=1}^m \rho_k^1 \rangle = m_j a_j + \ell \equiv \ell \bmod 2$$

we obtain that $\hat\rho_j^1 + \hat\rho_1^1$ is a cocycle. From

$$\langle \hat\rho_j^1 + \hat\rho_1^1, \sum_{i=2}^m \ell_i \rho_i^1 \rangle = \ell_j$$

it follows that the cohomology classes $([\hat\rho_2^1 + \hat\rho_1^1], \ldots, [\hat\rho_m^1 + \hat\rho_1^1])$ give the bases dual to $([\rho_2^1], \ldots, [\rho_m^1])$ under the duality

$$H_1(M; \mathbf{Z}_2) \to \operatorname{Hom}(H_1(M; \mathbf{Z}_2), \mathbf{Z}_2) \cong H^1(M; \mathbf{Z}_2).$$

In dimension 2 we have

$$\langle \hat\mu_j^2, \sigma_1^2 + \sigma_i^2 + \mu_i^2 + \mu_1^2 \rangle = \delta_{ji} \quad \text{for } 2 \leq j, i \leq m$$

and

$$\langle \hat\sigma_j^2, \sigma_1^2 + \sigma_i^2 + \mu_i^2 + \mu_1^2 \rangle = \delta_{ij} \quad \text{for } 2 \leq i \leq m.$$

Hence $([\hat\mu_2^2] = [\hat\sigma_2^2], \ldots, [\hat\mu_m^2] = [\hat\sigma_m^2])$ is the Hom-dual (in the sense of homology classes) of $([\sigma_2^2 + \mu_1^2 + \mu_2^2], \ldots, [\sigma_m^2 + \mu_1^2 + \mu_m^2])$.

**4.3. Intersection properties.** The curves $\rho_2^1, \ldots, \rho_m^1$ represent a free system of generators of $H_1(M; \mathbf{Z}_2)$. Next we construct surfaces $F_2^2, \ldots, F_m^2 \subset M$ as follows: multiplying $\sigma_j^1$ by $S^1$ gives the annulus $\sigma_j^2$ such that $\partial\sigma_j^2$ consists of two curves on $\partial V_j^3$ and $\partial V_1^3$, respectively. Let $F_j^2$ be the union of $\sigma_j^2$, $\mu_j^2$ and $\mu_1^2$. Now $\partial F_j^2 = 0 \bmod 2$. The surfaces $F_2^2, \ldots, F_m^2$ represent the basis of $H_2(M; \mathbf{Z}_2)$ from above.

Direct consequences of the construction are the following equations on intersection numbers:

$$\begin{aligned} \rho_j^1 \circ \rho_k^1 &= 0, \quad 2 \le j, k \le m, \\ \rho_j^1 \circ F_k^2 &= \delta_{jk}, \quad 2 \le j, k \le m, \end{aligned}$$

here the operator $\circ$ denotes the intersection of the two arguments. Next we want to determine the intersection $F_j^2 \circ F_k^2$. Of course, first one has to bring the surfaces into general position (what postulates only some deformations in $V_j^3$). Since $\partial V_j^3 \cap F_j^2$ is the fiber, $F_j^2$ intersects $a_j$ times the meridian $\partial \mu_j^2 = a_j \rho_j^1 + b_j \eta_j^1$. Hence, $\mu_j^2 \cap F_j^2$ is like the star consisting of the $a_j$ segments joining 0 with $e^{2\pi i r / a_j}$, $0 \le r \le a_j - 1$. Taking $a_j$ points near to the above $a_j$ ones, a point near 0 and joining them in a similar form by curves near the segments, we obtain exactly $\binom{a_j}{2}$ intersection points between the two stars. A consequence is that the intersection of two such copies consists of a curve which runs $\binom{a_j}{2}$ times parallel to the core of $V_j^3$ and, thus, is mod 2 parallel to $\rho_j^1$. This argument can also be done in $V_1^3$. The result now is:

$$F_j^2 \circ F_k^2 = \binom{a_1}{2} \rho_1^1 + \delta_{jk} \binom{a_j}{2} \rho_j^1.$$

**4.4. Cup products.** Let $K$ be a cell complex associated to $M$ and $DK$ the dual complex obtained from $K$ after a barycentric subdivision. The cup product is obtained by composition of the following homomorphisms:

$$H^1(K) \otimes H^1(K) \xrightarrow{1} H_1(K) \otimes H_1(K) \xrightarrow{2} H_2(DK) \otimes H_2(DK) \xrightarrow{3} H_1(DK) \xrightarrow{4} H_2(K) \xrightarrow{5} H^2(K)$$

where the maps $\xrightarrow{1}$ and $\xrightarrow{5}$ are given by the Hom-dual, the maps $\xrightarrow{2}$ and $\xrightarrow{4}$ by the barycentric dual, the map $\xrightarrow{3}$ by the intersection of the surfaces.

To calculate $[\hat{\rho}_j^1 + \hat{\rho}_1^1] \cup [\hat{\rho}_k^1 + \hat{\rho}_1^1]$, $2 \le j, k \le m$ we take the Hom-duals $\rho_j^1$ and $\rho_k^1$ of $\hat{\rho}_j^1$ and $\hat{\rho}_1^1$, resp. By Poincaré duality (in a homological form via a dual complex, for example) we take the duals $F_j^2$, $F_k^2$ and look at their intersection

$$F_j^2 \circ F_k^2 = \binom{a_1}{2} \rho_1^1 + \delta_{jk} \binom{a_j}{2} \rho_j^1.$$

Again, by Poincaré duality (in the homological form) we have the 2-cycle

$$\binom{a_1}{2} F_1^2 + \delta_{jk} \binom{a_j}{2} F_j^2 \quad \text{where } F_1^2 = F_2^2 + \ldots + F_m^2.$$

Going to the Hom-dual, we obtain

$$\binom{a_1}{2} \hat{\mu}_1^2 + \delta_{jk} \binom{a_j}{2} \hat{\mu}_j^2 \quad \text{where } \hat{\mu}_1^2 = \hat{\mu}_2^2 + \ldots \hat{\mu}_m^2.$$

