

AMPLE AND SPANNED VECTOR BUNDLES OF TOP CHERN NUMBER TWO ON PROJECTIVE VARIETIES

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1. Introduction

We work over an algebraically closed field k of characteristic zero. Let X be a normal projective variety of dimension n . Let E be an ample and spanned vector bundle of rank $r \geq n$ on X . Here E is said to be *ample* if the tautological line bundle $\mathcal{O}_{\mathbb{P}}(1)$ of the projective bundle $\mathbb{P} = \mathbb{P}_X(E)$ is ample. And E is said to be *spanned* if E is generated by its global sections.

The purpose here is to classify ample and spanned vector bundles E of top Chern number two when X is a smooth projective variety of arbitrary dimension and when X is a normal Gorenstein surface: for the first case, we give only an outline of a proof here and for the details we refer the reader to [11]; for the second case, we give a proof in detail.

There are several studies on the problem of classifying ample and spanned vector bundles of small top Chern number on normal projective varieties.

In case $c_n(E) = 1$, Lanteri-Sommese [9] have shown that $(X, E) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$ when X is a normal surface or a Gorenstein 3-fold with only isolated singularities. In the higher dimensional case, Wiśniewski [14] has shown that the same is true when X is a smooth projective variety. Fujita has pointed out that the same result for a variety X with only log terminal singularities also follows from Zhang's result [15] together with Lanteri-Sommese's argument in [9] (see Proposition 3.1). These results for $c_n(E) = 1$ have been proved in this way: if $c_n(E) = 1$, then the adjoint bundle $K_X \otimes \det E$ is not nef by

Lanteri-Sommese [9], namely there is a curve C with $(K_X \otimes \det E, C) < 0$; hence theorems on bundles with non-nef adjoint bundles in [14] and [15] imply $(X, E) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$.

In case $c_2(E) = 2$, the structure of (X, E) has been described explicitly when X is a smooth surface, along the idea of Ballico [1] ([10], see also [8]).

Our main results are the following:

Theorem 1.1. ([11, Theorem 1.1]). *Let X be a smooth projective variety of dimension $n \geq 2$, and E an ample and spanned vector bundle of rank r with $r \geq n$ and $c_n(E) = 2$ on X . Then $r = n$ and (X, E) is one of the following:*

- (1) *There exists a finite morphism $f: X \rightarrow \mathbb{P}^n$ of degree 2 over a projective space \mathbb{P}^n , and $E \cong f^*\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n}$.*
- (2) *$(X, E) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-1})$.*
- (3) *$(X, E) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus n})$, where \mathbb{Q}^n is a smooth quadric hypersurface in \mathbb{P}^{n+1} , and $\mathcal{O}_{\mathbb{Q}^n}(1)$ is the hyperplane line bundle.*
- (4) *X is isomorphic to a projective space bundle $\mathbb{P}_C(\mathcal{F})$ over an elliptic curve C with the projection $\pi: \mathbb{P}_C(\mathcal{F}) \rightarrow C$ and with the tautological line bundle $\mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$, and $E \cong \pi^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$. Here \mathcal{F} and \mathcal{E} are indecomposable rank- n vector bundles of degree 1 on C .*
- (5) *X is isomorphic to a projective space bundle $\mathbb{P}_C(\mathcal{F})$ over a hyperelliptic curve C of genus $g \geq 2$ with the projection $\pi: \mathbb{P}_C(\mathcal{F}) \rightarrow C$ and with the tautological line bundle $\mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$, and $E \cong \pi^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$. Here \mathcal{F} and \mathcal{E} are vector bundles of rank n on C such that the system $|\det \mathcal{F} \otimes \det \mathcal{E}|$ is the hyperelliptic pencil g_2^1 .*

Conversely, every bundle in the all cases but (5) is ample and spanned with $c_n(E) = 2$.

Remark. When $n = 2$, the case (5) is void (see Theorem 1.2 below and [10,

Theorem 6.1 and Proposition 5.4]). When $n \geq 3$, I do not know whether a bundle in the case (5) exists or not.

