SOME HOMOTOPY GROUPS OF THE DOUBLE SUSPENSION OF THE REAL PROJECTIVE SPACE $\mathbf{R}P^6$

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Introduction

Before we state our result, let us fix some notations. Let X be a based finite CW-complex and

$$\Sigma^n X = X \wedge S^n$$

be the n-fold reduced suspension of X. Let ι_X be the homotopy class of the identity map of X. The order of the element $\Sigma\iota_X = \iota_{\Sigma X}$ in $[\Sigma X, \Sigma X]$ is called the suspension order ([13]) or the characteristic ([2]) of X.

We denote by $\mathbf{R}P^n$ the real n-dimensional projective space. The purpose of the present note is to prove the following.

Theorem 1. *The suspension order of* $\Sigma^2\mathbf{R}P^6$ *is* 8.

As a direct consequence of this theorem, we have

Theorem 2. *The suspension order of* $\Sigma^2\mathbf{R}P^{2n}$ *is* $2^{\varphi(2n)}$, *where* $\varphi(m)$ *stands for the number of integers* k *satisfying* $1 \leq k \leq m$ *and* $k \equiv 0, 1, 2$ *or* $4 \bmod 8$.

We set

$$M^n = \Sigma^{n-2}\mathbf{R}P^2 \ (n \geq 2).$$

This is a Moore space of type $(\mathbf{Z}_2, n-1)$. To prove Theorem 1, we need to determine the following unstable homotopy groups

$$\pi_i(\Sigma^2\mathbf{R}P^6) \text{ and } [M^i, \Sigma^2\mathbf{R}P^6] \ (i \leq 8).$$

*The research was partially supported by FAPESP grant No. 96/1644-1

Our method is the composition methods developed by Toda([12]). We shall also use the homotopy excision theorem ([5]) about the metastable relative homotopy groups $\pi_k(X, A)$, where A is a finite CW-complex of dimension less than $n-1$ and $X = A \cup e^n$.

The results and notations of [12] will be used freely. It is assumed that we are especially familiar with the results and relations in $\pi_{n+k}(S^n)$ $(k \leq 6)$. We will also deal with the Whitehead products ([3], [1]) freely.

Let $\iota_n = \iota_{S^n}, \eta_2$ be the Hopf map, $\eta_n = \Sigma^{n-2}\eta_2$ and $\eta_n^2 = \eta_n \circ \eta_{n+1}$. Let $\nu' \in \pi_6(S^3)$ be a generator of the 2-primary component $\mathbf{Z}_4$ of $\pi_6(S^3)$, $\nu_4 \in \pi_7(S^4)$ be the Hopf map and $\nu_n = \Sigma^{n-4}\nu_4$ $(n \geq 4)$.

1. Recalling some fundamental results

We set $\mathrm{P}^n = \mathbf{R}\mathrm{P}^n$ for simplicity. We set $\mathrm{P}^n_k = \mathrm{P}^n/\mathrm{P}^{k-1}$ $(k \leq n)$. Let $\gamma_n : S^n \longrightarrow \mathrm{P}^n$ be the projection. We denote by $i_{n-1} : S^{n-1} \hookrightarrow M^n$ an inclusion and by $p_n : M^n \to S^n$ a collapsing map. Let $X = A \cup CB$ be a mapping cone of a mapping from B to A, where $B = S^{n-1}$ or M^{n-1}. For an element $\alpha \in \pi_{k-1}(B)$, we denote by $\hat{\alpha} \in \pi_k(X, A)$ an element satisfying $p'_*\hat{\alpha} = \Sigma\alpha$, where $p' : (X, A) \to (\Sigma B, *)$ is a collapsing map. We recall

$$\Sigma\gamma_2 = 2i_2\eta_2;\ \Sigma^2\gamma_2 = 0,$$

and so we have

$$\Sigma^2\mathrm{P}^3 = M^4 \vee S^5.$$

Let

$$k_1 : M^4 \hookrightarrow M^4 \vee S^5 \text{ and } k_2 : S^5 \hookrightarrow M^4 \vee S^5$$

be the inclusions, respectively. Then, by the cell structure of $\Sigma^2\mathrm{P}^4$, we have

$$\Sigma^2\gamma_3 = \pm k_1\tilde{\eta}_3 + 2k_2.$$

Here $\tilde{\eta}_3 \in \pi_5(M^4)$ is a coextension of η_3 satisfying $p_{4*}\tilde{\eta}_3 = \eta_4$. We recall ([10]) that

$$\pi_4(M^3) = \mathbf{Z}_4\{\alpha\},\ \text{where } \Sigma\alpha = \tilde{\eta}_3.$$

We denote by $p'_n : \mathrm{P}^{2n+2} \longrightarrow \mathrm{P}^{2n+2}_{2n+1} = M^{2n+2}$ a map given by pinching P^{2n} to a single point. We set

$$\tau_n = \Sigma\gamma_{2n}.$$

We know that τ_n is of order 2. We define an extension

$$\bar{\tau}_n \in [M^{2n+2}, \Sigma\mathrm{P}^{2n}]$$

of τ_n as the third map of the cofiber sequence

$$\mathrm{P}^{2n} \xrightarrow{i} P^{2n+2} \xrightarrow{p'_n} M^{2n+2} \longrightarrow \Sigma\mathrm{P}^{2n}.$$

By abuse of notation, we often use p to represent a collapsing map and use i to represent an inclusion map. We define an element $\beta \in \pi_5(\Sigma^2\mathrm{P}^4)$ as the composite ik_2 for the inclusion $i : \Sigma^2\mathrm{P}^3 \hookrightarrow \Sigma^2\mathrm{P}^4$. From the definition of β, we have $p'_1\beta = i_5 \in \pi_5(M^6)$ and $\Sigma\beta$ is taken as a coextension $\tilde{\imath}_5$ of i_5. Let $\bar{\eta}_3 \in [M^5, S^3]$ be an extension of η_3, set $\bar{\eta}_n = \Sigma^{n-3}\bar{\eta}_3 \in [M^{n+2}, S^n]$ and $\tilde{\eta}_n = \Sigma^{n-3}\tilde{\eta}_3 \in \pi_{n+2}(M^{n+1})$ $(n \geq 3)$. We recall ([7]) that $\pi_6(\Sigma^3\mathrm{P}^4) = \mathbf{Z}_8\{\tilde{\imath}_5\}$ and the following.