Finally we recover a result of theorem 3.2, Case(A): *for* $2 \leq j, k \leq m$,

$$[\hat{\rho}_j^1 + \hat{\rho}_1^1] \cup [\hat{\rho}_k^1 + \hat{\rho}_1^1] = \binom{a_1}{2} [\hat{\mu}_1^2] + \delta_{jk} \binom{a_j}{2} [\hat{\mu}_j^2],$$

similarly:

$$[\hat{\rho}_j^1 + \hat{\rho}_1^1] \cup [\hat{\mu_k^2}] = [\rho_j^1] \circ [F_k^2] = \delta_{jk}.$$

## 5. A chain approximation to the diagonal for infinite fundamental group.

If an irreducible 3-manifold $M$ has an infinite fundamental group then it is an Eilenberg-MacLane space $K(\Pi, 1)$; thus $H^*(M; A) = H^*(\Pi; A)$ and the computation of the cup products in $H^*(M; A)$ can be transformed into a purely algebraic calculation in group cohomology. Our method to effect this calculation depends on finding a free $R$-resolution of $\mathbf{Z}$ where $R = \mathbf{Z}\Pi$ is the integral group ring of $\Pi$ and then determining an appropriate chain approximation to the diagonal. One such resolution is the equivariant chain complex, that is, the chains of the universal cover $\tilde{M}$. Since there most arguments are in $\tilde{M}$ we drop in this section the tilde for cells of $\tilde{M}$ and underline cells from the base $M$.

**5.1. Equivariant chain complex on the solid torus $\underline{V} = D^2 \times S^1$.** Let $\underline{\mu}_*^1 = \partial D^2 \times \{1\}$ and $\underline{\lambda}_*^1 = \{1\} \times S^1$ be, respectively, the standard meridian and longitude of $\underline{V}$ (on the boundary $\partial \underline{V}$ of $\underline{V}$). Let $\underline{\rho}_*^1$, $\underline{\eta}_*^1$ be a pair of simple closed curves on $\partial \underline{V}$ which cut the torus into a disk with base point $\underline{\sigma}^0 = \underline{\rho}_*^1 \cap \underline{\eta}_*^1 = \underline{\mu}_*^1 \cap \underline{\lambda}_*^1$. Then $\underline{\mu}_*^1 \sim a\underline{\rho}_*^1 + b\underline{\eta}_*^1$, $\underline{\lambda}_*^1 \sim c\underline{\rho}_*^1 + d\underline{\eta}_*^1$ on $\partial \underline{V}$ for suitable integers $a, b, c, d$ with $ad - bc = \pm 1$. Assume that $ad - bc = 1$, then there exists a map

$\varphi\colon D^2 \to \underline{V}$ such that $\varphi|$ restricted to the interior of $D^2$ is an embedding of the interior of the disk into the interior of $\underline{V}$. Furthermore, $\varphi(\partial D^2) \subset \underline{\rho}^1_* \cup \underline{\eta}^1_*$ and $\varphi|_{(\partial D^2)} \sim a\underline{\rho}^1_* + b\underline{\eta}^1_*$ on $\partial V$. This defines a 2-cell $\underline{\mu}^2 := \mathrm{Im}(\varphi) \subset \underline{V}$ and the complement of $\underline{\mu}^2$ in the interior of $V$ is an open 3-cell, denoted by $\underline{\sigma}^3$.

Let $V$ be the universal cover of $\underline{V}$. For every cell $\underline{\gamma}^k \in \underline{V}$ fix a cell $\gamma^k \in V$ over $\underline{\gamma}^k$. Let $t$ be the generator of $\pi_1(V)$ and define $s = [\rho^1_*] = t^{-b}$, and $h = [\eta^1_*] = t^a$. Then $V$ has a cell decomposition consisting of the cells $t^k\sigma^0, t^k\rho^1, t^k\eta^1, t^k\rho^2, t^k\mu^2, t^k\sigma^3$ for $k \in \mathbf{Z}$, which are the coverings of the cells of $\underline{V}$. After puncturing $\partial\underline{V}$ we obtain a free group $\Gamma$ of rank 2 with free generators $s$, $h$ corresponding to the curves $\rho^1_*$, $\eta^1_*$. The class of the simple closed curve $\partial\mu^2$ is represented by some word $W_{a,b}(s,h)$ of the free group $\langle s,h\rangle$ which is uniquely determined up to conjugacy ([5], [6]). Thus $\partial\sigma^0 = 0$, $\partial\rho^1 = (s-1)\sigma^0$, $\partial\eta^1 = (h-1)\sigma^0$, $\partial\rho^2 = (1-s)\eta^1 + (h-1)\rho^1$, $\partial\mu^2 = F\rho^1 + G\eta^1$, $\partial\sigma^3 = \rho^2 + (1-t)\mu^2$. Here $F$ and $G$ are polynomials in $t, t^{-1}$; of course, they depend on the map used to attach $\underline{\mu}^2$ to $\partial V$. The coefficients $F$ and $G$ can be found, for instance, using the fundamental formula of the Fox calculus [5]: $W_{a,b}(s,h) - 1 = \frac{\partial W_{a,b}}{\partial s}(s-1) + \frac{\partial W_{a,b}}{\partial h}(h-1)$. This is a consequence of the geometric interpretation of $W_{a,b}(s,h)$ as the class of the boundary of a disk (cf. [6, 2.3]); this gives the solution $F = t^{a-1} + \ldots + 1$ , $G = t^{-1} + \ldots + t^{-b}$. For all possible attaching maps these have the smallest differences between the highest and lowest powers of $t$ appearing in the coefficients of $\rho^1$ and $\eta^1$; these differences are $a$ and $|b|$, respectively for $F$ and $G$.

