FIXED POINTS ON SURFACE FIBER BUNDLES

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Abstract

This work studies fixed points of fiber preserving maps over a space B, where the fiber is a compact surface different from the 2-sphere S^2 and from the projective plane RP^2. We define and study an invariant called abelianized obstruction. Using Fox's calculus we provide a method to decide whether such obstruction vanishes or not. For the case in which the fibration is a T-principal fiber bundle, where T is the torus, we compare this invariant with other invariants, which are very much related with our geometric problem. We also describe the minimal number of connected components of $Fix_B(f)$,the fixed point set of f over B, in terms of some of the invariants mentioned above.

Resumo

Neste artigo estudamos pontos fixos de aplicações fibradas que cobrem a identidade da base, onde os espaços em questão são variedades compactas sem bordo e cujas fibras são superfícies diferentes da esfera S^2 e do plano projetivo $RP(2)$.Na tentativa de deformar uma função dada a uma função livre de pontos fixos, por homotopias que preservam a fibra, deparamos com algumas obstruções.Uma delas, a obstrução abelianizada, é efetivamente descrita usando o cálculo diferencial livre de Fox. No caso do fibração ser um toro fibrado principal, nós consideramos outros invariantes algébricos bastante relacionados com o problema geométrico e ainda dar uma estimativa para o número mínimo de componentes conexas do conjunto de pontos fixos de qualquer função fibramente homotópica a função dada f.

0. Introduction

Let $F \to M \to B$ be a smooth fiber bundle, where F, M and B are smooth connected compact manifolds without boundary. In [F-H-80] Fadell and Husseini consider the problem of defining invariants which detect fixed points of

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fiber preserving maps $f : M \to M$ over B. They make use of obstruction theory restricted to the case where the fiber F has dimension greater than or equal to 3. Our purpose is to look at the situation where the fiber is a surface of genus greater than or equal to 1. The fact that now F is a $K(\pi, 1)$ space provides some simplification. On the other hand, the fact that the dimension of F is 2 introduces some extra difficulties if compared with [F-H-80]. We will start by showing that the problem of deforming f over B to a fixed point free map is equivalent to an algebraic problem involving the fundamental group. This is proposition 1.5. We will define a cohomology class called the abelianized obstruction to deform f over B to a fixed point free map, in the same fashion as in [F-H-80]. This is definition 1.4. In a principal bundle, it should be easier to compute such classes than to solve the algebraic problem which comes from proposition 1.5. In general, the abelianized obstruction being equal to zero is not equivalent to the map being deformed, over B, to a fixed point free map, even if B is a point. In such a situation, the examples correspond to finding a self-map f on a surface with Nielsen number $N(f) = 0$, with f not being deformable to a fixed point free map. In [Ji-85] one finds such examples. Nevertheless, by [Pe-88], if B is the circle S^1 and F is the torus then the abelianized obstruction vanishes if, and only if, the map can be deformed over B to a fixed point free map. Then we move to principal bundles with the fiber being a torus. We refer to this as T-principal fiber bundle. In this situation, using Fox's calculus we provide a method to compute whether the abelianized abstruction invariant is zero or not. We also define other relevant invariants and we exhibit some relations among them. We also estimate the minimal number of connected components of $Fix_B(f)$ in terms of some of these invariants.

This paper is divided in 5 sections. In section 1 we quickly review the basic results of [F-H-80] which we need. We also show that the geometric problem is equivalent to an algebraic one, and we define the abelianized obstruction. In section 2 we move to T-principal fiber bundles. Then we show that the abelianized obstruction for a T-principal fiber bundles is equal to the abelianized obstruction to compress a certain map $f\colon M \to T$ into $T - e$, where e is the identity of T; this is theorem 2.2. In section 3 we define new invariants. Under certain hypotheses we estimate a lower bound for the number of connected components of $Fix_B(f)$; this is theorem 3.2. In section 4 we provide a method to decide when the abelianized obstruction is different from zero; this is theorem 4.2. Finally in section 5 we give two examples. The first one has to do with the question of equivalence of the invariants. The second one shows that even if all of our invariants are zero, the map may not be deformed over B to a fixed point free map.

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1. Preliminary and General Results

We will consider in this paper the following setting. Throughout this work we consider the smooth fiber bundle $F \to M \xrightarrow{p} B$, where F, M and B are smooth connected compact manifolds without boundary.