Lemma 1.1. $\Sigma\bar{\tau}_1 = \tilde{\eta}_3 p \in [M^5, M^4]$ *and* $\Sigma^2\bar{\tau}_2 = \tilde{\imath}_5\bar{\eta}_6 \in [M^8, \Sigma^3\mathrm{P}^4]$.

We have the following.

Lemma 1.2. $\pi_5(\Sigma^2\mathrm{P}^4) = \mathbf{Z}_8\{\beta\}$, *where* $2\beta = \pm i'\tilde{\eta}_3$ *and* $H(\beta) = i \in \pi_5(\Sigma(\Sigma\mathrm{P}^4 \wedge \Sigma\mathrm{P}^4))$.

Proof. We consider the exact sequence

$$\pi_6(\Sigma^2\mathrm{P}^4, \Sigma^2\mathrm{P}^3) \xrightarrow{\partial} \pi_5(\Sigma^2\mathrm{P}^3) \xrightarrow{i_*} \pi_5(\Sigma^2\mathrm{P}^4) \longrightarrow 0.$$

We have $\pi_6(\Sigma^2\mathrm{P}^4, \Sigma^2\mathrm{P}^3) = \mathbf{Z}\{\omega\}$, $\pi_5(\Sigma^2\mathrm{P}^3) \cong \pi_5(M^4) \oplus \pi_5(S^5)$, where ω is the characteristic map of the 6-cell of $\Sigma^2\mathrm{P}^4$. We have $\partial\omega = \Sigma^2\gamma_3 = \pm k_1\tilde{\eta}_3 + 2k_2$. From the definition of β, we have the result apart from the last assertion.

In the EHP sequence

$$\pi_4(\Sigma\mathrm{P}^4) \xrightarrow{\Sigma} \pi_5(\Sigma^2\mathrm{P}^4) \xrightarrow{H} \pi_5(\Sigma(\Sigma\mathrm{P}^4 \wedge \Sigma\mathrm{P}^4)) \xrightarrow{\Delta} \pi_4(\Sigma\mathrm{P}^4),$$

we have $\pi_5(\Sigma(\Sigma P^4 \wedge \Sigma P^4)) \cong \pi_5(\Sigma(M^3 \wedge M^3)) \cong \pi_5(M^6) \cong \mathbf{Z}_2$ and $\Delta i = i''\Delta\iota_5 = 2i\eta_2 = i''\Sigma\gamma_2 = 0$, where $i'' : M^3 \hookrightarrow \Sigma P^4$ is an inclusion satisfying $\Sigma i'' = i'$. So H is an epimorphism and $H(\beta) = i$. This completes the proof.

□

We denote by ι'_n the identity map of M^n. We recall that $M^2 \wedge M^2 = M^3 \cup_{2\iota'_3} CM^3$ and that $2\iota'_3 = i_2\eta_2 p_3 \in [M^3, M^3]$.

We have the following([10]) ; $\pi_3(M^2 \wedge M^2) = \mathbf{Z}_8\{\tilde{i}'_2\}$ and $\pi_{2n}(\Sigma(M^n \wedge M^n)) = \mathbf{Z}_4\{\Sigma^{2n-3}\tilde{i}'_2\}$ for $n \geq 3$, where $\tilde{i}'_2$ is a coextension of i_2 and it satisfies $2\tilde{i}'_2 = i\eta_2$. We also recall ([10]) that $\pi_6(M^4) = \mathbf{Z}_4\{\delta\} \oplus \mathbf{Z}_2\{\tilde{\eta}_3\eta_5\}$, where $\delta \in \pi_6(M^4)$ is the attaching map in the Stiefel manifold $V_{5,2} = M^4 \cup_\delta e^7$. We have the relations $2\delta = i_3\nu' = \Delta(\Sigma^5\tilde{i}'_2)$, $\Sigma\delta = 2i_4\nu_4$ and $H(\delta) = \Sigma^3\tilde{i}'_2$. We show

Lemma 1.3. $\pi_6(\Sigma^2 P^4) = \mathbf{Z}_4\{i'\delta\} \oplus \mathbf{Z}_2\{\beta\eta_5\}$
and $\beta\eta_5 = \Sigma^2\gamma_4 + 2i'\delta \in \pi_6(\Sigma^2 P^4)$ *so that* $\pi_6(\Sigma^2 P^5) = \mathbf{Z}_4\{i'\delta\}$.

Proof. In the exact sequence

$$\pi_7(\Sigma^2 P^4, \Sigma^2 P^3) \xrightarrow{\partial} \pi_6(\Sigma^2 P^3) \xrightarrow{i_*} \pi_6(\Sigma^2 P^4) \longrightarrow 0,$$

we have

$$\pi_7(\Sigma^2 P^4, \Sigma^2 P^3) = \mathbf{Z}_2\{\hat{\eta}_5\} \text{ and } \pi_6(\Sigma^2 P^3) \cong \pi_6(M^4) \oplus \pi_6(S^5).$$

Since $\partial\hat{\eta}_5 = \Sigma^2\gamma_3 \circ \eta_5 = k_1\tilde{\eta}_3\eta_5$, we have the first half.

We consider the EHP sequence

$$\pi_7(\Sigma^2 P^4) \xrightarrow{\Sigma} \pi_8(\Sigma^3 P^4) \xrightarrow{H} \pi_8(\Sigma(\Sigma^2 P^4 \wedge \Sigma^2 P^4)) \xrightarrow{\Delta} \pi_6(\Sigma^2 P^4) \longrightarrow \cdots.$$

Since

$$\pi_8(\Sigma(\Sigma^2 P^4 \wedge \Sigma^2 P^4)) = \pi_8(\Sigma(\Sigma^2 P^3 \wedge \Sigma^2 P^3)) \cong \pi_8(\Sigma(M^4 \wedge M^4)) = \mathbf{Z}_4\{\Sigma^5\tilde{i}'_2\},$$

we have an isomorphism $\Sigma(i' \wedge i')_* : \pi_8(\Sigma(M^4 \wedge M^4)) \to \pi_8(\Sigma(\Sigma^2 P^4 \wedge \Sigma^2 P^4))$. Lemmas 1.1 implies that $\Sigma^3\gamma_4 = (\Sigma\beta)\eta_6 \in \pi_7(\Sigma^3 P^4)$ so that $\Sigma^2\gamma_4 \equiv \beta\eta_5 \bmod \Delta(\Sigma(i' \wedge i')\Sigma^5\tilde{i}'_2) = 2i'\delta$. Since $H(\beta\eta_5) = i\eta_5$ and $2H(\delta) = H(i_3\nu') = i\eta_5$, we have the relation. The structure of $\pi_6(\Sigma^3 P^5)$ follows, completing the proof. □

Let $i'_n : M^{2n-1} \longrightarrow M^n \wedge M^n$ be the inclusion and set $i''_n = \Sigma i'_n$. By inspecting the proof of Theorem 2.4 of Chapter 12 of [14], by [1] and by Remark in §1 of [10], we have the following.