Below we will give an augmented $\mathbf{Z}\Pi$-equivariant chain complex for an orientable Seifert manifold $\underline{M}$ with infinite fundamental group and orientable base-surface. Notice that the augmented $R = \mathbf{Z}\Pi$-equivariant chain complex $\mathcal{C} = (C_*(M;\mathbf{Z}), \partial)$ is isomorphic to $(C_*(\underline{M};\mathbf{Z}\Pi), \partial)$, however this isomorphism is not natural and depends on the choice of generators. $\mathcal{C}$ is a free resolution and this suffices to find the additive structure of $H^*(\underline{M};A)$. However, to determine the ring structure (i.e. the cup products), we make $\mathcal{C} \otimes \mathcal{C}$ into a $R$-chain complex by setting $\partial(x \otimes y) = \partial x \otimes y + (-1)^{\deg(x)} x \otimes \partial y$, and $(nu + mv)(x \otimes y) = n(ux \otimes uy) + m(vx \otimes vy)$ for $m$, $n \in \mathbf{Z}$, $u$, $v \in \Pi$, $x$,

$y \in \mathcal{C}$. Then we seek a diagonal approximation $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ such that $\Delta$ is a $R$-chain map and preserves augmentation. That such a diagonal map $\Delta$ exists is a consequence of the acyclic model theory, but it must be found explicitly (it is not unique). Of course, it suffices to know $\Delta$ on the (free) generators of the complex $\mathcal{C}$.

**5.2. The equivariant chain complex for Seifert manifolds.** The equivariant chain complex $\mathcal{C}$ for the universal cover $M$ consists of free $R-$modules $C_i$ in dimensions 0,1,2,3 with free generators

$(G_0)$ $\dim 0 :$ $\sigma_0^0, \ldots, \sigma_m^0;$

$(G_1)$ $\dim 1 :$ $\sigma_1^1, \ldots, \sigma_m^1; \rho_0^1, \ldots, \rho_m^1; \nu_1^1, \omega_1^1, \ldots, \nu_g^1, \omega_g^1; \eta_0^1, \ldots, \eta_m^1;$

$(G_2)$ $\dim 2 :$ $\sigma_1^2, \ldots, \sigma_m^2; \rho_0^2, \ldots, \rho_m^2; \nu_1^2, \omega_1^2, \ldots, \nu_g^2, \omega_g^2; \mu_0^2, \ldots, \mu_m^2; \delta^2;$

$(G_3)$ $\dim 3 :$ $\sigma_0^3, \ldots, \sigma_m^3; \delta^3.$

The definition of the boundary map $\partial$ of the chain complex $\mathcal{C}$ requires the following conventions and definitions in the group ring $R$. First of all, in addition to the list of generators given in $(G_1)$, $(G_2)$ adopt the notation $\sigma_0^1 = 0, \sigma_0^2 = 0$. Next let $r_j = s_0 s_1 \ldots s_j$ for $-1 \le j \le m$ and $r_{m+j} := s_0 \ldots s_m \prod_{k=1}^{j} [v_k, w_k]$ for $1 \le j \le g$ Observe that $r_{m+g} = 1$. Given relatively prime integers $a_j > 0$, $b_j > 0$, choose integers $c_j > 0$, $d_j > 0$ so that $\begin{vmatrix} a_j & b_j \\ c_j & d_j \end{vmatrix} = 1$ and let $t_j = s_j^{c_j} h^{d_j}$. Then $s_j = t_j^{-b_j}$ and $h = t_j^{a_j}$. When $j = 0$ set $a_0 = 1$, $b_0 = e$, so that $s_0 = h^{-e}$. Now define the Laurent polynomials

$$
\begin{aligned}
f_{l,j} &= 1 + t_j + \ldots + t_j^{l-1}, \quad l \ge 1, & f_{a_j,j} &= F_j = \tfrac{t_j^{a_j} - 1}{t_j - 1}, \\
g_{l,j} &= t_j^{-1} + t_j^{-2} + \ldots + t_j^{-l}, \quad l \ge 1, & g_{b_j,j} &= G_j = \tfrac{1 - t_j^{-b_j}}{t_j - 1}, \\
P_j &= 1 + t_j^{-b_j} + \ldots + t_j^{-b_j(c_j - 1)}, \\
Q_j &= 1 + t_j^{a_j} + \ldots + t_j^{a_j(d_j - 1)}.
\end{aligned}
$$

In particular, $F_0 = 1$ and $G_0 = (1 - h^{-e})/(h - 1)$. Finally, define the chains:

$$
\begin{aligned}
\pi_j^1 &:= r_{j-1} \left( \sigma_j^1 + \rho_j^1 \right) - r_j \sigma_j^1, \\
\pi_{m+j}^1 &:= r_{m+j-1} \left( 1 - v_j w_j v_j^{-1} \right) \nu_j^1 + \left( r_{m+j-1} v_j - r_{m+j} \right) \omega_j^1 ,
\end{aligned}
$$

$$\begin{aligned}
\pi_j^2 &:= -r_{j-1}\left(\sigma_j^2+\rho_j^2\right)+r_j\sigma_j^2,\\
\pi_{m+j}^2 &:= r_{m+j-1}\left(v_jw_jv_j^{-1}-1\right)\nu_j^2+\left(r_{m+j}-r_{m+j-1}v_j\right)\omega_j^2 .
\end{aligned}$$

