ON PROJECTIVE MANIFOLDS WITH DEGENERATE SECANT VARIETIES

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0. Introduction

Let X be an n-dimensional nondegenerate (i.e., not contained in a hyperplane) projective manifold in $\mathbf{P}^N$ over an algebraically closed field k of characteristic 0. Let Sec$X$ denote the secant variety of X in $\mathbf{P}^N$. We have always $\dim \mathrm{Sec}X \leq \min\{2n+1, N\}$. If $\dim \mathrm{Sec}X < \min\{2n+1, N\}$, we say that Sec$X$ is degenerate. The Linear Normality Theorem [Z, Chap.2, Corollary 2.17] implies that if Sec$X$ is degenerate then $\dim \mathrm{Sec}X \geq (3n+2)/2$. If equality holds, X is called a Severi variety. Severi varieties were completely classified by F. L. Zak [Z, Chap.4, Th. 4.7]. Zak also generalized the class of Severi varieties to a class of manifolds, named Scorza varieties, and classified Scorza varieties [Z, Chap.6]. In this paper, we propose a new class of projective manifolds with degenerate secant varieties, which is wider than the class of Scorza varieties, and investigate some properties of this class of manifolds.

Suppose that Sec$X$ is degenerate. Let $\varepsilon = 2\dim \mathrm{Sec}X - 3n - 2$. Let Sm(Sec$X$) denote the smooth locus of Sec$X$, and $\gamma : \mathrm{Sm}(\mathrm{Sec}X) \to G(\dim \mathrm{Sec}X, \mathbf{P}^N)$ the Gauss map $u \mapsto T_u\mathrm{Sec}X$ of Sm(Sec$X$). Then we have the following proposition.

Proposition 0.1. $\dim \mathrm{Im}(\gamma) = 2(\dim \mathrm{Sec}X - n - 1 - c)$ *for some integer* c $(0 \leq c \leq \varepsilon)$.

Note that if $X \subset \mathbf{P}^N$ is a Scorza variety then the integer c in Proposition 0.1 is zero ([Z, Chap.6, (1.4.11)]) but the converse is not true. Note also that, to the best of my knowledge, all examples of $c > 0$ are constructed from those of $c = 0$. In this paper we classify low dimensional projective manifolds with degenerate secant varieties satisfying $c = 0$.

The main result of this paper is the following.

Theorem 0.2. *Suppose that* $\mathrm{Sec}X$ *is degenerate and of dimension* $2n$ *and that* $\dim \mathrm{Im}(\gamma) = 2(n-1)$.
If $n = 4$*, then* $(X, \mathcal{O}_X(1))$ *is one of the following.*
1) $(\mathbf{P}_{\mathbf{P}^l}(\mathcal{E}), H(\mathcal{E}))$*, where* $\mathcal{E} = \mathcal{O}(1)^{\oplus(4-l)} \oplus \mathcal{O}(2)$ $(l = 2, 3, 4)$*;*
2) $(\mathbf{P}_{\mathbf{P}^2}(\mathcal{E}), H(\mathcal{E}))$*, where* $\mathcal{E} = \mathcal{O}(1) \oplus T_{\mathbf{P}^2}$*.*
If $n = 5$*, then* $(X, \mathcal{O}_X(1))$ *is one of the following.*
1) $(\mathbf{P}_{\mathbf{P}^l}(\mathcal{E}), H(\mathcal{E}))$*, where* $\mathcal{E} = \mathcal{O}(1)^{\oplus(5-l)} \oplus \mathcal{O}(2)$ $(l = 2, 3, 4, 5)$*;*
2) $(\mathbf{P}_{\mathbf{P}^l}(\mathcal{E}), H(\mathcal{E}))$*, where* $\mathcal{E} = \mathcal{O}(1)^{\oplus(6-2l)} \oplus T_{\mathbf{P}^l}$ $(l = 2, 3)$*;*
3) $X \subset \mathbf{P}^N$ *is a linear section of* $G(1, \mathbf{P}^5) \subset \mathbf{P}^{14}$ *a section cut out by codimension* 3 *linear subspace of* $\mathbf{P}^{14}$*;*
4) $(\Sigma_{10}, \mathcal{O}(1))$*, where* Σ_{10} *is the adjoint manifold of the simple algebraic group of exceptional type* G_2 *and* $\mathcal{O}(1)$ *is the fundamental line bundle on it. (In other words* Σ_{10} *is the* 5*-dimensional Mukai manifold of genus* 10 *([Mu]).)*