Lemma 1.4. $\Delta(i''_{n+1}) = [\iota'_{n+1}, \iota'_{n+1}]i''_n = [\iota'_{n+1}, i_n] \in [M^{2n}, M^{n+1}]$ *for* $n \geq 3$.

Next we show

Lemma 1.5.

i) $[M^5, M^4] = \mathbf{Z}_2\{i_3\bar{\eta}_3\} \oplus \mathbf{Z}_2\{\tilde{\eta}_3 p_5\}$
and $\Sigma : [M^5, M^4] \to [M^6, M^5]$ *is an isomorphism.*

ii) $[M^4, M^3] = \mathbf{Z}_2\{\bar{\tau}_1\} \oplus \mathbf{Z}_2\{\alpha p_4\}$ *and* $\Sigma\bar{\tau}_1 = \Sigma(\alpha p_4) = \tilde{\eta}_3 p_5$.

iii) $[M^7, M^5] = \mathbf{Z}_2\{i_4\eta_4\bar{\eta}_5\} \oplus \mathbf{Z}_2\{\tilde{\eta}_4\eta_6 p_7\} \oplus \mathbf{Z}_2\{i_4\nu_4 p_7\}$;
$[M^6, M^4] = \mathbf{Z}_2\{\delta p_6\} \oplus \mathbf{Z}_2\{i_3\eta_3\bar{\eta}_4\} \oplus \mathbf{Z}_2\{\tilde{\eta}_3\eta_5 p_6\}$;
$[M^8, M^5] = \mathbf{Z}_4\{\tilde{\eta}_4\bar{\eta}_6\} \oplus \mathbf{Z}_2\{[\iota'_5, i_4]\} \oplus \mathbf{Z}_2\{i_4\nu_4\eta_7 p_8\}$
and $[M^7, M^4] = \mathbf{Z}_4\{\tilde{\eta}_3\bar{\eta}_5\} \oplus \mathbf{Z}_2\{\delta\eta_6 p_7\} \oplus \mathbf{Z}_2\{\overline{i_3\nu'}\}$, *where* $\overline{i_3\nu'}$ *is an extension of* $i_3\nu' \in \pi_6(M^4)$.

iv) $\Delta(i''_4) = [\iota'_4, i_3] = \delta p_6 \in [M^6, M^4]$; $H[\iota'_5, i_4] = (\Sigma^5\tilde{i}'_2)p_8 \in [M^8, \Sigma(M^4 \wedge M^4)]$
and $\Sigma\overline{i_3\nu'} = i_4\nu_4\eta_7 p_8 \in [M^8, M^5]$.

Proof. By making use of the cofibration $i_4 : S^4 \hookrightarrow M^5$, an EHP sequence and the fact $[\iota_3, \iota_3] = 0$, we have i). By making use of the cofibration starting with $2\iota_3$ and by Lemma 4.1 of [10], we have ii).

By making use of cofibrations starting with $2\iota_n$ for $n = 5$, 6 and 7 and by Lemma 2.2 of [7], we have iii). We note that $2\overline{i_3\nu'} = i_3\nu'\eta_6 p_7 = 2\delta\eta_6 p_7 = 0$.

In the EHP sequence

$$[M^7, M^4] \xrightarrow{\Sigma} [M^8, M^5] \xrightarrow{H} [M^8, \Sigma(M^4 \wedge M^4)] \xrightarrow{\Delta} [M^6, M^4] \xrightarrow{\Sigma} [M^7, M^5],$$

we have ([10]) $[M^8, \Sigma(M^4 \wedge M^4)] = \mathbf{Z}_2\{i''_4\} \oplus \mathbf{Z}_2\{(\Sigma^5\tilde{i}'_2)p_8\}$. Since $\Sigma(\delta p_6) = 2i_4\nu_4 p_7 = 0, \Delta((\Sigma^5\tilde{i}'_2)p_8) = 2\delta p_6 = 0$ and $H(i_4\nu_4\eta_7 p_8) = i\eta_7 p_8 = 2(\Sigma^5\tilde{i}'_2)p_8 = 0$, we have iv) by iii) and Lemma 1.4. This completes the proof.

$\square$

2. Some homotopy groups of $\Sigma^2\mathrm{P}^6$

By making use of the cofibration $i_5 : S^5 \hookrightarrow M^6$ and by Lemmas 1.2 and 1.3, we have the following.

Lemma 2.1. *We have the following isomorphisms;*

$$[M^6, \Sigma^2\mathrm{P}^4] = \mathbf{Z}_2\{i\eta_3\bar{\eta}_4\} \oplus \mathbf{Z}_2\{i'\delta p_6\} \oplus \mathbf{Z}_2\{\beta\eta_5 p_6\};$$

$$[M^6, \Sigma^2\mathrm{P}^6] = \mathbf{Z}_2\{i\eta_3\bar{\eta}_4\} \oplus \mathbf{Z}_2\{i'\delta p_6\}.$$

We show

Lemma 2.2.

i) $\pi_7(\Sigma^2\mathrm{P}^4) = \mathbf{Z}_2\{\beta\eta_5^2\} \oplus \mathbf{Z}_2\{[i,\beta]\}$ *and* $[i,\beta]$ *has an extension* $[i',\beta] \in [M^8, \Sigma^2\mathrm{P}^4]$, *where* $i' : M^4 \hookrightarrow \Sigma^2\mathrm{P}^4$ *is the inclusion.*

ii) $\pi_7(\Sigma^2\mathrm{P}^6) = \mathbf{Z}_2\{i'[i,\beta]\}$.

iii) $[M^7, \Sigma^2\mathrm{P}^6] = \mathbf{Z}_2\{i'\overline{\imath_3\nu'}\} \oplus \mathbf{Z}_2\{i'[i,\beta]p_7\}$.