Let $f : M \to M$ be a fiber-preserving map, i.e. $p \circ f = p$. We want know when the map f is deformable, by fiberwise homotopy, to a fixed point free map g.

Let $M \times_B M$ be the total space of the induced fiber map from p by p and let Δ be the diagonal in $M \times_B M$.We replace the inclusion $i : M \times_B M - \Delta \to M \times_B M$ by the equivalent fiber map $q : E_B(M) \to M \times_B M$, whose the fiber we denoted by $\mathcal{F}$.

With these notations, Fadel and Husseini proved the following result in [F-H-80]:

Proposition 1.1 *f is deformable to a fixed point free map g over B if, and* *only if, there exists the lift s in diagram (I) below:*

where $q_f : E_B(f) \to M$ is the induced fiber bundle from $(1, f)$ by q.

$\square$

Fadell an Husseini in [F-H-80] have related the homotopy groups of $\mathcal{F}$ with homotopy groups of F. More precisely:

Proposition 1.2. *The fiber $\mathcal{F}$ has homotopy groups*

$$\Pi_{j-1}(\mathcal{F}) \cong \Pi_j(M \times_B M, M \times_B M - \Delta) \cong \Pi_j(F, F - y)$$

where y is a point in F.

$\square$

They have also concluded that $\mathcal{F}$ is simply connected when the dimension of F is greater than 2. In this case, they have used the standard obstruction theory and defined:

Definition 1.3. Let $\{\Pi_{k-1}(\mathcal{F})\}$ be the induced local system of coefficients on M and let $q_f : E_B(f) \to M$ be the induced fiber map from $(1, f)$ by q . The primary obstruction for finding a cross section,

$$\mathcal{O}_B(f) \in H^k(M; \{\Pi_{k-1}(\mathcal{F})\}),$$

is *the primary obstruction fixed point index of f.*

This class is the obstruction to construct an extension of a cross section from the $k-1$ skeleton to the k skeleton of M. There may be other obstructions that preclude finding a global cross section.

Now, if F is a surface, different from S^2 and $RP(2)$, then the fiber $\mathcal{F}$ is not abelian. So the standard obstruction theory cannot be applied, but we can replace the group $\Pi_1(\mathcal{F})$ by its abelianized group.

Definition 1.4. Let $\{H_1(\mathcal{F})\}$ be the induced abelianized local system of coefficients on M. The primary obstruction $\mathcal{A}_B(f) \in H^2(M; \{H_1(\mathcal{F})\})$ is the abelianized obstruction to deforming f over B to a fixed point free map.

The class $\mathcal{A}_B(f)$ is the only necessary condition to find a cross section. In fact, there is only an obstruction to extending f from the 1-skeleton to the 2-skeleton. In this case, the group $\Pi_1(\mathcal{F})$ injects in $\Pi_1(E_B(f))$, so we can apply theorem 4.4.3, page 265, [Ba-77] and conclude:

Proposition 1.5. *There exists a cross section of the fiber bundle $q_f : E_B(f) \to M$ if, and only if, the homomorphism q_π, on the fundamental group, admits a right inverse homomorphism.*

$\square$

2. Principal Torus Fiber Bundle

Observe that $(x, y) \in M \times_B M$ if, and only if, x and y are in the same fiber.When we consider a principal torus fiber bundle $T \to M \xrightarrow{p} B$ the fiber T is a group, so, given $(x, y) \in M \times_B M$ there is an element $\theta(x, y) \in T$ such that $y = \theta(x, y) \cdot x$ where the symbol $'\cdot'$ denotes the operation in T.

If $f : M \to M$ is a continuos map over B, then $(x, f(x)) \in M \times_B M$. Thus $f(x) = \theta(x, f(x)) \cdot x = \theta \circ (1, f)(x)$. Setting $\theta \circ (1, f) = \theta_f$ we have $f(x) = \theta_f(x) \cdot x$. In this manner, given $f : M \to M$ over B we can construct $\theta_f : M \to T$ which satisfies $f(x) = \theta_f(x) \cdot x$.

On the other hand any map $\varphi : M \to T$ induces a map $f_\varphi : M \to M$ over B, defined by $f_\varphi(x) = \varphi(x) \cdot x$. Hence:

Proposition 2.1. *If $T \to M \xrightarrow{p} B$ is a principal torus fiber bundle then:*

(1) *There is a bijection between homotopy classes of preserving fiber maps of*

M and homotopy classes of M in T.