Theorem 1.2. *Let E be an ample and spanned vector bundle of rank $r \geq 2$ on a normal Gorenstein projective surface X . If $c_2(E) = 2$, then $r = 2$ and (X, E) is one of the following :*

- (1) *There exists a finite morphism $f : X \rightarrow \mathbb{P}^2$ of degree 2 whose branch divisor has no double components and $E \cong f^* \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$.*
- (2) *$(X, E) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$.*
- (3) *$(X, E) \cong (\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1)^{\oplus 2})$.*
- (4) *$(X, E) \cong (\mathbb{Q}_0^2, \mathcal{O}_{\mathbb{Q}_0^2}(1)^{\oplus 2})$, where \mathbb{Q}_0^2 is an (integral) quadric cone in \mathbb{P}^3 .*
- (5) *X is isomorphic to a geometrically ruled surface $\mathbb{P}_C(\mathcal{F})$ over an elliptic curve C with the projection $\pi : \mathbb{P}_C(\mathcal{F}) \rightarrow C$ and with the tautological line bundle $\mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$, and $E \cong \pi^* \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$. Here \mathcal{F} and \mathcal{E} are indecomposable rank-2 vector bundles of degree 1 on C .*

Conversely, every bundle above is ample and spanned with $c_2(E) = 2$.

To prove both theorems, the essential case is $\text{rank } E = \dim X$. In this case, an invariant $\text{sp}(E)$ of Ballico [1] divides our situation into two cases: $\text{sp}(E) = n$ and $\text{sp}(E) > n$. When $\text{sp}(E) = n$, Ballico's theorem [1] and, for a smooth higher dimensional variety X , Zhang's theorem [15] together with Lanteri-Sommese's argument [9] imply the precise structure. When $\text{sp}(E) > n$, a key step is to show that the adjoint bundle $K_X \otimes \det E$ fails to be ample. Then Fujita's theorem [4] for smooth X , and a normal surface version of Lanteri-Maeda's theorem for a normal Gorenstein projective surface X tell us the structure of (X, E) . We pick up the bundles in our case and determine the precise structure.

Our exposition proceeds as follows. In §2, we recall the definition of Ballico's invariant $\text{sp}(E)$ and some results on it. In §3, we outline a proof of Theorem

1.1. In §4, we study the adjoint bundle of an ample and spanned vector bundle on a normal Gorenstein projective surface. In §5, we give a proof of Theorem 1.2.

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2. Ballico's invariant $\text{sp}(E)$

In this section, we recall the definition of the invariant $\text{sp}(E)$ of Ballico [1] (see also [10]).

Let X be a normal projective variety of dimension n , and E a spanned vector bundle of rank n with $c_n(E) > 0$ on X . By $|E|$ we denote $\mathbb{P}(H^0(X, E)^\vee) = (H^0(X, E) \setminus \{0\})/k^*$ and by $[t]$ the point of $|E|$ corresponding to $t \neq 0 \in H^0(X, E)$. We consider the family of zeros $F = \{([t], p) \in |E| \times X; t(p) = 0\}$, the projective space bundle $\mathbb{P}_X(\mathcal{F}_E)$ associated with the vector bundle \mathcal{F}_E whose dual bundle is the kernel of the evaluation map $\text{ev} : H^0(X, E) \otimes \mathcal{O}_X \rightarrow E$. Hence we get the following diagram:

$$\begin{array}{ccccc} |E| \times X & \supset & F & \xrightarrow{\Psi} & X \\ & & \phi \downarrow & & \\ & & |E|. & & \end{array} \quad (2.1)$$

Here $\Phi : F \rightarrow |E|$ and $\Psi : F \rightarrow X$ denote the first and second projections. Then Φ is a finite flat morphism of degree $c_n(E)$ over a dense and open subset

$$|E|_{\text{reg}} := \{[t] \in |E|; \dim(t)_0 = 0 \text{ and } (t)_0 \text{ is on the Cohen-Macaulay locus of } X\}.$$

Thus we have a morphism $\tau : |E|_{\text{reg}} \rightarrow \text{Hilb}_X^{c_n(E)}$ by the universality of the Hilbert scheme $\text{Hilb}_X^{c_n(E)}$ of $c_n(E)$ points in X . By considering the norm morphism $\nu : \text{Hilb}_X^{c_n(E)} \rightarrow S^{c_n(E)}(X)$, we also have a morphism $\rho = \nu \circ \tau : |E|_{\text{reg}} \rightarrow S^{c_n(E)}(X)$ to the $c_n(E)$ th symmetric product of X .