Proof. Since $\Sigma^2\mathrm{P}^3 = M^4 \vee S^5$, we have a split exact sequence

$$0 \longrightarrow \pi_8(M^4 \times S^5, \Sigma^2\mathrm{P}^3) \xrightarrow{\partial} \pi_7(\Sigma^2\mathrm{P}^3) \xrightarrow{i_*} \pi_7(M^4 \times S^5) \longrightarrow 0.$$

We know $M^4 \times S^5 = (M^4 \vee S^5) \bigcup_{[k_1,k_2]} CM^8$, $\pi_8(M^4 \times S^5, \Sigma^2\mathrm{P}^3) = \mathbf{Z}_2\{\hat{\imath}_7\}$ and $\partial\hat{\imath}_7 = [k_1, k_2] \circ i_7$.

We consider the exact sequence

$$\pi_8(\Sigma^2\mathrm{P}^4, \Sigma^2\mathrm{P}^3) \xrightarrow{\partial} \pi_7(\Sigma^2\mathrm{P}^3) \xrightarrow{i_*} \pi_7(\Sigma^2\mathrm{P}^4) \xrightarrow{j_*} \pi_7(\Sigma^2\mathrm{P}^4, \Sigma^2\mathrm{P}^3) \xrightarrow{\partial} \pi_6(\Sigma^2\mathrm{P}^3).$$

By the homotopy excision theorem, we have

$$\pi_8(\Sigma^2\mathrm{P}^4, \Sigma^2\mathrm{P}^3) = \mathbf{Z}_2\{\widehat{\eta_5^2}\} \oplus \mathbf{Z}_2\{[\omega, i]\} \text{ for } i = k_1 i_3$$

and $\pi_7(\Sigma^2\mathrm{P}^4, \Sigma^2\mathrm{P}^3) = \mathbf{Z}_2\{\hat{\eta}_5\}$. By Lemma 1.5.iv), we have

$$\partial\hat{\eta}_5 = \Sigma^2\gamma_3 \circ \eta_5 = k_1\tilde{\eta}_3\eta_5,\ \partial\widehat{\eta_5^2} = \Sigma^2\gamma_3 \circ \eta_5^2 = k_1\tilde{\eta}_3\eta_5^2$$

and

$$\begin{aligned} \partial[\omega, i] = [\Sigma^2\gamma_3, i] &= [2k_2, i] + [k_1\tilde{\eta}_3, i] \\ &= k_1[\tilde{\eta}_3, i_3] \\ &= k_1[\iota'_4, i_3]\tilde{\eta}_5 \\ &= k_1\delta\eta_6. \end{aligned}$$

So i_* is an epimorphism and $i[k_1, k_2]i_7 = [i', \beta]i_7 = [i, \beta]$. This leads us to i).

In the homotopy exact sequence

$$\pi_8(\Sigma^2\mathrm{P}^6, \Sigma^2\mathrm{P}^4) \xrightarrow{\partial} \pi_7(\Sigma^2\mathrm{P}^4) \xrightarrow{i_*} \pi_7(\Sigma^2\mathrm{P}^6) \xrightarrow{j_*} \pi_7(\Sigma^2\mathrm{P}^6, \Sigma^2\mathrm{P}^4) \xrightarrow{\partial} \pi_6(\Sigma^2\mathrm{P}^4),$$

we have

$$\pi_8(\Sigma^2\mathrm{P}^6, \Sigma^2\mathrm{P}^4) = \mathbf{Z}_2\{\hat{\imath}_6\hat{\eta}_6\} \text{ and } \pi_7(\Sigma^2\mathrm{P}^6, \Sigma^2\mathrm{P}^4) = \mathbf{Z}_2\{\hat{\imath}_6\}.$$

By Lemma 1.3, $\partial\hat{\imath}_6 = \Sigma^2\gamma_4$ and $\partial(\hat{\imath}_6\hat{\eta}_6) = \Sigma^2\gamma_4 \circ \eta_6 = \beta\eta_5^2$. So we have ii).

By ii) and Lemma 1.3, $[M^7, \Sigma^2\mathrm{P}^6]$ is generated by $i'\overline{i_3\nu'}$ and $i''[i, \beta]p_r$. This leads us to iii) and completes the proof.

□

Remark. The result ii) of Lemma 2.2 was pointed out by Milgram ([6]).

Next we shall determine $\pi_8(\Sigma^2\mathrm{P}^n)$. Since $\eta_3\nu_4 = \nu'\eta_6$, $\tilde{\eta}_3\nu_5 \in \pi_8(M^4)$ is a coextension of $\nu'\eta_6$.

Lemma 2.3. *$\tilde{\eta}_3\nu_5$ is of order 2 and $\tilde{\eta}_4\nu_6 = i_4\nu_4\eta_7^2$.*

Proof. We have $2\tilde{\eta}_3\nu_5 = i_3\eta_3^2\nu_5 = i_3(\eta_3\Sigma\nu')\eta_7 = 0$. So we have the first half. From the definition of the Toda bracket, we have

$$\begin{aligned} \overline{i_3\nu'}\tilde{\eta}_6 &\in \{i_3\nu', 2\iota_6, \eta_6\} \\ &\subset \{i_3, 2\nu', \eta_6\} \\ &\supset \{i_3, 2\iota_3, \nu'\eta_6\} \\ &\ni \tilde{\eta}_3\nu_5. \end{aligned}$$

So we have $\tilde{\eta}_3\nu_5 \equiv \overline{i_3\nu'}\tilde{\eta}_6 \mod i_{3*}\pi_8(S^3) + \pi_7(M^4)\circ\eta_7 = \{\delta\eta_6^2\}$. We have $\Sigma(\delta\eta_6^2) = 2i_4\nu_4\eta_7^2 = 0$. So, by Lemma 1.5.iv), we have $\tilde{\eta}_4\nu_6 = \Sigma(\overline{i_3\nu'}\tilde{\eta}_6) = i_4\nu_4\eta_7^2$. This completes the proof.