(2) *$Fix_B(f) = {\theta_f}^{-1}(e)$, where e is the identity of T.*

$\square$

If we replace the inclusion $j : (T-e) \to T$ by the fiber map $q_1 : E(T-e) \to T$ and let $E(\theta)$ denote the induced fiber bundle on $M \times_B M$ from θ by q_1,we have:

Theorem 2.2. *If $T \to M \xrightarrow{p} B$ is a T-principal fiber bundle then:*

(1) *The fiber map $q_f : E_B(f) \to M$ is equivalent to fiber map $E(\theta_f) \to M$,the induced fiber bundle from $\theta_f = \theta \circ (1, f)$ by $q_1 : E(T-e) \to T$.*

(2) *$\mathcal{A}_B(f) = \mathcal{A}(\theta_f)$.*

Proof. Note that $E_B(M)$ consists of pairs of paths (α, β) in M such that $p \circ \alpha = p \circ \beta$ with $\alpha(0) \neq \beta(0)$ and $E(\theta)$ consists of triples (x, y, γ), such that $(x, y) \in M \times_B M$ and γ is a path on T with $\gamma(0) \neq e$.

Now we consider the following maps

$$\Phi : E_B(M) \to E(\theta) \quad \text{and} \quad \Psi : E(\theta) \to E_B(M)$$

given by

$$\Phi(\alpha, \beta) = (\alpha(1), \beta(1), \gamma_{\alpha\beta}) \quad \text{and} \quad \Psi(x, y, \gamma) = (\bar{x}, \gamma \cdot \bar{x}),$$

where $\gamma_{\alpha\beta}(t) = \theta(\alpha(t), \beta(t))$, $\bar{x}(t) = x$ and $(\gamma \cdot \bar{x})(t) = \gamma(t) \cdot x$, for all t.

It is immediate that $\Phi \circ \Psi = I_{E(\theta)}$.

The homotopy $H_s(\alpha, \beta) = (\alpha_s, \beta_s)$ where $\alpha_s(t) = \alpha(\,(1-s)t + s\,)$ and $\beta_s(t) = \theta(\alpha(t), \beta(t)) \cdot \alpha_s(t)$, gives us that $\Psi \circ \Phi \simeq I_{E_B(M)}$ over $M \times_B M$. So $E_B(M)$ and $E(\theta)$ have the same homotopy type,that is $E_B(M) \simeq E(\theta)$.

Thus we have reduced the problem of finding the lift s on diagram (I) to finding the lift $s' : M \to E(T-e)$ for $\theta_f = \theta \circ (1, f)$. As explained before the problem is equivalent to looking for the lift on the fundamental group of the 2-skeleton of M. Note that the fiber of $E(T-e) \to T$ is the same $\mathcal{F}$.

Now, we consider the abelianized obstruction for θ_f; this is the cohomological class $\mathcal{A}(\theta_f) \in H^2(M; \{H_1\mathcal{F}\})$. So, from the equivalence of fiber bundles on (II) and from the naturality of the abelianized obstruction we conclude the proof.

□

Definition 2.3. When detecting fixed points over B, in the case of the principal torus fiber bundle, we list some important algebraic invariants :

(1) The induced map of θ_f on the second homology group:

$$(\theta_f)_2 : H_2(M;\mathbb{Z}) \to H_2(T;\mathbb{Z}).$$

(2) The index of the image of $(\theta_f)_\pi : \Pi_1(M) \to \Pi_1(T)$ which we denote by $i(f)$:

$$i(f) = [\Pi_1(T) : (\theta_f)_\pi \Pi_1(M)].$$

(3) The abelianized obstruction $\mathcal{A}(\theta_f) = \mathcal{A}_B(f)$.

3. Algebraic Invariant on T-Fiber Bundle

This section is a straightforward adaptation of the corresponding results for S^1-fibrations given on [Go-87].

Proposition 3.1. *If $(\theta_f)_2 \neq 0$ then $i(f)$ is finite.*

Proof. If $i(f) = \infty$ then the rank of the image $(\theta_f)_\pi \Pi_1(M)$ is 1 or zero. Since $\Pi_1(T-e) \to \Pi_1(T)$ is surjective and $\Pi_1(T-e)$ is a free group with two generators, we can construct a lift $s' : M \to E(T-e)$ of θ_f, by Theorem 4.3.1, page 265 of [Ba-77]. This fact provides a contradiction, since $(\theta_f)_2 = (q_1 \circ s')_2 = 0$, because $H_2(E(T-e)) \cong H_2(T-e) = 0$. □

It is interesting to remark that the converse of this proposition fails. The counterexample is in section 5, but in the case of a S^1-fibration it is true [Go-87].