Then we define $\text{sp}(E)$ by

$$\text{sp}(E) = \dim \tau(|E|_{\text{reg}}).$$

Since ν is isomorphic over the open subset parameterizing distinct smooth $c_n(E)$ points of X , we have $\text{sp}(E) = \dim \rho(|E|_{\text{reg}})$. Hence this invariant is Ballico's in [1]. By definition, we have $n \leq \text{sp}(E) \leq n \cdot c_n(E)$. By [10, Proposition 1.1], for every $[t] \in |E|_{\text{reg}}$, the fibre $\tau^{-1}(\tau([t]))$ of τ over $\tau([t])$ is $|E \otimes \mathcal{I}_{(t)_0}| \cap |E|_{\text{reg}}$, where $|E \otimes \mathcal{I}_{(t)_0}| := \mathbb{P}(H^0(X, E \otimes \mathcal{I}_{(t)_0})^\vee) \subseteq |E|$. Hence, for a general section $t \in H^0(X, E)$, it holds that

$$\text{sp}(E) = \dim_k H^0(X, E) - \dim_k H^0(X, E \otimes \mathcal{I}_{(t)_0}). \quad (2.2)$$

If $\text{sp}(E) = n$ and if E is ample, the finite morphism from X to the Grassmann of n -quotients associated with the evaluation map $H^0(X, E) \otimes \mathcal{O}_X \rightarrow E$ maps $(t)_0$ to one point. Thus we have the following proposition.

Proposition 2.3 (Ballico [1, Theorem 4.1], see also [11, Proposition 4.1]). *Let E be an ample and spanned vector bundle of rank n on a normal projective variety X of dimension n . Assume that $\text{sp}(E) = n$. Then there exist an ample and spanned vector bundle E' of rank n with $c_n(E') = 1$ on a normal projective variety X' of dimension n , and a finite morphism $f : X \rightarrow X'$ of degree $c_n(E)$ such that $E \cong f^*(E')$. Moreover if $n = 2$, then $X' \cong \mathbb{P}^2$ and $E' \cong \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$.*

3. Outline of proof of Theorem 1.1

Now we outline a proof of Theorem 1.1. First we assume that $r = n$. In view of Ballico's invariant $\text{sp}(E)$, we divide the proof into two cases; $\text{sp}(E) = n$ and $\text{sp}(E) > n$.

If $\text{sp}(E) = n$, by Proposition 2.3, X has a double cover $f : X \rightarrow X'$ over a normal projective variety X' of dimension n and E is the pull-back of an ample and spanned vector bundle E' with $c_n(E') = 1$ on X' . Since X and X' are normal and $\deg f = 2$, there exists the involution ι on X such that the quotient $X/\langle \iota \rangle$ is X' , and hence, X' has only log terminal singularities. Thus

by the following Proposition, which is a consequence of Zhang's Theorem [15, Theorem 1] along with Lanteri-Sommese's argument [9], we have $X' \cong \mathbb{P}^n$ and $E' \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n}$; this is the case (1). (The above argument determining the structure of (X', E') is due to Fujita).

Proposition 3.1 (See [11, Proposition 5.1]). *Let X be a projective variety of dimension n with only log terminal singularities and E an ample and spanned vector bundle of rank n with $c_n(E) = 1$ on X . Then $(X, E) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$.*

If $\text{sp}(E) > n$, for a general point $p \in X$ and a general line $\ell \subset |E \otimes \mathcal{I}_{\{p\}}|$,

$$C := \overline{\{q \in X \mid \{p, q\} = (t)_0 \text{ for some } t \in \ell \cap |E|_{\text{reg}}\}}$$