The free resolution $\mathcal{C}$ is given by the exact sequence

$$\mathcal{C}: 0 \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbf{Z} \longrightarrow 0$$

and the differentials are defined by

$$\begin{aligned}
&(R_{1,1}) \quad \partial\sigma_j^1 = \sigma_j^0-\sigma_0^0, \quad 1\le j\le m,\\
&(R_{1,2}) \quad \partial\rho_j^1 = (s_j-1)\sigma_j^0, \quad 0\le j\le m,\\
&(R_{1,3}) \quad \partial\eta_j^1 = (h-1)\sigma_j^0, \quad 0\le j\le m,\\
&(R_{1,4}) \quad \partial\nu_j^1 = (v_j-1)\sigma_0^0, \quad \partial\omega_j^1=(w_j-1)\sigma_0^0, \quad 1\le j\le g,\\
&(R_{2,1}) \quad \partial\sigma_j^2 = \eta_0^1-\eta_j^1+(h-1)\sigma_j^1, \quad 1\le j\le m,\\
&(R_{2,2}) \quad \partial\rho_j^2 = (1-s_j)\eta_j^1+(h-1)\rho_j^1, \quad 0\le j\le m,\\
&(R_{2,3}) \quad \partial\nu_j^2 = (1-v_j)\eta_0^1+(h-1)\nu_j^1, \quad \partial\omega_j^2=(1-w_j)\eta_0^1 \quad +(h-1)\omega_j^1,\\
&\qquad\qquad 1\le j\le g,\\
&(R_{2,4}) \quad \partial\delta^2 = \sum_{j=0}^{m}\pi_j^1+\sum_{j=1}^{g}\pi_{m+j}^1,\\
&(R_{2,5}) \quad \partial\mu_j^2 = F_j\cdot\rho_j^1+G_j\cdot\eta_j^1 \quad 0\le j\le m,\\
&(R_{3,1}) \quad \partial\sigma_j^3 = \rho_j^2+(1-t_j)\mu_j^2, \quad 0\le j\le m,\\
&(R_{3,2}) \quad \partial\delta^3 = (1-h)\delta^2-\sum_{j=0}^{m}\pi_j^2-\sum_{j=1}^{g}\pi_{m+j}^2.
\end{aligned}$$

Observe that $\pi_0^1=\rho_0^1$, $\pi_0^2=-\rho_0^2$, and $\partial\pi_j^1=(r_j-r_{j-1})\,\sigma_0^0$, $0\le j\le m+g$, $\partial\pi_{m+j}^2=(r_j-r_{j-1})\,\eta_0^1+(1-h)\,\pi_{m+j}^1$, $0\le j\le m+g$. For the central result we introduce the 1-chain $\tau_j^1=P_j\rho_j^1+t_j^{-b_jc_j}Q_j\eta_j^1$.

**5.3. Diagonal Approximation Theorem.** *A diagonal approximation to the equivariant chain complex is defined on the generators of the chain complex* $\mathcal{C}$

*as follows:*

$$\begin{array}{rcll}
\Delta(\sigma_j^0) & = & \sigma_j^0 \otimes \sigma_j^0, & \Delta(\sigma_j^1) = \sigma_j^1 \otimes \sigma_j^0 + \sigma_0^0 \otimes \sigma_j^1, \\
\Delta(\rho_j^1) & = & s_j\sigma_j^0 \otimes \rho_j^1 + \rho_j^1 \otimes \sigma_j^0, & \Delta(\eta_j^1) = h\sigma_j^0 \otimes \eta_j^1 + \eta_j^1 \otimes \sigma_j^0, \\
\Delta(\nu_j^1) & = & \nu_j^1 \otimes \sigma_0^0 + v_j\sigma_0^0 \otimes \nu_j^1, & \Delta(\omega_j^1) = \omega_j^1 \otimes \sigma_0^0 + w_j\sigma_0^0 \otimes \omega_j^1 \\
\Delta(\sigma_j^2) & = & h\sigma_0^0 \otimes \sigma_j^2 - h\sigma_j^1 \otimes \eta_j^1 + \sigma_j^2 \otimes \sigma_j^0 + \eta_0^1 \otimes \sigma_j^1 & \\
\Delta(\rho_j^2) & = & \rho_j^2 \otimes \sigma_j^0 + s_j\eta_j^1 \otimes \rho_j^1 - h\rho_j^1 \otimes \eta_j^1 + hs_j\sigma_j^0 \otimes \rho_j^2 & \\
\Delta(\nu_j^2) & = & \nu_j^2 \otimes \sigma_0^0 + v_j\eta_0^1 \otimes \nu_j^1 - h\nu_j^1 \otimes \eta_0^1 + hv_j\sigma_0^0 \otimes \nu_j^2 & \\
\Delta(\omega_j^2) & = & \omega_j^2 \otimes \sigma_0^0 + w_j\eta_0^1 \otimes \omega_j^1 - h\omega_j^1 \otimes \eta_0^1 + hw_j\sigma_0^0 \otimes \omega_j^2 &
\end{array}$$

$$\begin{aligned}
\Delta(\mu_j^2) &= \mu_j^2 \otimes t_j^{-b_j}\sigma_j^0 + t_j^{a_j-b_j}\sigma_j^0 \otimes \mu_j^2 \\
&\quad - \sum_{k=0}^{a_j-1} \sum_{l=-b_j}^{k-1} t_j^k\rho_j^1 \otimes t_j^l\tau_j^1 - \sum_{l=0}^{a_j-1} \sum_{k=l-b_j}^{a_j-b_j-1} t_j^k\tau_j^1 \otimes t_j^l\rho_j^1 \\
&\quad - \sum_{k=1-b_j}^{-1} \sum_{l=-b_j}^{k-1} t_j^k\eta_j^1 \otimes t_j^l\tau_j^1 \sum_{l=1-b_j}^{-1} \sum_{k=a_j-b_j}^{a_j+l-1} t_j^k\tau_j^1 \otimes t_j^l\eta_j^1 \\
&\quad - G_j \sum_{r=1}^{a_j-1} t_j^r\tau_j^1 \otimes f_{r,j}\tau_j^1 - F_j \sum_{r=1}^{b_j} t_j^{-r}\tau_j^1 \otimes g_{r,j}\tau_j^1 \\
\Delta(\delta^2) &= A + B
\end{aligned}$$

where

$$\begin{aligned}
A &= \delta^2 \otimes s_0\sigma_0^0 + r_{m-1}\sigma_0^0 \otimes \delta^2 + \pi_m^1 \otimes \pi_m^1 + \rho_0^1 \otimes \rho_0^1 + \pi_m^1 \otimes \rho_0^1 \\
&\quad - \sum_{j=2}^{m-1} \sum_{i=1}^{j-1} \pi_j^1 \otimes \pi_i^1 - \sum_{j=1}^{m} \pi_j^1 \otimes r_{j-1}\sigma_j^1 + \sum_{j=1}^{m} r_j\sigma_j^1 \otimes r_{j-1}\rho_j^1
\end{aligned}$$