It is easy to verify that $(\theta_f)_2 \neq 0$ implies $Fix_B(g) \neq \emptyset$ for every $g \simeq f$ over B. The next theorem gives us a decomposition of the set of fixed points of $g \simeq f$ over B in at least $i(f)$ components.

Theorem 3.2. *If $(\theta_f)_2 \neq 0$ then there exist at least $i(f)$ connected compact components $C_1, C_2, \ldots, C_{i(f)}$, with Čech cohomology $\check{H}^{n-2}(C_j; K) \neq 0$, where K is $\mathbb{Z}$ or $\mathbb{Z}_2$, depending on whether M is an orientable or non-orientable n-dimensional compact manifold.*

Proof. If $(\theta_f)_2 \neq 0$ then, by proposition 3.1 , the rank of $(\theta_f)_\pi(\Pi_1 M)$ is two. Let $\rho : T \to T$ be the $i(f)$-fold covering map corresponding to the subgroup $(\theta_f)_\pi(\Pi_1 M)$.

For every $g \simeq f$ over B, there is the lift $\tilde{\theta}_g : M \to T$ such that $\rho \circ \tilde{\theta}_g = \theta_g$.

We consider the following commutative diagram in which the coefficients of the homological groups are taken in $\mathbb{Z}$:

The compositions of vertical arrows are the induced homomorphisms on homological groups of θ_f and their restrictions.

Since that $(\theta_f)_2 \neq 0$, with $\mathbb{Z}$-coefficients,then from the Universal Coefficients theorem,this homology with coefficients in $\mathbb{Q}$ is non trivial :

$$H_2(M, M - (\rho \circ \tilde{\theta}_g)^{-1}(e); \mathbb{Q}) \neq 0$$

Furthermore

$$H_2(M, M - (\rho \circ \tilde{\theta}_g)^{-1}(e); \mathbb{Z}_2) \neq 0$$

Now we have the following isomorphisms, where the first is given by Poincaré duality; the homology and the cohomology have coefficients in $\mathbb{Z}$ or Z_2, according to M is the n-dimensional manifold being orintable or not.

$$\begin{aligned} H_2(M, M-(\rho\circ\tilde{\theta}_g)^{-1}(e)) &\cong \check{H}^{n-2}\left((\rho\circ\tilde{\theta}_g)^{-1}(e)\right) = \check{H}^{n-2}\left(Fix_B(g)\right) \\ &\cong \check{H}^{n-2}\left(\tilde{\theta}_g^{-1}(e_1)\cup\tilde{\theta}_g^{-1}(e_2)\cup\cdots\cup\tilde{\theta}_g^{-1}(e_{i(f)})\right) \\ &\cong \bigoplus_{J=1}^{i(f)} \check{H}^{n-2}\left(\tilde{\theta}_g^{-1}(e_j)\right) \end{aligned}$$

where $e_j \in \rho^{-1}(e)$.

Again, from $(\theta_f)_2 \neq 0$ and by the diagram below:

$$\begin{array}{ccc} H_2(M) & \longrightarrow & H_2(M, M-\tilde{\theta}_g^{-1}(e_j)) \\ \tilde{\theta}_g \downarrow & & \downarrow \\ H_2(T) & \xrightarrow{\cong} & H_2(T, T-e) \end{array}$$

we conclude that $H_2(M, M-\tilde{\theta}_g^{-1}(e_j)) \neq 0$.

Therefore from Poincaré duality, it follows that $\check{H}^{n-2}(\tilde{\theta}_g^{-1}(e_j)) \neq 0$. If we write $\tilde{\theta}_g^{-1}(e_j)$ as a disjoint union of connected components then for one of them, for instance C_j, we have $\check{H}^{n-2}(C_j) \neq 0$ and so the resut follows.

□

Note that if we consider the differential structure, we can take e_j as a regular value and so C_j is a $(n-2)$ submanifold.

Finally, we put the realization problem i.e.:

If $(\theta_f)_2 \neq 0$, is there a map g homotopic to f over B such that $Fix_B(g)$ has exactly $i(f)$ connected components?