$\square$

Lemma 2.4. $\pi_8(M^4) = \mathbf{Z}_2\{\tilde{\eta}_3\nu_5\} \oplus \mathbf{Z}_2\{\delta\eta_6^2\} \oplus \mathbf{Z}_2\{[\delta, i_3]\}$.

Proof, By the exact sequence

$$\pi_7(V_{5,2}, M^4) \xrightarrow{\partial} \pi_6(M^4) \xrightarrow{i_*} \pi_6(V_{5,2}) \longrightarrow 0,$$

we have $\pi_6(V_{5,2}) = \mathbf{Z}_2\{i\tilde{\eta}_3\eta_5\}$.

We consider the homotopy exact sequence for the S^3-bundle $V_{5,2}$ over S^4:

$$\pi_{10}(S^4) \xrightarrow{\Delta} \pi_9(S^3) \xrightarrow{i'_*} \pi_9(V_{5,2}) \xrightarrow{p'_*} \pi_9(S^4) \xrightarrow{\Delta} \pi_8(S^3) \xrightarrow{i'_*} \pi_8(V_{5,2}) \xrightarrow{p'_*} \pi_8(S^4)$$
$$\xrightarrow{\Delta} \pi_7(S^3) \xrightarrow{i'_*} \pi_7(V_{5,2}) \xrightarrow{p'_*} \pi_7(S^4) \xrightarrow{\Delta} \pi_6(S^3) \xrightarrow{i'_*} \pi_6(V_{5,2}) \xrightarrow{p'_*} \pi_6(S^4).$$

Since the last p'_* is an isomorphism and $\Delta(\Sigma\nu') = 2\nu'$, we have $\Delta(\nu_4) = \pm\nu'$. Hence, by chasing the diagram, we have the following result which overlaps with the results of [11]:

$$\pi_7(V_{5,2}) \cong \mathbf{Z} \oplus \mathbf{Z}_2,\ \pi_8(V_{5,2}) = \mathbf{Z}_2\{i\tilde{\eta}_3\nu_5\} \text{ and } \pi_9(V_{5,2}) = \mathbf{Z}_2\{i\tilde{\eta}_3\nu_5\eta_8\}.$$

We consider the exact sequence

$$\pi_9(M^4) \xrightarrow{i_*} \pi_9(V_{5,2}) \xrightarrow{j_*} \pi_9(V_{5,2}, M^4) \xrightarrow{\partial} \pi_8(M^4)$$
$$\xrightarrow{i_*} \pi_8(V_{5,2}) \xrightarrow{j_*} \pi_8(V_{5,2}, M^4) \xrightarrow{\partial} \pi_7(M^4).$$

There exists an element $\tilde{\eta}_3\nu_5\eta_8 \in \pi_9(M^4)$ of order 2, because $p_{4*}\tilde{\eta}_3\nu_5\eta_8 = \eta_4\nu_5\eta_8 = (\Sigma\nu')\eta_7^2 \neq 0$. So, by the above result and by Lemma 2.3, both i_* are split epimorphisms, respectively. Therefore we have a split exact sequence

$$0 \longrightarrow \pi_9(V_{5,2}, M^4) \xrightarrow{\partial} \pi_8(M^4) \xrightarrow{i_*} \pi_8(V_{5,2}) \longrightarrow 0.$$

By the homotopy excision theorem, we have $\pi_9(V_{5,2}, M^4) = \mathbf{Z}_2\{\widehat{\eta_6^2}\} \oplus \mathbf{Z}_2\{[\omega, i_3]\}$, where ω is the charactesitic map of the 7-cell of $V_{5,2}$. We have $\partial\widehat{\eta_6^2} = \delta\eta_6^2$ and $\partial[\omega, i_3] = [\delta, i_3]$. This completes the proof.

$\square$

Finally we show the main result in this section.

Lemma 2.5.

i) $\pi_8(\Sigma^2\mathrm{P}^4) = \mathbf{Z}_4\{\beta\nu_5\} \oplus \mathbf{Z}_2\{i'[\delta, i_3]\} \oplus \mathbf{Z}_2\{[i,\beta]\eta_7\}$ *and* $2\beta\nu_5 = i'\tilde{\eta}_3\nu_5$.

ii) $[M^8, \Sigma^2\mathrm{P}^4] = \mathbf{Z}_4\{[i',\beta]\} \oplus \mathbf{Z}_2\{\beta\eta_5\bar{\eta}_6\} \oplus \mathbf{Z}_2\{i'[\delta, i_3]p_8\} \oplus \mathbf{Z}_2\{\beta\nu_5 p_8\}$ *and* $2[i',\beta] = [i,\beta]\eta_7 p_8$.

iii) $\pi_8(\Sigma^2\mathrm{P}^5) = \mathbf{Z}_4\{i''\beta\nu_5\} \oplus \mathbf{Z}_2\{i'[\delta, i_3]\}$, $i''[i,\beta]\eta_7 = 0$ *and* $\pi_8(\Sigma^2\mathrm{P}^6) \cong \pi_8(\Sigma^2\mathrm{P}^5)$.

iv) $[M^8, \Sigma^2\mathrm{P}^6] = \mathbf{Z}_2\{i''[i',\beta]\} \oplus \mathbf{Z}_2\{i'[\delta, i_3]p_8\} \oplus \mathbf{Z}_2\{i''\beta\nu_5 p_8\}$.

Proof. In the exact sequence

$$\pi_9(\Sigma^2\mathrm{P}^4, \Sigma^2\mathrm{P}^3) \xrightarrow{\partial} \pi_8(\Sigma^2\mathrm{P}^3) \xrightarrow{i_*} \pi_8(\Sigma^2\mathrm{P}^4) \xrightarrow{j_*} \cdots,$$

we have $\pi_8(\Sigma^2\mathrm{P}^3) \cong \pi_8(M^4) \oplus \pi_8(S^5) \oplus \pi_9(M^4 \times S^5, M^4 \vee S^5) \cong (\mathbf{Z}_2)^3 \oplus \mathbf{Z}_{24} \oplus \mathbf{Z}_2$ and i_* is an epimorphism by the proof of Lemma 2.2.i). We have

$$\pi_9(\Sigma^2\mathrm{P}^4, \Sigma^2\mathrm{P}^3) = \mathbf{Z}_{24}\{\hat{\nu}_5\} \oplus \mathbf{Z}_2\{[\omega, i]\hat{\eta}_7\}.$$

Since $\partial\hat{\nu}_5 = \Sigma^2\gamma_3 \circ \nu_5 = k_1\tilde{\eta}_3\nu_5 + 2k_2\nu_5$ and $\partial([\omega, i]\hat{\eta}_7) = k_1\delta\eta_6^2$, we have i).