is a complete curve on X , where $\bar{\cdot}$ denotes the closure. Assume that

$$\ell \subset |E|_{\text{reg}}. \quad (*)$$

Then we have $(K_X \otimes \det E, C) \leq 0$: Indeed, Φ is a double cover over ℓ , and hence $\Phi^{-1}(\ell) = \ell \times \{p\} \cup \tilde{\ell}$, where $\tilde{\ell} := \{(q, t) \in F \mid t \in \ell, (t)_0 = \{p, q\}\}$. Note that $\Phi(\tilde{\ell}) = C$. Thus for the canonical line bundle K_F of F , we have $(K_F, \tilde{\ell}) = (K_F, \ell \times \{p\})$. On the other hand, $(\mathcal{O}_F(1), \tilde{\ell}) = (\mathcal{O}_{|E|}(1), \ell) = (\mathcal{O}_F(1), \ell \times \{p\})$. Since $K_F \cong \Psi^*(K_X \otimes \det E) \otimes \mathcal{O}_F(-N + n - 1)$, we have $(\Psi^*(K_X \otimes \det E), \tilde{\ell}) = (\Psi^*(K_X \otimes \det E), \ell \times \{p\})$. Since the right hand side is zero, we have $(K_X \otimes \det E, C) = (\Psi^*(K_X \otimes \det E), \tilde{\ell}) = 0$ by the projection formula.

Unfortunately, the assumption (*) above is not true in general (for example, the case (4) in Theorem 1.1): namely, Φ has a positive dimensional fibre on ℓ even if $p \in X$ and $\ell \subset |E \otimes \mathcal{I}_{\{p\}}|$ are general, hence we cannot apply the above argument directly. But we can modify it so that we have the following proposition (see [11, Proposition 7.1] for the details).

Proposition 3.2. *Let X be a normal projective variety of dimension n and E a spanned vector bundle of rank n with $c_n(E) = 2$ on X . Assume that $\text{sp}(E) > n$ and that X is Gorenstein. Then there exists a projective integral curve C on*

X through a general point of X such that $(K_X \otimes \det E, C) \leq 0$. In particular, $K_X \otimes \det E$ is not ample.

Therefore the adjoint bundle $K_X \otimes \det E$ is not ample if $\text{sp}(E) > n$. Hence by Fujita's theorem [4], the following three cases fit our situation:

(a) $(X, E) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-1})$;

(b) $(X, E) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus n})$;

(c) *There exist vector bundles \mathcal{F} and \mathcal{E} of rank n on a smooth projective curve C , such that X is isomorphic to a projective space bundle $\mathbb{P}_C(\mathcal{F})$ over C with the projection $\pi : \mathbb{P}_C(\mathcal{F}) \rightarrow C$ and with the tautological line bundle $\mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$, and $E \cong \pi^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$.*

The cases (a) and (b) correspond to (2) and (3) respectively. If the case is (c), we have (4) if $g(C) \leq 1$ and (5) if $g(C) \geq 2$, and conversely, we can check that the bundles but in (5) are ample and spanned with $c_n(E) = 2$. (For the details, see [11, §8]).

To complete our proof, we have to show that there is no bundle when $r > n$, by using a standard argument (see, [8], [11]): If $r > n$, by Serre's lemma, we have an exact sequence of vector bundles

$$0 \rightarrow \mathcal{O}_X^{\oplus r-n} \rightarrow E \rightarrow E' \rightarrow 0,$$

since E is spanned. Then E' is an ample and spanned vector bundle of rank n with $c_n(E) = 2$. By Theorem 1.1 for $r = n$, we easily check in each case that $H^1(X, E'^{\vee}) = 0$. Therefore the exact sequence above is split in each case. This contradicts the ampleness of E . This completes the proof of Theorem 1.1.

4. The adjoint bundle of an ample and spanned bundle on a normal surface

To prove Theorem 1.2, we need the following proposition.

Proposition 4.1 (Normal surface version of Theorem of Lanteri-Maeda [7]).
Let E be an ample and spanned vector bundle of rank $r \geq 2$ on a normal projective surface X . Then the adjoint system $|K_X \otimes \det E|$ on the Gorenstein locus $\text{Gor}(X)$ of X has no base points on $\text{Gor}(X)$ unless $(X, E) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2})$.

Before proving Proposition 4.1, we need the following three lemmas.

Lemma 4.2. *Let E be an ample and spanned vector bundle of rank $r \geq 2$ on a projective curve C . Then E contains a subline bundle of degree ≥ 1 and hence $\deg(E) \geq r$.*

Proof. See [7, Lemma 2].