We do not know the answer to this question. In part III of [Go-87], Gonçalves gives us the affirmative answer when we have a S^1-fibration with orientable total space.

4. Abelianized Obstruction

Let X be a 0-conneted CW complex whose fundamental group has the following presentation:

$$\Pi_1(X) \cong <\alpha_1, \alpha_2 \ldots, \alpha_n \ ; \ \beta_1, \beta_2, \ldots, \beta_m>,$$

where the α_i are the generators and β_j are the relations.

Let $\varphi : X \to T$ be a map with induced homomorphism on the fundamental group given by $\varphi_\pi(\alpha_i) = a^{x_i} b^{y_i}$, where a and b are the generators of $\Pi_1(T)$.

In this section, we calculate the cochain which represents the abelianized obstruction $\mathcal{A}(\varphi) \in H^2(M, \{H_1\mathcal{F}\})$, obtained from induced fiber bundle $\mathcal{F} \to E(T-e) \to T$ by φ. We also give conditions for the vanishing of this obstruction in terms of matricial equations in a group ring. For this we make use of the Fox's calculus in [Fo-53].

Proposition 4.1.

(1) $H_1(\mathcal{F}) \cong \mathbb{Z}\Pi_1(T) \cong \bigoplus_{\substack{x\in\mathbb{Z}\\ y\in\mathbb{Z}}} [B_{a^x b^y}]$, *where* $B_{a^x b^y} = B_{(x,y)}$ *is a generator of a copy of* $\mathbb{Z}$.

(2) If, by means of this isomorphism, we identify $w \in H_1(\mathcal{F})$ *with a word* w *in* $\{a, b, a^{-1}, b^{-1}\}$ *then the action of* $\alpha \in \Pi_1(T)$ *on* w *is the product* $\alpha \cdot w$.

Proof. Let $p : \mathbb{R}^2 \to T$ be the universal covering map and let $\tilde{x}_0$ and x_0 be the base points of $\mathbb{R}^2 - p^{-1}(e)$ and $T - e$, respectively. From proposition 1.2, $\Pi_1(\mathcal{F}) \cong \Pi_2(T, T-e; x_0)$.

Take $\alpha' \in \Pi_1(T-e)$ such that $i_\pi(\alpha') = \alpha$ where i_π is the induced epimorphism of the inclusion $i : (T-e) \to T$.We consider the following diagram:

The map h_α is the deck transformation corresponding to α and $y_0 = h_\alpha(\tilde{x}_0)$. The isomorphism $\tau_{\alpha'}$ is the action of α'.

From these isomorphisms, we conclude that $H_1(\mathcal{F})$ is the abelianized group of $\Pi_1(\mathbb{R}^2, \mathbb{R}^2 - p^{-1}(e); x_0)$; therefore item (i) is proved.

Note that the left face of diagram is not commutative but if we look at this diagram on homology groups that face is commutative.

Geometrically, if we suppose $\alpha = a^m b^n$, then h_α translates a small circle around the point of coordinates $(x, y) \in p^{-1}(e)$, the corresponding generator $B_{(x,y)} \in H_1(\mathbb{R}^2 - p^{-1}(e))$, to a circle around the point $(x+m, y+n)$. When we identify $B_{(x,y)}$ in $\mathbb{Z}\Pi_1(T)$, this corresponds to the product $\alpha \cdot (a^x b^y)$.

$\square$

Now, we are going to look in detail at the abelianized obstruction $\mathcal{A}(\varphi)$.

We can construct the lift over the 1-skeleton X_1 of X because this is 0-connected.

If $\varphi : X \to T$ is a map in which the induced homomorphism $\varphi_\pi : \Pi_1(X) \to \Pi_1(T)$ is given by $\varphi_\pi(\alpha_i) = a^{x_i} b^{y_i}$, then we consider a lift $g_1 : X_1 \to E(T - e)$ satisfying $(g_1)_\pi(\alpha_i) = a^{x_i} b^{y_i}$, where the generators a and b are in $\Pi_1(T - e) \cong \Pi_1(E)$ where E denotes the total space $E(T - e)$.

Let $\hat{g}_1 : (X_2, X_1) \to (\hat{E}, E)$ be the extension of g_1 to the 2-skeleton X_2 of X, where $\hat{E}$ is the mapping cylinder of E.