In the exact sequence

$$0 \longleftarrow \pi_7(\Sigma^2\mathrm{P}^4) \xleftarrow{i_7{}^*} [M^8, \Sigma^2\mathrm{P}^4] \xleftarrow{p_8{}^*} \pi_8(\Sigma^2\mathrm{P}^4) \xleftarrow{2} \pi_8(\Sigma^2\mathrm{P}^4),$$

we find elements $[i',\beta]$ and $\beta\eta_5\bar{\eta}_6$ in $[M^8, \Sigma^2\mathrm{P}^4]$ by Lemma 2.2.i). We have $2[i',\beta] = [i',\beta]i_7\eta_7 p_8 = [i,\beta]\eta_7 p_8$. This determine the group extension of ii).

In the exact sequence

$$\pi_9(\Sigma^2\mathrm{P}^5, \Sigma^2\mathrm{P}^4) \xrightarrow{\partial} \pi_8(\Sigma^2\mathrm{P}^4) \xrightarrow{i_*} \pi_8(\Sigma^2\mathrm{P}^5) \xrightarrow{j_*} \pi_8(\Sigma^2\mathrm{P}^5, \Sigma^2\mathrm{P}^4) \xrightarrow{\partial} \pi_7(\Sigma^2\mathrm{P}^4),$$

we have $\pi_8(\Sigma^2\mathrm{P}^5, \Sigma^2\mathrm{P}^4) = \mathbf{Z}_2\{\hat{\eta}_6\}$ and $\partial\hat{\eta}_6 = \Sigma^2\gamma_4 \circ \eta_6 = \beta\eta_5^2$ by Lemma 1.3. So i_* is an epimorphism. We have

$$\pi_9(\Sigma^2\mathrm{P}^5, \Sigma^2\mathrm{P}^4) = \mathbf{Z}_2\{\widehat{\eta_6^2}\} \oplus \mathbf{Z}_2\{[\omega, i]\}.$$

Since $\partial\widehat{\eta_6^2} = \beta\eta_5^3 = 4\beta\nu_5 = 2i'\tilde{\eta}_3\nu_5 = 0$ and $\partial[\omega, i] = [\beta\eta_5, i] + [2i'\delta, i] = [i, \beta]\eta_7$, we have the result of iii) apart from the last assertion.

In the exact sequence

$$\pi_9(\Sigma^2\mathrm{P}^6, \Sigma^2\mathrm{P}^5) \xrightarrow{\partial} \pi_8(\Sigma^2\mathrm{P}^5) \xrightarrow{i_*} \pi_8(\Sigma^2\mathrm{P}^6) \longrightarrow 0,$$

we have $\pi_9(\Sigma^2\mathrm{P}^6, \Sigma^2\mathrm{P}^5) = \mathbf{Z}_2\{\hat{\eta}_7\}$.

Since $\Sigma^2\gamma_5 \in \{i'', \Sigma^2\gamma_4, 2\iota_6\}$, we have, by Lemma 1.3,

$$\begin{aligned} \partial\hat{\eta}_7 = \Sigma^2\gamma_5 \circ \eta_7 &\in \{i'', \Sigma^2\gamma_4, 2\iota_6\} \circ \eta_7 \\ &= -i''\{\Sigma^2\gamma_4, 2\iota_6, \eta_6\} \\ &\subset -i''\{i\nu', 2\iota_6, \eta_6\} - i''\{\beta\eta_5, 2\iota_6, \eta_6\}. \end{aligned}$$

By Lemma 2.2, we have $\{i\nu', 2\iota_6, \eta_6\} \subset \{i, 2\nu', \eta_6\} \supset i'\{i_3, 2\iota_3, \nu'\eta_6\}$. So we have $i'\tilde{\eta}_3\nu_5 \in \{i\nu', 2\iota_6, \eta_6\}$ mod $i_*\pi_8(S^3) + \pi_7(\Sigma^2\mathrm{P}^4) \circ \eta_7 = \{[i, \beta]\eta_7\}$ and $\{\beta\eta_5, 2\iota_6, \eta_6\} \ni 2\beta\nu_5$ mod $\pi_7(\Sigma^2\mathrm{P}^4) \circ \eta_7 = \{[i, \beta]\eta_7\}$. Therefore we have $\partial\hat{\eta}_7 \equiv i''(i'\tilde{\eta}_3\nu_5 + 2\beta\nu_5) = 0$ mod $i''_*[i, \beta]\eta_7 = 0$. This leads us to iii).

In the exact sequence for $X = \Sigma^2\mathrm{P}^6$:

$$\pi_7(X) \xleftarrow{2} \pi_7(X) \xleftarrow{i^*} [M^8, X] \xleftarrow{p^*} \pi_8(X) \xleftarrow{2} \pi_8(X),$$

we have $i''[i', \beta] \circ i = i''[i, \beta]$ by the proof of Lemma 2.2.i). We also have $2i''[i', \beta] = i''[i, \beta]\eta_7 p_8 = 0$ by ii) and iii). This completes the proof.

□

3. The suspension order of $\Sigma^2\mathrm{P}^6$

Let $\varphi \in [M^5, M^4]$ be the attaching map in $\mathrm{P}_3^6 = M^4 \bigcup_\varphi CM^5$. We consider the cofiber sequence

$$M^4 \xrightarrow{i} \mathrm{P}_3^6 \xrightarrow{p'} M^6 \xrightarrow{\Sigma\varphi} M^5. \tag{3.1}$$

The squaring operation

$$\mathrm{Sq}^2 : \tilde{H}^i(\mathrm{P}_3^6; \mathbf{Z}_2) \to \tilde{H}^{i+2}(\mathrm{P}_3^6; \mathbf{Z}_2)$$

is nontrivial for $i = 3$ and trivial for $i = 4$. So, by Lemma 1.5.i), we can take $\varphi = i_3\bar{\eta}_3$.