If $\mathrm{Sec}X$ is degenerate and of dimension $2n$, then $n \geq 2$ by the Linear Normality Theorem. If $n = 3$, then T. Fujita ([F. Th. (2.1)]) showed that $(X, \mathcal{O}_X(1))$ is one of the following: $(\mathbf{P}_{\mathbf{P}^l}(\mathcal{O}(1)^{\oplus 3-l} \oplus \mathcal{O}(2)), H(\mathcal{O}(1)^{\oplus 3-\ell} \oplus \mathcal{O}(2)))$ where $(l = 2, 3)$, or $(\mathbf{P}(T_{\mathbf{P}^2}), H(T_{\mathbf{P}^2}))$.

Notation and conventions

We work over an algebraically closed field k of characteristic 0. We follow the notation and terminology of [H]. We use the word *manifold* to mean a smooth variety. For a manifold X, we denote by $\kappa(X)$ the Kodaira dimension of X. We use the word *line* to mean a smooth rational curve of degree 1. Given two distinct points x, y on $\mathbf{P}^N$, let $x * y$ denote the line joining them. For subsets X, Y of $\mathbf{P}^N$, let $X * Y$ be the closure of the union of all lines $x * y$ joining two distinct points $x \in X$ and $y \in Y$. For a vector bundle E of rank $e+1$ on a variety X, we define the i-th Segre class $s_i(E)$ of E by the formula $s_i(E) \cap \alpha = p_*(c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(1)^{e+i} \cap p^*\alpha)$ where α is a k-dimensional cycle modulo rational equivalence and $p : \mathbf{P}(E) \to X$ is the projection. We also define the total Segre class $s(E)$ to be $1 + s_1(E) + s_2(E) + \cdots$. The total Chern class $c(E) = 1 + c_1(E) + c_2(E) + \cdots$ is defined by the formula $c(E)s(E) = 1$. These definitions of $s_i(E)$ and $c_i(E)$ are the same as those of [Fl]. By abuse of notation, we simply write $s_n(E)$ for $\deg s_n(E)$ when $n = \dim X$. We denote also by $H(E)$ the tautological line bundle $\mathcal{O}_{\mathbf{P}(E)}(1)$ on $\mathbf{P}(E)$. For a linear system Λ, $\mathrm{Bs}\Lambda$ denotes the base locus of Λ. Let $[r]$ denote the greatest integer not greater than r for a real number r.

1. Preliminaries and Proof of Proposition 0.1

Let X be an n-dimensional nondegenerate closed submanifold in $\mathbf{P}^N$. Let B be the blowing-up of $X \times X$ along the diagonal Δ, and let $S_0 = \{(x, y, u) \in (X \times X \setminus \Delta) \times \mathbf{P}^N | x, y, \text{ and } u \text{ are collinear}\}$. Since $\mathbf{P}(\Omega_X)$ is the exceptional divisor of B, we can identify $X \times X \setminus \Delta$ with $B \setminus \mathbf{P}(\Omega_X)$. Thus S_0 can be identified with a closed submanifold of $(B \setminus \mathbf{P}(\Omega_X)) \times \mathbf{P}^N$. We define S to be the closure of S_0 in $B \times \mathbf{P}^N$. We call S the complete secant bundle of X. Let $p : S \hookrightarrow B \times \mathbf{P}^N \to B$ be the first projection and $\sigma : S \hookrightarrow B \times \mathbf{P}^N \to \mathbf{P}^N$ the second projection. Then $\mathrm{Sec}X = \mathrm{Im}(\sigma)$. For a point $u \in \mathrm{Sec}X$, let $\Sigma_u = \sigma(p^{-1}(p(\sigma^{-1}(u))))$, $Q_u = \Sigma_u \cap X$, and $\theta_u = \{x \in X | u \in T_xX\}$. We call Σ_u the secant cone, Q_u the secant locus, and θ_u the tangent locus, with respect to u. Let $\mathrm{Sm}(\mathrm{Sec}X)$ denote the smooth locus of $\mathrm{Sec}X$. Let $\gamma : \mathrm{Sm}(\mathrm{Sec}X) \to G(\dim \mathrm{Sec}X, \mathbf{P}^N)$ be the Gauss map of $\mathrm{Sm}(\mathrm{Sec}X)$. For a point $u \in \mathrm{Sm}(\mathrm{Sec}X)$, C_u denotes the closure of $\gamma^{-1}(\gamma(u))$ in $\mathrm{Sec}X$. We call C_u the contact locus of $T_u\mathrm{Sec}X$ with $\mathrm{Sec}X$. We fix and will use these notations in the following sections.