and

$$\begin{aligned}
B &= \sum_{j=1}^{g} \pi_{m+j}^1 \otimes \rho_0^1 + \sum_{j=1}^{g} \sum_{k=0}^{j-1} \pi_{m+k}^1 \otimes \pi_{m+j}^1 + \sum_{j=1}^{g} r_{m+j-1}\nu_j^1 \otimes \pi_{m+j}^1 \\
&\quad + \sum_{j=1}^{g} r_{m+j-1}v_j\tau_j^1 \otimes \pi_{m+j}^1 - \sum_{j=1}^{g} r_{m+j-1}v_j\tau_j^1 \otimes r_{m+j-1}\nu_j^1 \\
&\quad + \sum_{j=1}^{g} r_{m+j-1}v_jw_jv_j^{-1}\nu_j^1 \otimes r_{m+j}\tau_j^1 \\
\Delta(\sigma_j^3) &= \sigma_j^3 \otimes \sigma_j^0 + t_j^{a_j-b_j}\sigma_j^0 \otimes \sigma_j^3 - t_j\mu^2 \otimes t_j^{-b_j}\tau^1 - t_j^{a_j-b_j}\tau_j^1 \otimes t_j\mu_j^2 \\
&\quad + t_j\mu_j^2 \otimes G_j\tau_j^1 - \mu_j^2 \otimes G_j\tau_j^1 - t_j^{-b_j}P_j\mu_j^2 \otimes (\rho_j^1 + G_j\tau_j^1) \\
&\quad - t_j^{a_j}(\rho_j^1 + G_j\tau_j^1) \otimes P_j\mu_j^2 \\
\Delta(\delta^3) &= A' + B'
\end{aligned}$$

where

$$\begin{aligned} A' &= \delta^3 \otimes s_0\sigma_0^0 + r_{m-1}h\sigma_0^0 \otimes \delta^3 - h\delta^2 \otimes s_0\eta_0^1 - r_{m-1}\eta_0^1 \otimes \delta^2 - \rho_0^2 \otimes \rho_0^1 \\ &+h\rho_0^1 \otimes \rho_0^2 + \pi_m^2 \otimes \pi_m^1 - h\pi_m^1 \otimes \pi_m^2 + \pi_m^2 \otimes \rho_0^1 + h\pi_m^1 \otimes \rho_0^2 \\ &- \sum_{j=2}^{m-1}\sum_{i=1}^{j-1} \pi_j^2 \otimes \pi_i^1 + \sum_{j=2}^{m-1}\sum_{i=1}^{j-1} h\pi_j^1 \otimes \pi_i^2 - \sum_{j=1}^{m} \pi_j^2 \otimes r_{j-1}\sigma_j^1 \\ &- \sum_{j=1}^{m} h\pi_j^1 \otimes r_{j-1}\sigma_j^2 - \sum_{j=1}^{m} r_j\sigma_j^2 \otimes r_{j-1}\rho_j^1 + \sum_{j=1}^{m} r_jh\sigma_j^1 \otimes r_{j-1}\rho_j^2 \end{aligned}$$

and

$$\begin{aligned} B' = &\sum_{j=1}^{g} \pi_{m+j}^2 \otimes \rho_0^1 + \sum_{j=1}^{g}\sum_{k=0}^{j-1} \pi_{m+k}^2 \otimes \pi_{m+j}^1 - \sum_{j=1}^{g}\sum_{k=0}^{j-1} h\pi_{m+k}^1 \otimes \pi_{m+j}^2 \\ &- \sum_{j=1}^{g} r_{m+j-1}\nu_j^2 \otimes \pi_{m+j}^1 - \sum_{j=1}^{g} r_{m+j-1}h\nu_j^1 \otimes \pi_{m+j}^2 - \sum_{j=1}^{g} r_{m+j-1}hv_j\omega_j^1 \otimes \pi_{m+j}^2 \\ &- \sum_{j=1}^{g} r_{m+j-1}v_j\omega_j^2 \otimes \pi_{m+j}^1 + \sum_{j=1}^{g} r_{m+j-1}v_j\omega_j^2 \otimes r_{m+j-1}\nu_{m+j}^1 \\ &- \sum_{j=1}^{g} r_{m+j-1}v_jh\omega_j^1 \otimes r_{m+j-1}\nu_j^2 - \sum_{j=1}^{g} r_{m+j-1}v_jw_jv_j^{-1}\nu_j^2 \otimes r_{m+j}\omega_j^1 \\ &+ \sum_{j=1}^{g} r_{m+j-1}v_jw_jv_j^{-1}h\nu_j^1 \otimes r_{m+j}\omega_j^2 + \sum_{j=1}^{g} h\pi_{m+j}^1 \otimes \rho_0^2. \end{aligned}$$

**Sketch of the proof of theorem 3.2.** By definition $\alpha_j = [\hat{\rho}_j^1 + \hat{\rho}_1^1]$. It follows that for any $z \in C_2$ $\alpha_j \cup \alpha_k = [(\hat{\rho}_j^1 + \hat{\rho}_1^1) \smile= (\hat{\rho}_k^1 + \hat{\rho}_1^1)]$, where:

$$\left((\hat{\rho}_j^1 + \hat{\rho}_1^1) \smile (\hat{\rho}_k^1 + \hat{\rho}_1^1)\right)(z) = \times\left(\left(\hat{\rho}_j^1 \otimes \hat{\rho}_k^1 + \hat{\rho}_j^1 \otimes \hat{\rho}_1^1 + \hat{\rho}_1^1 \otimes \hat{\rho}_k^1 + \hat{\rho}_1^1 \otimes = \hat{\rho}_1^1\right)(\Delta z)\right)$$

and $\times : \mathbf{Z}/2 \otimes \mathbf{Z}/2 \to \mathbf{Z}/2$ is the multiplication map. Then for some $\kappa$, $\kappa_i^\sigma$, $\kappa_i^\rho$, $\kappa_i^\mu \in \mathbf{Z}/2$,

$$\left(\hat{\rho}_j^1 + \hat{\rho}_1^1\right) \smile \left(\hat{\rho}_k^1 + \hat{\rho}_1^1\right) = \kappa\hat{\delta}^2 + \sum_{i=0}^{m} \kappa_i^\sigma \hat{\sigma}_i^2 + \sum_{i=0}^{m} \kappa_i^\rho \hat{\rho}_i^2 + \sum_{i=0}^{m} \kappa_i^\mu \hat{\mu}_i^2.$$