In the next diagram we use the following conventions:

(1) the label Y^{ab} denotes the abelianized group of Y and the label $f^{ab} : Y^{ab} \to Z^{ab}$ denotes the induced homomorphism from $f : Y \to Z$.

(2) Δ is the composition $\Pi_2(\hat{E}, E) \underset{\cong}{\to} \Pi_2(\hat{\mathcal{F}}, \mathcal{F}) \underset{\cong}{\xrightarrow{\delta}} \Pi_1(\mathcal{F})$.These isomorphisms are related in [Wh-78] page 195.

(3) we also denote by $p : \tilde{X}_2 \to X_2$ the restriction of the universal covering map $p : \tilde{X} \to X$.

(4) ρ is the homomorphism of Hurewicz.

According to lemma 5.3, page 293 and theorem 4.9, page 288, chapter VI of [Wh-78], the equivariant homomorphism

$$c(\varphi) = \Delta \circ (\hat{g}_1)_\pi^{ab} \circ p_\pi^{ab} \circ \rho^{-1} \in Hom^{\pi_1(X)}(\Gamma_*(\tilde{X}); H_1(\mathcal{F}))$$

is the cocycle on the equivariant cellular chain complex denoted by $\Gamma_*(\tilde{X})$. This cocycle represents the abelianized obstruction class $\mathcal{A}(\varphi)$ and it is zero if, and only if, there is an equivariant homomorphism Ψ as in diagram (IV).

Let $c_i : (\Delta_1, \dot{\Delta}_1) \to (\tilde{X}_1, \tilde{X}_0)$ be a lifting of the generator $\alpha_i \in \Pi_1(X)$ starting in $\tilde{x}_0$ and let $\tilde{x}_{\alpha_i}$ be its endpoints. These paths are the generators of $H_1(\tilde{X}_1, \tilde{X}_0)$ as a $\Pi_1(X)$-module. Let $e_{\beta_j} \in H_2(\tilde{X}_2, \tilde{X}_1)$ be generators such that $\delta \circ p_\pi \circ \rho^{-1}(e_{\beta_j}) = \beta_j \in \Pi_1(X_1)$, the relations in $\Pi_1(X)$. These are also the generators of $H_2(\tilde{X}_2, \tilde{X}_1)$ as a $\Pi_1(X)$-module.

We note that $(g_1)_\pi(\beta_j) \in \Pi_1(E)$ is carried by the homomorphism $\Pi_1(E) \to \Pi_1(T)$ into zero ; so we have $(g_1)_\pi(\beta_j) \in \Pi_1(\mathcal{F})$. We denote this element in $H_1(\mathcal{F})$ by $\varphi^{ab}(\beta_j)$.

With these generators we have:

Theorem 4.2.

(1) $c(\varphi)(e_{\beta_j}) = \varphi^{ab}(\beta_j)$

(2) $j \circ \partial_2(e_{\beta_j}) = \sum_{i=1}^{n} [\frac{\partial \beta_j}{\partial \alpha_i}] \cdot c_i$ *where* $\frac{\partial}{\partial \alpha_i}$ *are the free derivatives with respect*

to α_i in the sense of [Fo-53] and the simbol '$\cdot$' denotes the product as $\Pi_1(X)$-module on $H_1(\tilde{X}_1, \tilde{X}_0)$

(3) *There is an equivariant homomorphism Ψ in diagram (IV), i.e. $\mathcal{A}(\varphi) = 0$ if, only if, there exists a solution for the following matricial equation on $\mathbb{Z}\Pi_1(T)$:*

$$\begin{pmatrix} \dfrac{\partial\beta_1}{\partial\alpha_1} & \cdots & \dfrac{\partial\beta_1}{\partial\alpha_n} \\ \vdots & & \vdots \\ \dfrac{\partial\beta_m}{\partial\alpha_1} & \cdots & \dfrac{\partial\beta_m}{\partial\alpha_n} \end{pmatrix}^{\varphi} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \varphi^{ab}(\beta_1) \\ \vdots \\ \varphi^{ab}(\beta_m) \end{pmatrix}$$

Proof. The proof of (3) follows immediately from (1) and (2).