We first observe that for a general point $u \in \mathrm{Sec}X$ $\dim Q_u = 2n + 1 - \dim \mathrm{Sec}X$, $\Sigma_u = u * Q_u$, and C_u is a linear subspace in $\mathbf{P}^N$ (see, for example, [Z, Chap. 1, Th. 2.3 c)]). We also have $\Sigma_u \subseteq C_u$ for any general point $u \in \mathrm{Sm}(\mathrm{Sec}X)$ (see, for example, [Oh, Cor. 1.2]). Therefore $2n+2-\dim \mathrm{Sec}X = \dim \Sigma_u \leq \dim C_u$ and hence $\dim \mathrm{Im}(\gamma) \leq 2\dim \mathrm{Sec}X - 2n - 2$.

Now we give a proof of Proposition 0.1. Let $D'_u = \overline{\cup_{v\in C_u:\ \text{general}} Q_v}$. Then $C_u = \mathrm{Sec}D'_u$ and $1+2\dim D'_u = \dim Q_u + \dim C_u$ for any general point $u \in \mathrm{Sec}X$ (see [Oh, Lemma 1.4]). Let $X_u = \{x \in X | T_xX \subseteq T_u\mathrm{Sec}X\}$ for a point $u \in \mathrm{Sm}(\mathrm{Sec}X)$. Then $D'_u \subseteq X_u$ for any general point $u \in \mathrm{Sec}X$ by [Oh, Corollary 1.3]. Let $T(X_u, X) = \cup_{x\in X_u} T_xX$. Then $T(X_u, X) \subseteq T_u\mathrm{Sec}X$. Note that $T_u\mathrm{Sec}X \neq \mathbf{P}^N$ since $\mathrm{Sec}X$ is degenerate. Note also that $X \subseteq X_u * X$ and X is not contained in $T_u\mathrm{Sec}X$ since X is nondegenerate in $\mathbf{P}^N$ and $T_u\mathrm{Sec}X \neq \mathbf{P}^N$. Therefore $\dim X_u * X = \dim X_u + n + 1$ by [Z, Chap.1, Theorem 1.4]. Since $\dim X_u * X \leq \dim \mathrm{Sec}X$, we have $\dim D'_u \leq \dim \mathrm{Sec}X - n - 1$. Therefore we have $\dim C_u = 2n+2-\dim \mathrm{Sec}X + 2c$ for some integer c ($0 \leq c \leq \varepsilon$). This completes the proof.

Suppose that $\mathrm{Sec}X$ is degenerate in the following. Then we have the following proposition.

Proposition 1.1. *Assume that* $\dim \mathrm{Im}(\gamma) = 2(\dim \mathrm{Sec}X - n - 1)$. *Then the secant cone* Σ_u *is a linear subspace of* $\mathbf{P}^N$ *of dimension* $2n+2-\dim \mathrm{Sec}X$ *for any general point* $u \in \mathrm{Sec}X$. *Moreover the secant locus* Q_u *is a smooth hyperquadric in* Σ_u, *and the tangent locus* θ_u *is a smooth hyperplane section of* Q_u *for any general point* $u \in \mathrm{Sec}X$. *In particular* X *is rationally connected,* $\kappa(X) = -\infty$, *and* $h^i(\mathcal{O}_X) = 0$ *for all* $i > 0$.

Proof. The first statement follows immediately from the fact that $\Sigma_u \subseteq C_u$. For a proof of the second statement, note first that a linear subspace Σ_u contains Q_u as a hypersurface. Second note that X is not a hypersurface in $\mathbf{P}^N$ because $\mathrm{Sec}X$ is degenerate, so that the trisecant lemma [F, (1.6)] shows that Q_u is a hyperquadric in Σ_u. For the rest of the second statement, refer to the proof of Th. 3 in [F-R, p.964, l.15 – p.967, l.7], and make obvious adjustments. For the definition of rational connectedness, see [Ko-Mi-Mo, (2.2)]. Since general two points can be joined by a positive dimensional quadric Q_u, X is rationally connected. The rest of the assertion follows immediately from [M-M, Th. 1] and [Ko-Mi-Mo, (2.5.2)].

$\square$

By generalizing [F, Lemma (2.3)], where $n = 3$, to arbitrary dimension n, we have the following proposition.