□

Lemma 4.3. *Let E be an ample and spanned vector bundle of rank $r \geq 2$ on a Gorenstein projective surface X . Assume that $K_X \otimes \det E \cong \mathcal{O}_X$. Then (X, E) is isomorphic to one of the following.*

- (1) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$.
- (2) $(\mathbb{P}^2, T_{\mathbb{P}^2})$, where $T_{\mathbb{P}^2}$ is the tangent bundle of \mathbb{P}^2 .
- (3) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3})$.
- (4) $(\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1)^{\oplus 2})$ where \mathbb{Q}^2 is a smooth quadric surface in \mathbb{P}^3 .
- (5) $(\mathbb{Q}_0^2, \mathcal{O}_{\mathbb{Q}_0^2}(1)^{\oplus 2})$ where \mathbb{Q}_0^2 is an (integral) quadric cone in \mathbb{P}^3 .

Proof. By the assumption, X is a normal Del Pezzo surface (See Brenton [2], Hidaka-Watanabe [6]). Set $L = \det E$. Let $\pi : \widetilde{X} \rightarrow X$ be the minimal resolution. If \widetilde{X} is non-rational, by [6, Theorem 2.2], \widetilde{X} is a geometrically ruled surface $p : \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}) \rightarrow C$ over an elliptic curve C , where $\deg(\mathcal{L}) > 0$. And X is obtained by contracting the minimal section C_0 , and $K_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}(C_0) = \pi^* K_X$. Since $K_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-2C_0) \otimes p^* \mathcal{L}^{-1}$, we have $\pi^* L = \pi^* K_X^{-1} = \mathcal{O}_{\widetilde{X}}(C_0) \otimes p^* \mathcal{L}$. Hence for $x \in C$, we have $\pi^* L \cdot p^{-1}(x) = 1$. By (4.2), this is a contradiction.

Thus \widetilde{X} is rational. By [6, Theorem 3.4], X has only rational double points as singularities, and hence $K_{\widetilde{X}} = \pi^*K_X = \pi^*L^{-1}$. By (4.2), \widetilde{X} has no (-1) -curve. By [6, Theorem 3.4] again, $X \cong \mathbb{P}^2$, $X \cong \mathbb{Q}^2$, or $X \cong \mathbb{Q}_0^2$.

When $X \cong \mathbb{P}^2$, we have $L \cong \mathcal{O}_{\mathbb{P}^2}(3)$, and $r = 2$ or 3 , by (4.2). If $r = 2$, by a theorem of van de Ven [13], $(X, E) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ or $(\mathbb{P}^2, T_{\mathbb{P}^2})$, which is the case (1) or (2). If $r = 3$, by uniformity, $(X, E) \cong (\mathbb{P}^2, \mathcal{O}(1)^{\oplus 3})$, which is the case (3).

When $X \cong \mathbb{Q}^2$, we have $L = K_X^{-1} = \mathcal{O}_{\mathbb{Q}^2}(2)$. For every fibre $f \cong \mathbb{P}^1$ of the projection $p : \mathbb{Q}^2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$, we have $L|_f \cong \mathcal{O}_{\mathbb{P}^1}(2)$ and hence $E|_f \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ by ampleness. Thus $p_*(E \otimes \mathcal{O}_{\mathbb{Q}^2}(-1))$ is locally free of rank 2 on \mathbb{P}^1 and $E \otimes \mathcal{O}_{\mathbb{Q}^2}(-1) \cong p^*(p_*(E \otimes \mathcal{O}_{\mathbb{Q}^2}(-1)))$ by Grauert's theorem ([5, ch. III, Corollary 12.9]). Hence E is the direct sum of ample line bundles with $\det E \cong \mathcal{O}_{\mathbb{Q}^2}(2)$. Therefore we have $(X, E) \cong (\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1)^{\oplus 2})$, which is the case (4).