The proof of (1) is a consequence of the definition of

$$c(\varphi) = \Delta \circ (\hat{g}_1)_\pi^{ab} \circ p_\pi^{ab} \circ \rho^{-1} \in Hom^{\pi_1(X)}(\Gamma_*(\tilde{X}); H_1(\mathcal{F})),$$

the choice of e_{β_j}, the extension $\hat{g}_1$ of the lift g_1 and the commutativity of diagram IV.

To prove (2), we represent the relation β_j simply by β and we write this in the reduced word $\prod_{i=1}^{k} \alpha_{J_i}^{\varepsilon_i}$, where $\varepsilon_i = \pm 1$ and if $j_i = j_{i+1}$,then $\varepsilon_i + \varepsilon_{i+1} \neq 0$.

We consider the r-initial word of β i.e. $\gamma_r = \prod_{i=1}^{r} \alpha_{j_i}^{\varepsilon_i}$ and we put γ_0 as being the unit element $1 \in \Pi_1(X_1, x_0)$.

So we take paths u_r in $\tilde{X}_1$ from $\tilde{x}_{\gamma_{r-1}}$ to $\tilde{x}_{\gamma_r}$, for $r = 1, \cdots, k$. Therefore the lift of β on $\tilde{X}_1$ is the justaposed path $\tilde{\beta} = u_1 u_2 \ldots u_k$.

When we look at the lift $\tilde{\beta}$ as a class in $H_1(\tilde{X}_1, \tilde{X}_0)$ by the composition

$$\Pi_1(\tilde{X}_1) \to H_1(\tilde{X}_1) \to H_1(\tilde{X}_1, \tilde{X}_0)$$

we can write $\tilde{\beta} = u_1 + u_2 + \ldots + u_k$. Now if we consider h_{γ_r}, the corresponding deck transformation of γ_r, we have:

$$u_r = \begin{cases} h_{\gamma_{r-1}}(\epsilon_r c_{j_r}) & \text{if } \epsilon_r = 1 \\ h_{\gamma_r}(\epsilon_r c_{j_r}) & \text{if } \epsilon_r = -1 \end{cases}$$

Now if we consider the structure of $\mathbb{Z}\Pi_1(X)$-module on $H_1(\tilde{X}_1, \tilde{X}_0)$ we write: $u_r = \epsilon_r \beta_{(r)} \cdot c_{j_r}$ where $\beta_{(r)}$ is the inicial section of reduced word β (formula 2.7 of [Fo-53]). If we use the formula 2.8 in [Fo-53], we have:

$$\begin{aligned} u_1 + u_2 + \ldots + u_k &= \sum_{r=1}^{k} \epsilon_r \beta_{(r)} \cdot c_{j_r} \\ &= (\sum_{j_r=1} \epsilon_r \beta_{(r)}) \cdot c_1 + (\sum_{j_r=2} \epsilon_r \beta_{(r)}) \cdot c_2 + \ldots + (\sum_{j_r=n} \epsilon_r \beta_{(r)}) \cdot c_n \\ &= \frac{\partial \beta}{\partial \alpha_1} \cdot c_1 + \frac{\partial \beta}{\partial \alpha_2} \cdot c_2 + \ldots + \frac{\partial \beta}{\partial \alpha_n} \cdot c_n \ . \end{aligned}$$

From the choice of $\beta = \beta_j$ and the commutativity of the diagram, we conclude that

$$j \circ \partial_2(e_{\beta_j}) = \sum_{i=1}^{n} \frac{\partial \beta_j}{\partial \alpha_i} \cdot c_i$$

$\square$

Remark 4.3. The theorem is also true if we replace the map $\varphi : X \to T$ by $\varphi : X \to S$, where S is a surface and if we replace $E(T - e))$ by $E(S - point)$.

5. Examples

Example 1. We will construct a trivial principal torus fiber bundle where $\mathcal{A}(\theta_f) \neq 0$, $(\theta_f)_2 = 0$ and $(\theta_f)_\pi$ is surjective.

Take the Hopf fibration $q : S^3 \to S^2$ and a map $g : T \to S^2$ whose degree is one. Let $q_g : M \to T$ be the induced fiber map from q by g.

We can prove that M is a $K(\pi, 1)$ space with the fundamental group having the following presentation:

$$\Pi_1(M) = < \alpha_1, \alpha_2 \ ; \ \beta_1 = [\alpha_1, [\alpha_1, \alpha_2] \ ] \ , \ \beta_2 = [\alpha_2, [\alpha_1, \alpha_2] \ ] >$$

where $[\alpha_i, \alpha_j] = \alpha_i \alpha_j \alpha_i^{-1} \alpha_j^{-1}$. The induced homomorphism $(q_g)_\pi : \Pi_1(M) \to \Pi_1(T)$ is given by $(q_g)_\pi(\alpha_1) = a$ and $(q_g)_\pi(\alpha_2) = b$ where a and b are the generators of $\Pi_1(T)$.