It is clear that the coefficients

$$\begin{aligned}\kappa_i^\sigma &= \times\left(\left(\hat\rho_j^1\otimes\hat\rho_k^1+\hat\rho_j^1\otimes\hat\rho_1^1+\hat\rho_1^1\otimes\hat\rho_k^1+\hat\rho_1^1\otimes\hat\rho_1^1\right)\left(\Delta\sigma_i^2\right)\right) \quad \text{and}\\ \kappa_i^\rho &= \times\left(\left(\hat\rho_j^1\otimes\hat\rho_k^1+\hat\rho_j^1\otimes\hat\rho_1^1+\hat\rho_1^1\otimes\hat\rho_k^1+\hat\rho_1^1\otimes\hat\rho_1^1\right)\left(\Delta\rho_i^2\right)\right)\end{aligned}$$

are zero, since the expressions for $\Delta\sigma_i^2$ and $\Delta\rho_i^2$, given in the Diagonal Approximation Theorem, do not involve any terms of the form $\rho_j^1\otimes\rho_k^1$, $\rho_j^1\otimes\rho_1^1$, $\rho_1^1\otimes\rho_k^1$, or $\rho_1^1\otimes\rho_1^1$. Furthermore, since $\delta^2\sim 0\in H^2(M;\mathbf{Z}/2)$, the coefficient $\kappa$ of $\delta^2$ is immaterial.

Finally, consider $\kappa_l^\mu = \times\left(\hat\rho_j^1\otimes\hat\rho_k^1+\hat\rho_j^1\otimes\hat\rho_1^1+\hat\rho_1^1\otimes\hat\rho_k^1+\hat\rho_1^1\otimes\hat\rho_1^1\right)(\Delta\mu_l^2)$. Observe that the only terms of $\Delta\mu_l^2$ which contribute to $\kappa_i^\mu$ are of the form $\rho_l^1\otimes\rho_l^1$ (for $j=k=l$) and $\rho_1^1\otimes\rho_1^1$ (for all $j$, $k$). To complete the calculation of $\alpha_j\cup\alpha_k$, it suffices to count the number of terms of the form $\rho_l^1\otimes\rho_l^1$ in $\Delta\mu_l^2$ for $1\le j=k=l\le n$, modulo 2. Thus, when $1\le l\le n$, the number of terms of the form $\rho_l^1\otimes\rho_l^1$ in $\Delta\mu_l^2$ is

$$\begin{aligned}&\sum_{i=0}^{a_l-1}\sum_{j=-b_l}^{i-1}(c_l)+\sum_{j=0}^{a_l-1}\sum_{i=j-b_l}^{a_l-b_l-1}(c_l)+b_l\sum_{r=1}^{a_l-1}\left(rc_l^2\right)+a_l\sum_{r=1}^{b_l}\left(rc_l^2\right)\\ &= c_l\left[\sum_{i=0}^{a_l-1}(i+b_l)+\sum_{j=0}^{a_l-1}(a_l-j)+b_lc_l\binom{a_l}{2}+a_lc_l\binom{b_l+1}{2}\right].\end{aligned}$$

Since $b_l\equiv c_l\equiv 1 \bmod 2$, and $a_l\equiv 0 \bmod 2$ it follows that

$$c_l\left[\binom{a_l}{2}+a_lb_l+a_l^2-\binom{a_l}{2}+b_lc_l\binom{a_l}{2}+a_lc_l\binom{b_l+1}{2}\right]\equiv\binom{a_l}{2} \bmod 2.$$

## 6. Extension of the previous results when the fundamental group is finite.

In this section we describe results of Kerstin Aaslepp [1]. Heuristically her method is the following. Assume that there are two spaces $X, Y$ such that the intersection is 'nice' and that there are given two cohomology classes $\Phi,\Psi\in H^*(X)$ and $\Phi',\Psi'\in H^*(Y)$ which can be represented by cocycles which vanish on the complements of $X \cap Y$ in $X$ and $Y$, respectively, and are the same on $X \cap Y$. Then one expects that the cup product $\Phi \cup \Psi$ can be calculated if $\Phi' \cup \Psi'$ is known. Following this line K. Aaslepp proves some results on the cup product for cellular cohomology on the level of cocycles. For the extension of the results on the cohomology of Seifert manifolds with infinite fundamental group to those with finite fundamental group we formulate and prove her main tool using only cohomology classes and groups. We thank T. Birk, M. Drawe, C. Sczesny and S. Skaberna who find a mistake in the proof of the theorem 6.2 in our first version and adapt the lemma 6.1 below.

For simplicity we consider cellular cohomology. Let $A, A', B$ be cell complexes such that $A \cap B = A' \cap B$ is a subcomplex of each of them. Then $M = B \cup A$ and $M' = B \cup A'$ are also cell complexes; let $i: B \hookrightarrow M$, $j: M \to (M, A)$ $i': B \hookrightarrow M'$ and $j': M' \to (M', A')$ be the inclusions. We say that a *cellular cochain* $\varphi \in C^p(M)$ *vanishes on* $A$ if $\varphi(e^p) = 0$ for every $p$-cell $e^p$ of $A$; if a cohomology class $\Phi \in H^p(M)$ contains a representative $\varphi$ vanishing on $A$ then we also say that $\Phi$ vanishes on $A$. Clearly, if a cocycle of $M$ vanishes on $A$ it also is a cocycle of $(M, A)$; hence, to every $\Phi \in H^p(M)$ vanishing on $A$ there is a uniquely determined element of $H^p(M, A)$ which is mapped by $j^*$ to $\Phi$. We also denote it by $\Phi$.