Proposition 1.2. *Assume that* $\dim \mathrm{Im}(\gamma) = 2(\dim \mathrm{Sec}X - n - 1)$. *If* $\dim \mathrm{Sec}X = 2n$, *then* $K_X.Q_u = -n-1$ *for a general point* $u \in \mathrm{Sec}X$.

2. Proof of Theorem 0.2

In this section we give a proof of Theorem 0.2. First of all, we state a couple of Lemmas.

Lemma 2.1. *Let* $X \subseteq \mathbf{P}^N$ *be an* n*-dimensional projective manifold. Then* $\dim \mathrm{Sec}X \leq 2n$ *if and only if*

$$(\deg X)^2 - \sum_{j=0}^{n} \binom{2n+1}{j} c_1(\mathcal{O}_X(1))^j s_{n-j}(T_X) \cap [X] = 0.$$

For a proof, see, for example, [F, (1.5) and (1.7)].

Lemma 2.2. *Let* $X \subseteq \mathbf{P}^N$ *be an* n*-dimensional projective manifold,* $L = \mathcal{O}_X(1)$*, and assume that* $(X, L) \cong (\mathbf{P}_Y(\mathcal{E}), H(\mathcal{E}))$ *for some locally free sheaf* $\mathcal{E}$ *of rank* $n-m+1$ *on an* m*-dimensional projective manifold* Y*. Then*

$$\begin{aligned}
&(L^n)^2 - \sum_{j=0}^{n} \binom{2n+1}{j} c_1(L)^j s_{n-j}(T_X) \cap [X] \\
&= (s_m(\check{\mathcal{E}}))^2 - \sum_{j=0}^{n}\sum_{p=0}^{n-j}\sum_{l=0}^{p} \binom{2n+1}{j}\binom{-n+m-l-1}{p-l} s_{j+p-l-n+m}(\check{\mathcal{E}}) s_l(\check{\mathcal{E}}) s_{n-j-p}(T_Y) \\
&= \begin{cases} c_1(\mathcal{E})^2 - (2n+1)c_1(\mathcal{E}) - n(n+1)(g(Y)-1) & \text{if } m=1 \\[2ex] (L^2)^2 - (n^2+n+1)L^2 - (1/6)(2n+1)(n+1)nc_1(\mathcal{E})c_1(K_Y) \\ \quad - \binom{n+2}{4}(c_1(K_Y)^2 - c_2(T_Y)) - \binom{n+1}{2}c_1(\mathcal{E})^2 & \text{if } m=2. \end{cases}
\end{aligned}$$

If $Y = \mathbf{P}^1$, *then* $\dim \mathrm{Sec}X \leq 2n$ *if and only if* $\mathcal{E} \cong \mathcal{O}(1)^{\oplus n}$ *or* $\mathcal{O}(1)^{\oplus(n-1)} \oplus \mathcal{O}(2)$, *and under this equivalent condition we obtain* $\mathrm{Sec}X = \mathbf{P}(H^0(L))$.

Proof. We obtain these results by calculation and by Lemma 2.1.

□

We get the following lemma by calculation.

Lemma 2.3. *Let C be a smooth complete curve of genus g and $\mathcal{E}$ a vector bundle of rank n on C. Let X be a smooth irreducible effective Cartier divisor of $\mathbf{P}(\mathcal{E})$ such that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(X) \cong H(\mathcal{E})^{\otimes 2} \otimes \pi^* M$ for some line bundle M of degree m on C, where $\pi : \mathbf{P}(\mathcal{E}) \to C$ is the projection. Let $L = H(\mathcal{E}) \otimes \mathcal{O}_X$ and $d = L^n$. Then*

$$d^2 - \sum_{j=0}^{n} \binom{2n+1}{j} c_1(L)^j s_{n-j}(T_X) \cap [X] = d^2 - 4nd - m - 4n^2(g-1).$$

In the rest of this section, let X be an n-dimensional nondegenerate projective manifold in $\mathbf{P}^N$ with degenerate secant variety $\mathrm{Sec}X$ of dimension $2n$, and let $L = \mathcal{O}_X(1)$.