When $X \cong \mathbb{Q}_0^2$, we have $\widetilde{X} \cong \mathbb{F}_2 := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ and $\pi : \widetilde{X} \rightarrow X$ is the contraction of the minimal section C_0 . Let $p : \widetilde{X} = \mathbb{F}_2 \rightarrow \mathbb{P}^1$ be the projection. Since every fibre f of p is not mapped to a point by π , $\pi^*E|_f$ is ample and spanned. Since $\pi^*L = \pi^*K_X^{-1} = K_{\widetilde{X}}^{-1} = \mathcal{O}(2C_0 + 4f)$, we have $\pi^*E|_f \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Thus $\pi^*E \cong \mathcal{O}(C_0 + af) \oplus \mathcal{O}(C_0 + bf)$ with $a + b = 4$ by Grauert's theorem. Since $\pi^*(E)$ is spanned, $a = b = 2$. Since $\mathcal{O}_{\widetilde{X}}(C_0 + 2f) \cong \pi^*\mathcal{O}_{\mathbb{Q}_0^2}(1)$, we have $(X, E) \cong (\mathbb{Q}_0^2, \mathcal{O}_{\mathbb{Q}_0^2}(1)^{\oplus 2})$, which is the case (5).

□

Lemma 4.4. *Let E be an ample and spanned vector bundle of rank $r \geq 2$ on a normal projective surface X . Then $c_1^2(E) - c_2(E) \geq 3$ holds.*

Proof. By considering an exact sequence of vector bundles $0 \rightarrow \mathcal{O}_X^{\oplus(r-2)} \rightarrow E \rightarrow E'' \rightarrow 0$ for $r \geq 3$, we may assume that $r = 2$. Set $L = \det E$. Let $t \in H^0(X, E)$ be a general section so that the Koszul sequence $0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow L \otimes \mathcal{I}_{(t)_0} \rightarrow 0$ is exact and $(t)_0$ is a set of smooth $c_2(E)$ points. And $\widetilde{X} := \mathbf{Proj}(\bigoplus_{n \geq 0} \mathcal{I}_{(t)_0}^n)$ is the blowing-up of $(t)_0$ with the projection $\pi : \widetilde{X} \rightarrow X$

and with the exceptional Cartier divisor A defined by the ideal $\pi^{-1}\mathcal{I}_{(t)_0}\mathcal{O}_{\widetilde{X}}$. Let $\iota : \widetilde{X} \hookrightarrow \mathbb{P}_X(E)$ be the inclusion corresponding to $E \rightarrow L \otimes \mathcal{I}_{(t)_0}$. Set $\mathcal{O}_{\widetilde{X}}(1) = \mathcal{O}_{\mathbb{P}}(1)|_{\widetilde{X}}$. Then $\mathcal{O}_{\widetilde{X}}(1) = \pi^*L \otimes \mathcal{O}_{\widetilde{X}}(-A)$ and $\mathcal{O}_{\widetilde{X}}(1)^2 = c_1^2(E) - c_2(E) > 0$. We have to show that $\mathcal{O}_{\widetilde{X}}(1)^2 = 1, 2$ do not occur.

If $\mathcal{O}_{\widetilde{X}}(1)^2 = 1$, since $\mathcal{O}_{\widetilde{X}}(1)$ is ample and spanned, then the Δ -genus $\Delta(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1)) := \mathcal{O}_{\widetilde{X}}(1)^2 + 2 - h^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1)) = 0$ (see [3, (4.2)]), and hence $(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1)) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ by [3, (5.10) and (5.15)]. This contradicts the minimality of \mathbb{P}^2 .