We set $p'' = p_6 p' : \mathrm{P}_3^6 \longrightarrow S^6$. Since $[\mathrm{P}_3^6, M^6]$ is stable, we have $2p' = 2\iota_6 \circ p'$. So we have

$$2p' = i_5\eta_5 p'' \in [\mathrm{P}_3^6, M^6]. \tag{3.2}$$

Let $\bar{p} \in \{p, i\bar{\eta}_4, p'\} \subset [\Sigma \mathrm{P}_3^6, S^5]$ be an extension of $p = p_5 : M^5 \longrightarrow S^5$. Because $\iota_5 \in \{2\iota_5, p, i\}$, a bracket with indeterminancy $\{2\iota_5\}$, we have

$$2\bar{p} = \pm\bar{\eta}_5\Sigma p'. \tag{3.3}$$

We show

Lemma 3.1.

i) $[M^6, S^3] = \mathbf{Z}_2\{\eta_3\bar{\eta}_4\} \oplus \mathbf{Z}_2\{\nu'\Sigma p_6\}$;
$[M^7, S^3] = \mathbf{Z}_2\{\eta_3^2\bar{\eta}_5\} \oplus \mathbf{Z}_2\{\nu'\eta_6 p_7\}$;
$[M^8, S^3] = \mathbf{Z}_4\{\nu'\bar{\eta}_6\}$;
$[M^7, S^4] = \mathbf{Z}_2\{\eta_4\bar{\eta}_5\} \oplus \mathbf{Z}_2\{\nu_4 p_7\} \oplus \mathbf{Z}_2\{(\Sigma\nu')p_7\}$;
$[M^8, S^4] = \mathbf{Z}_2\{\eta_4^2\bar{\eta}_6\} \oplus \mathbf{Z}_2\{\nu_4\eta_7 p_8\} \oplus \mathbf{Z}_2\{(\Sigma\nu')\eta_7 p_8\}$.

ii) $[\Sigma^2\mathrm{P}_3^6, S^4] = \mathbf{Z}_2\{\eta_4^2\Sigma\bar{p}\} \oplus \mathbf{Z}_2\{\nu_4\eta_7\Sigma^2 p''\} \oplus \mathbf{Z}_2\{(\Sigma\nu')\eta_7\Sigma^2 p''\}$.

iii) $[\Sigma^2\mathrm{P}_3^6, S^3] = \mathbf{Z}_4\{\nu'\Sigma\bar{p}\}$.

Proof. i) is easily obtained. By use of the cofiber sequence (3.1), we have an exact sequence for $\gamma = \Sigma^2\varphi = i_5\bar{\eta}_5$:

$$[M^7, S^4] \xleftarrow{\gamma*} [M^6, S^4] \xleftarrow{i*} [\Sigma^2\mathrm{P}_3^6, S^4] \xleftarrow{\Sigma^2 p' *} [M^8, S^4] \xleftarrow{\Sigma\gamma *} [M^7, S^4].$$

By i), we have

$$\bar{\eta}_4\gamma = \eta_4\bar{\eta}_5,\ \eta_4\bar{\eta}_5\Sigma\gamma = \eta_4^2\bar{\eta}_6 \text{ and } \nu_4 p_7\Sigma\gamma = (\Sigma\nu')p_7\Sigma\gamma = 0.$$

This leads us to ii). By a parallel argument to ii) and by (3.3), we have iii). This completes the proof.

□

We set $\theta = i\eta_3\bar{\eta}_4 \in [M^6, \Sigma^2 P^4]$. By Lemma 1.2, we have

$$\theta \circ i\bar{\eta}_5 = i\eta_3^2\bar{\eta}_5 = 2i'\tilde{\eta}_3\bar{\eta}_5 = 4\beta\bar{\eta}_5 = 0 \in [M^7, \Sigma^2\mathrm{P}^4].$$

So we can define an extension $\bar{\theta} \in [\Sigma^2\mathrm{P}^6_3, \Sigma^2\mathrm{P}^4]$ of θ. We have

$$\bar{\theta} \in \{\theta, i_5\bar{\eta}_5, \Sigma p'\} \subset [\Sigma^2\mathrm{P}^6_3, \Sigma^2\mathrm{P}^4].$$

We show

Lemma 3.2.

i) $[\Sigma^2\mathrm{P}^6_3, \Sigma^2\mathrm{P}^6]$ *is generated by* $i'''\bar{\theta}, i'\delta\Sigma\bar{p}, i''\beta\nu_5\Sigma^2 p''$, $i'[\delta, i_3]\Sigma^2 p''$ *and* $i''[i', \beta]\Sigma^2 p'$.

ii) $2\bar{\theta} \equiv 0 \bmod \{[i, \beta]\eta_7\Sigma^2 p'', 2i'\delta\Sigma\bar{p}\}$, $2i'''\bar{\theta} \equiv 0 \bmod 2i'\delta\Sigma\bar{p}$ *and* $\Sigma\bar{\theta}$ *is of order* 2.

Proof. By (3.1), we have an exact sequence

$$[M^7, \Sigma^2\mathrm{P}^6] \xleftarrow{\gamma^*} [M^6, \Sigma^2\mathrm{P}^6] \xleftarrow{i^*} [\Sigma^2\mathrm{P}^6_3, \Sigma^2\mathrm{P}^6] \xleftarrow{\Sigma^2 p'^*} [M^8, \Sigma^2\mathrm{P}^6] \xleftarrow{\Sigma\gamma^*} [M^7, \Sigma^2\mathrm{P}^6].$$

By Lemma 2.1, there exist elements $\bar{\theta}$ with $\theta = i\eta_3\bar{\eta}_4$ and $i'\delta\Sigma\bar{p}$ in $[\Sigma^2\mathrm{P}^6_3, \Sigma^2\mathrm{P}^4]$. So we settle the generators of $[\Sigma^2\mathrm{P}^6_3, \Sigma^2\mathrm{P}^6]$ by Lemma 2.5.iii). This leads us to i).