Lemma 2.4. *If* $\dim \mathrm{Im}(\gamma) = 2(n-1)$, *then we have* $\mathrm{Bs}|K_X + (n-1)L| = \emptyset$ *for all $n \geq 3$.*

Proof. If $\mathrm{Bs}|K_X + (n-1)L| \neq \emptyset$, then $(X, L) \cong (\mathbf{P}_C(\mathcal{E}), H(\mathcal{E}))$ for some vector bundle $\mathcal{E}$ of rank n on a smooth curve C by [S-V, (0.1)] since $\mathrm{Sec}X \neq \mathbf{P}^N$ and $n \geq 3$. Because X is rationally connected by Proposition 1.1, so is C, and hence $C = \mathbf{P}^1$. Therefore $\mathrm{Sec}X = \mathbf{P}(H^0(L))$ by Lemma 2.2, which contradicts the hypothesis that $\mathrm{Sec}X \neq \mathbf{P}^N$.

□

In the following, we always assume that $n \geq 4$. Let $\phi : X \to \mathbf{P}(H^0(K_X + (n-1)L))$ be the adjunction map, and let $\phi = s \circ r$ ($r : X \to Y, s : Y \to \mathbf{P}(H^0(K_X + (n-1)L))$) be the Stein factorization of ϕ.

Theorem 2.5. *If* $\dim \mathrm{Im}(\gamma) = 2(n-1)$, *then there are the following possibilities.*

(1) *Y is a smooth rational projective surface, s is a closed immersion induced by $|K_Y + c_1(\mathcal{E})|$, $(K_Y + c_1(\mathcal{E}))^2 \le (n-3)^2$, $(X, L) \cong (\mathbf{P}_Y(\mathcal{E}), H(\mathcal{E}))$ for some vector bundle of rank $n-1$ on Y, and*

$$\begin{aligned}(L^2)^2 - (n^2+n+1)L^2 - (1/6)(2n+1)(n+1)nK_Y c_1(\mathcal{E}) \\ - \binom{n+2}{4}(K_Y^2 - c_2(T_Y)) - \binom{n+1}{2}c_1(\mathcal{E})^2 &= 0.\end{aligned}$$

Furthermore $(K_Y + c_1(\mathcal{E}))^2 \ge 5$ unless $(Y, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}(1)^{\oplus l} \oplus \mathcal{O}(2))$ or $(\mathbf{P}^2, \mathcal{O}(1)^{\oplus l-1} \oplus T_{\mathbf{P}^2})$ where $l = 2$ or 3;

(2) *Y is an n-dimensional rationally connected smooth projective variety, and r is the blowing-up of Y at a finite point set, and $L = r^*M - \Sigma E_i$ (E_i: exceptional divisors) for some ample line bundle M on Y, and $K_Y + (n-1)M$ is very ample. Moreover $(K_Y + (n-2)M)|_{r(Q_u)} \le n-5$ for a general point $u \in \mathrm{Sec}X$, and $K_Y + (n-2)M$ is nef if $n \ge 5$, and $(Y, M) \cong (\mathbf{P}^4, \mathcal{O}(2))$ if $n = 4$.*

Proof. First note that Y is rationally connected because so is X by Proposition 1.1. Since $(K_X + (n-1)L)|_{Q_u} = n-3 \ge 1$ by Proposition 1.2, we have $\dim \phi(X) \ge 1$. Assume that $\dim Y = 1$. Then r is a quadric fibration over Y by [S-V, (0.2)] and a contraction morphism of an extremal ray by [B-S-W, Th. (3.2.6)]. Therefore we can show, by the same argument as that in [Fb, p.100, l.10–l.27], that there exist a locally free sheaf $\mathcal{E}$ of rank $n+1$ on Y and a line bundle M on Y such that X is a Cartier divisor of $\mathbf{P}(\mathcal{E})$, that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(X) \cong H(\mathcal{E})^{\otimes 2} \otimes \pi^*M$, and that $L \cong H(\mathcal{E}) \otimes \mathcal{O}_X$, where $\pi : \mathbf{P}(\mathcal{E}) \to Y$ is the projection and $r = \pi|_X$. Since Y is rationally connected, Y is a smooth rational curve. Let $d = L^n$, $e = \deg c_1(\mathcal{E})$, and $m = \deg M$. Then we have $n - 3 = (K_X + (n-1)L)|_{Q_u} = (K_{\mathbf{P}(\mathcal{E})} + (n+1)H(\mathcal{E}) + \pi^*M)|_{Q_u} = \pi^*(\mathcal{O}_{\mathbf{P}^1}(e + m - 2))|_{Q_u}$, and hence $e + m \le n - 1$. On the other hand, $(L^n)^2 - \sum_{j=0}^{n} \binom{2n+1}{j} c_1(L)^j s_{n-j}(T_X) \cap [X] = (d-2n)^2 - m$ by Lemma 2.3, and therefore $\dim \mathrm{Sec}X = 2n$ implies that $(d-2n)^2 = m$. Let m' be a nonnegative integer such that $m = m'^2$. Then we have $e = n - (m'(m' \mp 1)/2)$ because

$d = 2e + m$. It follows from $e + m \le n - 1$ that $m'^2 \pm m' + 2 \le 0$, which is however a contradiction. Hence $\dim Y \ge 2$.