If $\mathcal{O}_{\widetilde{X}}(1)^2 = 2$, then $\Delta(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1)) = 0$ or 1. If $\Delta(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1)) = 0$, \widetilde{X} is isomorphic to an integral quadric surface in \mathbb{P}^3 , which has no (-1) -curve supported on the nonsingular locus, contradiction. If $\Delta(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1)) = 1$, then $h^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1)) = 3$, so $\psi = \phi_{|\mathcal{O}_{\widetilde{X}}(1)|} : \widetilde{X} \rightarrow \mathbb{P}^2$ is a flat morphism of degree 2, since \widetilde{X} is normal of $\dim \widetilde{X} = 2$. Hence \widetilde{X} is of the form $\text{Spec}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-g-1))$ for the sectional genus $g = g(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1))$ as a scheme over \mathbb{P}^2 (see [3, p.49, (6.11) and (6.12)]). In particular, \widetilde{X} is Gorenstein with the canonical line bundle $K_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(g-2)$. Since π is an isomorphism around the singular points, X is also Gorenstein. So $\mathbb{P}_X(E)$ is Gorenstein with the canonical line bundle $K_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-2) \otimes \pi^*(K_X \otimes \det E)$. Since $\widetilde{X} \in |\mathcal{O}_{\mathbb{P}}(1)|$, by the adjunction formula, we also have $K_{\widetilde{X}} = \pi^*K_X \otimes \mathcal{O}_{\widetilde{X}}(A)$. Calculating the intersection number $K_{\widetilde{X}} \cdot A$ by using the two expression of $K_{\widetilde{X}}$, we have $-c_2(E) = (g-2)c_2(E)$ and hence $g = 1$. Thus $\pi^*L^{-1} \otimes \mathcal{O}_{\widetilde{X}}(A) = \mathcal{O}_{\widetilde{X}}(-1) = K_{\widetilde{X}} = \pi^*K_X \otimes \mathcal{O}_{\widetilde{X}}(A)$. Hence $K_X \otimes L \cong \mathcal{O}_X$. But, by (4.3), we have $\mathcal{O}_{\widetilde{X}}(1)^2 = c_1^2(E) - c_2(E) > 2$, contradiction. Thus we have the claim. □

Proof of Proposition 4.1. Set $L = \det E$. As in (4.4), we may assume that $r = 2$. Note that $c_2(E) > 0$ since E is ample and spanned. If $c_2(E) = 1$, then by [9, Theorem 1], $(X, E) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2})$. If $c_2(E) \geq 2$, then $c_1^2(E) \geq 5$ by (4.4). Suppose to the contrary that $|K_X \otimes \det E|$ has a base point in $\text{Gor}(X)$. By Reider-type-Theorem of Sakai [12, Theorem 4], there exists a non-zero effective divisor B on X such that $B.L \leq 1$. This contradicts the ampleness

and spannedness of E by (4.2).

□

5. Proof of Theorem 1.2

Now we prove Theorem 1.2. If E is ample and spanned with $c_n(E) = 2$, and if $\text{rank} E = 2$, as in the proof of Theorem 1.1, we divide the proof into two cases, $\text{sp}(E) = 2$ and $\text{sp}(E) > 2$.

When $\text{sp}(E) = 2$, by (2.3), there exists a finite morphism $f : X \rightarrow \mathbb{P}^2$ of degree 2 and $E \cong f^*(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2})$, i.e., this is the case (1). Note that the condition on the branch divisor is a necessary and sufficient condition for a double cover X to be integral and normal.

Next we assume that $\text{sp}(E) > 2$. By (4.1), $\varphi = \phi_{|K_X \otimes \det E|} : X \rightarrow \mathbb{P}(H^0(K_X \otimes \det E))$ is a morphism defined everywhere on X . By (3.2), we have $\dim \varphi(X) = 0, 1$. If $\varphi(X)$ is a point, then $K_X \otimes \det E \cong \mathcal{O}_X$. By (4.3), the case (2), (3) or (4) occurs. If $\varphi(X) = C'$ is an integral curve, we take Stein factorization $\pi : X \rightarrow C$ and $C \rightarrow C'$ of $\varphi : X \rightarrow C'$. Then C is a nonsingular projective curve, and π is a flat morphism of connected fibres, and $K_X \otimes \det E$ is the pull-back of a line bundle on C . Since a general fibre f is smooth and irreducible and since

$$\deg K_f = (K_X + f) \cdot f = K_X \cdot f = -\det E \cdot f \leq -2,$$

we have $f \cong \mathbb{P}^1$. By the theorem of Noether-Enriques, X is ruled. Since $\deg(K_X|_{\pi^{-1}(x)}) = -2$, we have $\deg(\det E|_{\pi^{-1}(x)}) = 2$ for every $x \in C$. By (4.2), every fibre $\pi^{-1}(x)$ is irreducible and reduced. Since π is flat, we have $\pi^{-1}(x) \cong \mathbb{P}^1$. Therefore X is a (smooth) geometrically ruled surface over C . Thus by [10, Theorem 6.1] (see also [10, Proposition 5.4]), we have (5).

As in Theorem 1.1, there is no bundle with $r > 2$. Conversely every bundle in the cases is ample and spanned with $c_2(E) = 2$ as in [10] (see also [11]). This completes the proof of Theorem 1.2.

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