From the definition of $\bar{\theta}$ and by (3.2), we have

$$\begin{aligned} 2\bar{\theta} &\in \{i\eta_3\bar{\eta}_4, i_4\bar{\eta}_5, i_6\eta_6\Sigma p''\} \\ &\subset \{i\eta_3, \eta_4^2, \eta_6\Sigma p''\} \\ &\supset i'\{i_3\eta_3, \eta_4^2, \eta_6\Sigma p''\}. \end{aligned}$$

So we have

$$2\bar{\theta} \in i'\{i_3\eta_3, \eta_4^2, \eta_6\Sigma p''\} \bmod i\eta_3[\Sigma^2\mathrm{P}^6_3, S^4] + \pi_7(\Sigma^2\mathrm{P}^4) \circ \eta_7\Sigma^2 p''.$$

We also have

$$\{i_3\eta_3, \eta_4^2, \eta_6\Sigma p''\} \subset \{i_3, 2\nu', \eta_6\Sigma p''\} \supset \{i_3, 2\iota_3, \nu'\eta_6\Sigma p''\}.$$

So we have

$$\tilde{\eta}_3\nu_5\Sigma^2 p'' \in \{i_3\eta_3, \eta_4^2, \eta_6\Sigma p''\} \bmod i_{3*}[\Sigma^2 \mathrm{P}_3^6, S^3] + \pi_7(M^4)\circ\eta_7\Sigma^2 p''.$$

In $[\Sigma^2\mathrm{P}_3^6, \Sigma^2\mathrm{P}^4]$, we have $i'\tilde{\eta}_3\nu_5\Sigma^2 p'' = 2\beta\nu_5\Sigma^2 p'' = 0$. We have $\pi_7(M^4)\circ\eta_7\Sigma^2 p'' = \{\delta\eta_6^2\Sigma^2 p''\} = 0$, because $\delta\eta_6^2\Sigma^2 p'' = 2\delta\bar{\eta}_6\Sigma^2 p' = 4\delta\Sigma\bar{p} = 0$ by (3.2) and (3.3). By Lemma 3.1, we have $i_3[\Sigma^2\mathrm{P}_3^6, S^3] = \{i_3\nu'\Sigma\bar{p}\} = \{2\delta\Sigma\bar{p}\}$ and $i\eta_3[\Sigma^2\mathrm{P}_3^6, S^4] = 0$. By Lemmas 2.2.i) and 2.5.ii), we have $\pi_7(\Sigma^2\mathrm{P}^4)\circ\eta_7\Sigma^2 p'' = \{[i,\beta]\eta_7\Sigma^2 p''\}$. This completes the proof.

□

Remark. By use of the exact sequence induced from (3.1) and by Lemma 2.5.ii), we have

$$2\bar{\theta} \in \{[i',\beta], i'[\delta, i]p_8, \beta\eta_5\bar{\eta}_6 p_8, \beta\nu_5 p_8\}\circ\Sigma^2 p'.$$

By (3.3), we have $\beta\eta_5\bar{\eta}_6\Sigma^2 p'' = 0$. By Theorem 3.4 of [7], $\beta\nu_5\Sigma^2 p''$ survives in the stable range. So we have $2\Sigma\bar{\theta} = 0$.

Finally we prove Theorem 1. We consider the cofiber sequence

$$\mathrm{P}^2 \xrightarrow{i} \mathrm{P}^6 \xrightarrow{q} \mathrm{P}_3^6 \xrightarrow{\psi} M^3 \longrightarrow \cdots \tag{3.4}$$

and the exact sequence induced from (3.4):

$$[M^4, \Sigma^2\mathrm{P}^6] \xleftarrow{i^*} [\Sigma^2\mathrm{P}^6, \Sigma^2\mathrm{P}^6] \xleftarrow{\Sigma^2 q^*} [\Sigma^2\mathrm{P}_3^6, \Sigma^2\mathrm{P}^6] \xleftarrow{\Sigma^2\psi^*} [M^5, \Sigma^2\mathrm{P}^6].$$

By Lemmas 2.5 and 3.2, we have the relations for $X = \mathrm{P}^6$:

$$\begin{aligned} 4\Sigma^2\iota_X &\in \mathrm{Im}\ \Sigma^2 q^* \\ &= \{i'''\bar{\theta}, i'\delta\Sigma\bar{p}, i''\beta\nu_5\Sigma^2 p'', i'[\delta, i_3]\Sigma^2 p'', i''[i',\beta]\Sigma^2 p'\}\Sigma^2 q \end{aligned}$$

and

$$8\Sigma^2\iota_X \equiv 0 \bmod 2i'\delta\Sigma\bar{p}\Sigma^2 q.$$

So we have $8\Sigma^3\iota_X \equiv 0 \bmod 4i\nu_4\Sigma^2\bar{p}\Sigma^3 q = 0$. Hence we have

$$8\Sigma^3\iota_X = 0.$$

This completes the proof of Theorem 1.

The proof of Thorem 2 is immediately obtained from the method of the appendix of [8].

Conjecture. $i'''\bar{\theta}$ is of order 2 and the suspension order of ΣP^6 is 8.

Remark.

i) The author announced the result $\pi_{12}(M^5) \cong (\mathbf{Z}_2)^2$ ([10]). This result should be corrected as follows: $\pi_{12}(M^5) \cong (\mathbf{Z}_2)^3$([4]).

ii) The author obtained the result $\{2\iota_5, \eta_5, \nu_6\} = 0$ and $\{2\iota_5, \eta_5, \zeta_6\} = 0$ by Lemma 4.3 of [9]. By Lemma 2.3, we have $\tilde{\eta}_5\nu_7 = i\nu_5\eta_7^2$. So it should be corrected as follows: $\{2\iota_5, \eta_5, \nu_6\} \equiv \nu_5\eta_7^2 \bmod 0$ and $\{2\iota_5, \eta_5, \zeta_6\} \equiv \nu_5\eta_8\mu_9 \bmod 0$.

The author's mistake was induced from using the following wrong formula about the Toda bracket: $\{i, 2\iota_5, 0\}_2 \ni 0$. This formula should be corrected as follows:

$$\{i_5, 2\iota_5, 0\}_2 \subset \{i_5, 2\iota_5, 0\} \ni 0 \bmod i_5\nu_5\eta_7^2.$$

Acknowledgement. The author wishes to thank Professor Rigas for the kind hospitality given during the stay at Unicamp. In fact the above serious mistake was found during this stay.

The author also wishes to thank the referee for the kind and useful advices helping improve the manuscript.

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Received October 10, 1996