If $\dim Y = 2$, then Y is a smooth projective surface and $(X, L) \cong (\mathbf{P}_Y(\mathcal{E}), H(\mathcal{E}))$ for some vector bundle of rank $n-1$ on Y by [S-V, (0.2)]. Furthermore $K_Y + c_1(\mathcal{E})$ is very ample by [L-M, Th. B and Th. C] because $H(\mathcal{E})$ is very ample, so that s is a closed immersion. Note also that Y is rational since $\dim Y = 2$. For general three points $x, y, z \in X$, there exist two points $u, v \in \mathrm{Sec}X$ such that $x, y \in Q_u$ and $y, z \in Q_v$ and u, v are in general position. Since Q_u and Q_v are algebraically equivalent, so is $r_*(Q_u)$ and $r_*(Q_v)$. Since $r(y) \in r(Q_u) \cap r(Q_v)$ and $r(Q_u) \ne r(Q_v)$, we get $r(Q_u)^2 \ge 1$. We also have $(K_Y + c_1(\mathcal{E}))^2 r(Q_u)^2 \le ((K_Y + c_1(\mathcal{E}))|_{r(Q_u)})^2 \le (n-3)^2$ by the Hodge index theorem. Therefore $(K_Y + c_1(\mathcal{E}))^2 \le (n-3)^2$. If $(K_Y + c_1(\mathcal{E}))^2 = 1$, then $Y = \mathbf{P}^2$ and $c_1(\mathcal{E}) \cong \mathcal{O}(4)$. If the rank of $\mathcal{E}$ is 3, then $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)$ or $\mathcal{O}(1) \oplus T_{\mathbf{P}^2}$ by [E1]. In this case $\dim \mathrm{Sec}X = 8$ and $h^0(L) \ge 11$. The condition that $\dim C_u = 2$ is also satisfied. If the rank of $\mathcal{E}$ is 4, then $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 4}$. Hence $(X, L) \cong (\mathbf{P}^2 \times \mathbf{P}^3, \mathcal{O}(1) \otimes \mathcal{O}(1))$, which, however, does not satisfy the condition that $\dim \mathrm{Sec}X = 10$. If $(K_Y + c_1(\mathcal{E}))^2 \ge 2$, then $\mathrm{rk}\mathcal{E} \ge 4$ since $(n-3)^2 \ge 2$. If $(K_Y + c_1(\mathcal{E}))^2 = 2$, then $Y = \mathbf{P}^1 \times \mathbf{P}^1$ and $c_1(\mathcal{E}) \cong \mathcal{O}(3) \otimes \mathcal{O}(3)$. This contradicts the ampleness of $\mathcal{E}$. If $(K_Y + c_1(\mathcal{E}))^2 = 3$, then Y is either a cubic surface in $\mathbf{P}^3$ or $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(2))$ by [Fb, (17.2)]. If Y is cubic, then $c_1(\mathcal{E})|_l = 2$ for every l, one of the 27 lines on Y, which is a contradiction. For the scroll we have $c_1(\mathcal{E})|_f = 3$ where f is any fiber of the scroll, and this also contradicts the ampleness of $\mathcal{E}$. Suppose that $(K_Y + c_1(\mathcal{E}))^2 = 4$. Then Y is either a del Pezzo surface of degree 4, a scroll $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(3))$, a scroll $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(2))$, or a Veronese surface $\mathbf{P}^2 \subset \mathbf{P}^5$ by [Fb, (17.3)] since $\kappa(Y) = -\infty$. If Y is a del Pezzo surface, then $c_1(\mathcal{E})|_l = 2$ for any exceptional divisor l of Y, which is a contradiction. For the scrolls we have $c_1(\mathcal{E})|_f = 3$ where f is any fiber of the projection $Y \to \mathbf{P}^1$, and this is also a contradiction. If Y is a Veronese surface, we obtain $c_1(\mathcal{E}) \cong \mathcal{O}(5)$. If $\mathcal{E}$ is an ample vector bundle of rank 4, we have $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2)$ or $\mathcal{O}(1)^{\oplus 2} \oplus T_{\mathbf{P}^2}$ by [E2, Th. 5.1]. For both bundles, we have $\dim \mathrm{Sec}X = 10$ and $h^0(L) \ge 14$. The condition that $\dim C_u = 2$ is also satisfied.