**Lemma 6.1.** *Let $A, A', B, M, M'$ be as above. Assume that*

(1) *either $exc'(Ker\ j'^*) = exc(Ker\ j^*)$ and $j'^*$ is surjective or $j'^*$ injective, and*

(2) *$i'^*: H^q(M') \to Im\ i^*$ is surjective. Let $\Phi \in H^p(M)$, $\Psi \in H^q(M)$ be cohomology classes such that $\Phi$ vanishes on $A$. Following the maps described in the commutative diagram below, we get two elements $\Phi' \in H^p(M')$, $\Psi' \in H^q(M')$. Then $\Phi \cup \Psi \in H^{p+q}(M)$ is equal to the image of $\Phi' \cup \Psi' \in H^{p+q}(M')$ obtained by the maps described in the commutative diagram below.*

$$\begin{array}{cccccccc}
H^p(M') & \times & H^q(M') & \longrightarrow & H^{p+q}(M') & (\Phi', \Psi') & \longrightarrow & \Phi' \cup \Psi' \\
{\scriptstyle j'^*}\Big\uparrow & & \Big\uparrow{\scriptstyle =} & & {\scriptstyle j'^*}\Big\uparrow{\scriptstyle (1)} & \uparrow & & \uparrow \\
H^p(M', A') & \times & H^q(M') & \longrightarrow & H^{p+q}(M', A') & (\Phi', \Psi') & \longrightarrow & \Phi' \cup \Psi' \\
{\scriptstyle \mathrm{exc}'}\Big\downarrow{\scriptstyle \cong} & & {\scriptstyle i'^*}\Big\downarrow{\scriptstyle (2)} & & {\scriptstyle \mathrm{exc}'}\Big\downarrow{\scriptstyle \cong} & \downarrow & & \downarrow \\
H^p(B, B \cap A') & \times & H^q(B) & \longrightarrow & H^{p+q}(B, B \cap A') & (\Phi|B, \Psi|B) & \longrightarrow & \Phi|B \cup \Psi|B \\
\| & & \| & & \| & \| & & \| \\
H^p(B, B \cap A) & \times & H^q(B) & \longrightarrow & H^{p+q}(B, B \cap A) & (\Phi|B, \Psi|B) & \longrightarrow & \Phi|B \cup \Psi|B \\
{\scriptstyle \mathrm{exc}}\Big\uparrow{\scriptstyle \cong} & & {\scriptstyle i^*}\Big\uparrow & & {\scriptstyle \mathrm{exc}}\Big\uparrow{\scriptstyle \cong} & \uparrow & & \uparrow \\
H^p(M, A) & \times & H^q(M) & \longrightarrow & H^{p+q}(M, A) & (\Phi, \Psi) & \longrightarrow & \Phi \cup \Psi \\
{\scriptstyle j^*}\Big\downarrow & & \| & & {\scriptstyle j^*}\Big\downarrow & \downarrow & & \downarrow \\
H^p(M) & \times & H^q(M) & \longrightarrow & H^{p+q}(M) & (\Phi, \Psi) & \longrightarrow & \Phi \cup \Psi
\end{array}$$

*here* exc *and* exc$'$ *are the excision isomorphisms.*

We will apply this lemma to the case of Seifert manifolds with finite fundamental group. The genus of such a space vanishes and the number of exceptional fibers is smaller than 4. For to have a non-trivial first $\mathbf{Z}_2$-cohomology there must be two exceptional fibers with even order. (These properties are well know, see [8, 6.2], or can be verified by an easy computation.) We will now show that the result about the cup product quoted in theorem 3.2 is also valid for Seifert manifolds with finite fundamental group.

If there are two exceptional fibers the fundamental group is cyclic and the manifold is a lens space $L(p,q)$ with $H_1(L(p,q)) \cong \mathbf{Z}_p$, $H_2(L(p,q)) = 0$. If none of the orders of the exceptional fibers is divisible by 2 then $\gcd(2,p) = 1$. Thus, by the universal coefficient theorems, $H^1(L(p,q), \mathbf{Z}_2) = H^2(L(p,q), = \mathbf{Z}_2) = 0$ and there are only trivial cup products. If there are three exceptional fibers and the fundamental group is finite then at least one of them has order 2 because otherwise, factoring by the element representing the normal fiber, we obtain a crystallographic group with compact fundamental domain of the euclidean or hyperbolic plane which, thus, has infinite order.

**Theorem 6.2.** *Let $M$ be an orientable Seifert manifold $M$ with finite funda-*

*mental group and* $H^j(M,\mathbf{Z}_2) \neq 0$ *for* $j = 1$ *or* $j = 2$*. Then* $M$ *has* 2 *or* 3 *exceptional fibers where at least two of them have even order. Assume that the exceptional fibers are of type* $(a_1, b_1), \ldots, (a_m, b_m)$ *such that*

$$a_j \equiv 0 \bmod 2 \text{ for } 1 \leq j \leq n, \quad a_j \equiv 1 \bmod 2 \text{ for } n < j \leq m;$$

*here* $2 \leq n \leq m \leq 3$*. For* $2 \leq j,k \leq n$*, the cup products in* $H^*(M;\mathbf{Z}_2)$ *are given by:*

$$\alpha_j \cup \alpha_k = \binom{a_1}{2}\beta_1 + \delta_{jk}\binom{a_j}{2}\beta_j \quad \text{and} \quad \alpha_j \cup \beta_k = \delta_{jk}\gamma,$$