Let $p_1 : M \times T \to M$ be the trivial principal fiber bundle and consider the map $f : M \times T \to M \times T$ defined by $f(x, y) = (x, q_g(x).y)$. Here, $\theta_f = q_g$ which we denote by φ.

Note that $\varphi_2 = 0$ because $\varphi_2 = (g_2)^{-1} \circ q_2 \circ p_2'$ where $p' : M \to S^3$ and $H_2(S^3) = 0$.

In accordance with theorem 4.2, we have $\mathcal{A}(\varphi) \neq 0$ if the equation below has no solution

$$\begin{pmatrix} 1+\alpha_1\dfrac{\partial z}{\partial \alpha_1}-\alpha_1 z\alpha_1^{-1}-w_1\dfrac{\partial z}{\partial \alpha_1} & \alpha_1\dfrac{\partial z}{\partial \alpha_2}-w_1\dfrac{\partial z}{\partial \alpha_2} \\ \alpha_2\dfrac{\partial z}{\partial \alpha_1}-w_2\dfrac{\partial z}{\partial \alpha_1} & 1+\alpha_2\dfrac{\partial z}{\partial \alpha_2}-\alpha_2 z\alpha_2^{-1}-w_2\dfrac{\partial z}{\partial \alpha_2} \end{pmatrix}^{\varphi} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a-1 \\ b-1 \end{pmatrix}$$

where $z = [\alpha_1, \alpha_2]$; $w_1 = [\alpha_1, z]$ and $w_2 = [\alpha_2, z]$.

If we replace $\dfrac{\partial z}{\partial \alpha_1} = 1 - \alpha_1\alpha_2\alpha_1^{-1}$ and $\dfrac{\partial z}{\partial \alpha_2} = \alpha_1 - [\alpha_1, \alpha_2]$ in the equation we obtain:

$$\begin{pmatrix} (a-1)(1-b) & (a-1)(a-1) \\ (b-1)(1-b) & (b-1)(a-1) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a-1 \\ b-1 \end{pmatrix}$$

This is equivalent to

$$(1-b)x + (a-1)y = 1$$

Now if we apply the augmentation homomorphism on this equation, we conclude that it does not have solution.

Example 2. In this example, we will show that all invariants, in section 2.3 vanish but there is no lift on diagram (II).

Let M_1 be an orientable compact manifold without boundary with:

$$\Pi_1(M_1) = < \gamma_1, \gamma_2 ; [\,[\gamma_1, \gamma_2]\ ,\ \gamma_2[\gamma_1\ ,\ \gamma_2]\gamma_2^{-1}] = \delta > .$$

We consider the principal torus fiber bundle $M_1 \times T$ and $f_1 : M_1 \times T \to M_1 \times T$, given by $f_1(x, y) = (x, \varphi_1(x) \cdot y)$ where $(\varphi_1)_\pi(\gamma_1) = a$ and $(\varphi_1)_\pi(\gamma_2) = b$.

Since the equation

$$\begin{pmatrix} \dfrac{\partial \delta}{\partial \gamma_1} & \dfrac{\partial \delta}{\partial \gamma_2} \end{pmatrix}^{\varphi_1} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \varphi_1^{ab}(\delta) = 0$$

admits at least the trivial solution we conclude that $\mathcal{A}(\varphi_1) = 0$

Now, we can write the map φ_1 as $\varphi_1 = \varphi \circ \Psi$ where φ is defined in the first example and $\Psi : \Pi_1(M_1) \to \Pi_1(M)$ is given by $\Psi_\pi(\gamma_1) = \alpha_1$ and $\Psi_\pi(\gamma_2) = \alpha_2$. In this manner it is not difficult to prove that $(\varphi_1)_2 : H_2(M_1) \to H_2(T)$ is zero.

We claim that there is no lift for φ_1. Otherwise the rank of the image of this lift would be either one or zero because it is the only way to preserve the relation δ on $\Pi_1(T - e)$. But this is not possible because $(\varphi_1)_\pi$ is surjective.

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