If the rank of $\mathcal{E}$ is 5, then $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 5}$. Hence $(X, L) \cong (\mathbf{P}^2 \times \mathbf{P}^4, \mathcal{O}(1) \otimes \mathcal{O}(1))$, which however does not satisfy the condition that $\dim \mathrm{Sec} X = 12$. The rest of the assertion in the case $\dim Y = 2$ follows from Lemma 2.2.

If $\dim Y > 2$, then $\dim Y = n$ by [S-V, (0.2)], and Y is smooth, r is the blowing-up of Y at a finite point set, and $L = r^*M - \Sigma E_i$ (E_i: exceptional divisors) for some ample line bundle M on Y by [S-V, (0.3)]. Moreover $K_Y + (n-1)M$ is very ample by [S-V, Th. (2.1)]. Since $n-3 = (K_X + (n-1)L)|_{Q_u} = (K_Y + (n-1)M)|_{r(Q_u)}$ and $M|_{r(Q_u)} \geq L|_{Q_u} = 2$, we obtain $(K_Y + (n-2)M)|_{r(Q_u)} \leq n-5$. If $n \geq 5$, then we know that $K_Y + (n-2)M$ is nef by [Fb, (11.8)], taking account of the fact that $K_Y + (n-1)M$ is ample. If $n = 4$, then $(K_Y + 2M)|_{r(Q_u)} \leq -1$ and hence $K_Y + 2M$ is not nef. Therefore $(Y, M) \cong (\mathbf{P}^4, \mathcal{O}(2))$ by [Fb, (11.8)] because $K_Y + 3M$ is ample.

□

Now we give a proof of Theorem 0.2.

Proof of Theorem 0.2. Suppose first that $n = 4$. Then by Theorem 2.5, $\dim Y = 2$ or 4. If $\dim Y = 2$, then (X, L) is isomorphic to $(\mathbf{P}_{\mathbf{P}^2}(\mathcal{E}), H(\mathcal{E}))$, where $\mathcal{E} = \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)$ or $\mathcal{O}(1) \oplus T_{\mathbf{P}^2}$ by Theorem 2.5 (1). If $\dim Y = 4$, then $(Y, M) \cong (\mathbf{P}^4, \mathcal{O}(2))$ by Theorem 2.5 (2). If r is an isomorphism, then $(X, L) \cong (\mathbf{P}^4, \mathcal{O}(2))$. If r is not an isomorphism, then $(X, L) \cong (\mathbf{P}_{\mathbf{P}^3}(\mathcal{O}(1) \oplus \mathcal{O}(2))), H(\mathcal{O}(1) \oplus \mathcal{O}(2)))$. These polarized manifolds satisfy the assumptions that $\dim \mathrm{Sec} X = 10$ and that $\dim C_u = 2$.

Suppose in the following that $n = 5$. Then by Theorem 2.5, $\dim Y = 2$ or 5. If $\dim Y = 2$, then $(X, \mathcal{O}_X(1)) \cong (\mathbf{P}_{\mathbf{P}^2}(\mathcal{E}), H(\mathcal{E}))$, where $\mathcal{E} = \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2)$ or $\mathcal{O}(1)^{\oplus 2} \oplus T_{\mathbf{P}^2}$ by Theorem 2.5 (1). These two polarized manifolds satisfy the hypotheses that $\dim \mathrm{Sec} X = 10$ and that $\dim C_u = 2$.

Let us consider the case (2) of Theorem 2.5. Now we have $K_Y + 3M$ is nef and therefore $(K_Y + 3M)|_{r(Q_u)} = 0$. Since general two points can be joined by $r(Q_u)$, this implies that $K_Y + 3M = 0$ by [K-M-M, Th. 3-1-1 and Th. 3-2-1]. If M is not the fundamental line bundle, then $(Y, M) \cong (\mathbf{P}^5, \mathcal{O}(2))$. If r is an isomorphism, then $(X, L) \cong (\mathbf{P}^5, \mathcal{O}(2))$. If r is not an isomorphism, then