*where* $\beta_1 = \beta_2 + \ldots + \beta_n$*. Moreover, if* $2 \leq i \leq n$ *as well, then*

$$\alpha_i \cup \alpha_j \cup \alpha_k = \binom{a_1}{2}\gamma, \text{ if } i \neq j \text{ or } j \neq k, \qquad \alpha_i \cup \alpha_i \cup \alpha_i = \left[\binom{a_1}{2} + \binom{a_i}{2}\right]\gamma.$$

**Proof.** In $M$ we take a regular closed neighborhood of a normal fiber $A$; let $B$ be the closure of the complement, now $M = A \cup B$, $A \cap B = \partial A = \partial B \cong S^1 \times S^1$, where $[\hat{s}^1]$ is the generator of $H^1(\partial B)$ from the base-surface. Let $A'$ be a Seifert fiber space with $A' \cap B = \partial A' = \partial B$ and assume that on $\partial B$ the fibrations of $B$ and of $A'$ coincide. Moreover we assume that the fibration of $A'$ admits $4-r$ exceptional fibers of type $(a_{r+1}, b_{r+1}), \ldots, (a_4, b_4)$ where $a_{r+1}, \ldots, a_4 \geq 3$ are odd, while $b_{r+1}, \ldots, b_n$ are even, $\gcd(a_j, b_j) = 1$. Now $M' = A' \cup B$ is a Seifert manifold with four exceptional fibers and genus 0; hence, $H^*(M')$ is given in theorem 3.2. To elements of $H^*(M')$ we add a $'$ (like $\alpha'_j$).

For the computation of none-trivial cup-products it is sufficient to consider $p, q = 1$ *or* 2 and $p + q = 2$ *respectively* 3. The generators $\alpha_2, \ldots, \alpha_n$ and $\beta_2, \ldots, \beta_n$ vanish on $A$ since the representatives given in theorem 3.1 do so (i.e. for $k\colon A \hookrightarrow M$ we have $k^*(H^p(M)) = 0$ for $p = 1$ *or* 2; hence $j^*\colon H^p(M,A) \to H^p(M)$ is surjective.) Moreover $H^{p+q}(A') = 0$ for $p+q = 2$ *or* 3 and $H^2(A) = 0$. Hence, $j'^*\colon H^{p+q}(M',A') \to H^{p+q}(M')$ is surjective, and $Ker\ j'^* = Ker\ j^* = 0$ for $p+q = 3$. For $p + q = 2$ $Ker\ j'^*$ is generated by $\beta'_1 = [\hat{\mu}'^2_j]_{(M',A')}$, $Ker\ j^*$ respectively. Now we have $exc'(\beta'_1) = [\hat{\mu}'^2_1]_{(B,B\cap A')} = [\hat{\mu}^2_1]_{(B,B\cap A)} =$

$exc(\beta_1)$. Moreover $H^3(M', B) \cong H^3(M') \cong \mathbf{Z}_2$ implies $i'^*\colon H^2(M') \to H^2(B)$ is surjective. $H^1(M')$ is generated by $\alpha'_2, \dots, \alpha'_n$, $H^1(M)$ respectively. $H^1(B)$ is generated by $[\hat\rho^1_j + \hat\rho^1_1]_B, j = 2, \dots, n$ and $[\hat s^1 + \hat\rho^1_1]_B$. Hence, $i'^*\colon H^1(M') \to Im\ i^*$ is surjective as $[\hat s^1 + \hat\rho^1_1]_B \notin Im\ i^*$.

In the diagram of Lemma 6.1 we obtain that the elements of bases of $H^p(M)$ are mapped to the elements of the base of $H^p(M')$ having the same letter, for example:

$$\begin{aligned}
&((j'^* \circ (\mathrm{exc}')^{-1} \circ \mathrm{exc} \circ (j^*)^{-1})\,(\alpha_k)\,,\ ((i'^*)^{-1} \circ i^*)\,(\alpha_\ell))\\
&\quad = ((j'^* \circ (\mathrm{exc}')^{-1} \circ \mathrm{exc} \circ (j^*)^{-1})\,([\hat\rho^1_k + \hat\rho^1_1])\,,\ ((i'^*)^{-1} \circ i^*)\,([\hat\rho^1_\ell + \hat\rho^1_1]))\\
&\quad = ([\hat\rho^1_k + \hat\rho^1_1]',\ [\hat\rho^1_\ell + \hat\rho^1_1]')\\
&\quad = (\alpha'_k, \alpha'_\ell)
\end{aligned}$$

and

$$\begin{aligned}
[j^* \circ \mathrm{exc}^{-1} \circ \mathrm{exc}' \circ (j'^*)^{-1}]\,(\beta'_k) \quad &= [j^* \circ \mathrm{exc}^{-1} \circ \mathrm{exc}' \circ (j'^*)^{-1}]\,([\hat\mu^2_k]')\\
&= [\hat\mu^2_k]_M = \beta_k;
\end{aligned}$$

hence, $\alpha_k \cup \alpha_\ell = \begin{pmatrix} a_1 \\ 2 \end{pmatrix} \beta_1 + \delta_{k\ell} \begin{pmatrix} a_k \\ 2 \end{pmatrix} \beta_k$.

Using similar arguments we obtain the required formulae for the other cup products in $H^*(M)$ from those of $H^*(M')$.

**Remark.** Based on results like 6.2 Kerstin Aaslepp [1] also determines the cup product for the Seifert manifolds with a base of positive genus and exceptional fibers amalgamating the cohomology rings of locally trivial $S^1$-bundles over surfaces and Seifert manifolds with a base of genus 0 and she obtains the same results as Bryden-Zvengrowski [4].

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Claude Hayat-Legrand
U.R.A. n$^o$ 1408
Université Paul Sabatier
118, route de Narbonne
F-31062 Toulouse-Cedex
France
e-mail: Hayat@picard.ups-tlse.fr

Heiner Zieschang
Fakultät für Mathematik
Ruhr-Universität Bochum
D-44780 Bochum
Germany
marlene.schwarz@rz.ruhr-uni-bochum.de

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