$(X, L) \cong (\mathbf{P}_{\mathbf{P}^4}(\mathcal{O}(1) \oplus \mathcal{O}(2)), H(\mathcal{O}(1) \oplus \mathcal{O}(2)))$. These two polarized manifolds satisfy the hypotheses that $\dim \mathrm{Sec}X = 10$ and that $\dim C_u = 2$. Assume that M is the fundamental line bundle of Y. Then Y is a Fano manifold of coindex 3 and M is very ample because $K_Y + 4M$ is very ample, so that (Y, M) satisfies the hypothesis (ES) of [M]. If $B_2(Y) \geq 2$, then (Y, M) is either $(\mathbf{P}^2 \times Q^3, \mathcal{O}(1) \otimes \mathcal{O}(1))$, $(\mathbf{P}(T_{\mathbf{P}^3}), H(T_{\mathbf{P}^3}))$, or $(\mathbf{P}_{\mathbf{P}^3}(\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)), H(\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)))$ by [M, Th. 7]. Thus for every point $y \in Y$ there exists a line passing through y, which implies that r is an isomorphism by the ampleness of L. Since the secant variety of the manifold $(\mathbf{P}^2 \times Q^3, \mathcal{O}(1) \otimes \mathcal{O}(1))$ is 11-dimensional by [Z, Chap.3, Th. 1.6], (X, L) is either $(\mathbf{P}(T_{\mathbf{P}^3}), H(T_{\mathbf{P}^3}))$ or $(\mathbf{P}_{\mathbf{P}^3}(\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)), H(\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)))$. These polarized manifolds satisfy the hypothesis that $\dim C_u = 2$ and the condition that $\dim \mathrm{Sec}X = 10$.

Next let us consider the case that $B_2(Y) = 1$. Note that $g(Y, M) + 4 = h^0(M) \geq h^0(L) \geq \dim \mathrm{Sec}X + 2 = 12$.

Suppose that r is an isomorphism. Then we get $g(X, L) \geq 8$. Thus $X \subset \mathbf{P}^N$ is either a complete intersection of $G(1, \mathbf{P}^5) \subset \mathbf{P}^{14}$ and a codimension 3 linear subspace of $\mathbf{P}^{14}$ or the G_2 adjoint manifold $\Sigma_{10} \subset \mathbf{P}^{13}$ by [Mu, Th. 2], because $\mathrm{Sec}\Sigma_9 = \mathbf{P}^{13}$ by [K] and therefore the dimension of the secant variety of a general hyperplane section of Σ_9 is eleven, and all smooth hyperplane section of Σ_9 are isomorphic. A smooth complete intersection of $G(1, \mathbf{P}^5) \subset \mathbf{P}^{14}$ with a codimension 3 linear subspace of $\mathbf{P}^{14}$ satisfies the condition that $\dim \mathrm{Sec}X = 10$. The G_2 adjoint manifold $\Sigma_{10} \subset \mathbf{P}^{13}$ satisfies the assumptions that $\dim \mathrm{Sec}X = 10$ and that $\dim C_u = 2$ by [K-O-Y].

Suppose that r is not an isomorphism. Then $h^0(M) \geq h^0(L) + 1$ so that $g(Y, M) \geq 9$. Therefore Y is either a hyperplane section of $\Sigma_9 \subset \mathbf{P}^{13}$, or $\Sigma_{10} \subset \mathbf{P}^{13}$. For each point $y \in \Sigma_9$, Σ_9 contains a rational curve C passing through y such that $-K|_C \leq 7$ by [Ko, Chap.V, Th. 1.6.1]. Since the index of the Fano manifold Σ_9 is four, we have $-K|_C = 4$, and hence C is a line in Σ_9. Let $f : \mathbf{P}^1 \to C \hookrightarrow \Sigma_9$ be the normalization of C and let $f(0) = y$. Denote by ι the restriction of f to $\{0\}$. Then $\dim_{[f]} \mathrm{Hom}(\mathbf{P}^1, \Sigma_9; \iota) \geq -K|_C = 4$. On the other hand, we have $\dim \mathrm{Aut}(\mathbf{P}^1; 0) = 2$. Thus Σ_9 contains a closed cone of dimension ≥ 3 with vertex y. A hyperplane section Y of Σ_9 therefore contains a line passing through y for each point $y \in Y$. This contradicts the ampleness of L. For the G_2 adjoint variety $\Sigma_{10} \subset \mathbf{P}^{13}$ it follows from [Ko, Chap.V, Th. 1.15] that there exists a line C (i.e., $M|_C = 1$) on Σ_{10}. Hence for every point $y \in \Sigma_{10}$ there exists a line passing through y on Σ_{10}, which contradicts the ampleness of L.

$\square$

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