MULTIPLICITY ONE RESULTS FOR UNITARY GROUPS

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1. Introduction

Among the many modern techniques employed in the study of diophantine equations, the use of ζ-functions and L-functions, especially those associated to automorphic forms, seems to be crucial.

On the other hand, there is a standard way of passing from classical cusp forms to cusp forms defined on $GL_2(\mathbf{A}) = G(\mathbf{A})$, where $\mathbf{A}$ denotes the *adele ring* of $\mathbb{Q}$. We denote by $\mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbf{A}))$ (or simply $\mathcal{A}_0$ for short) the space of cusp forms on $G(\mathbf{A})$.

It is well-known that $G(\mathbf{A})$ can be written as the restricted direct product of the groups $G_p = GL_2(\mathbb{Q}_p)$ with respect to the compact groups K_p, where K_p equals $O(2)$ for the archimedean prime, and $K_p = GL_2(\mathbb{Z}_p)$ for finite p.

It is also known that any irreducible admissible representation π of $G(\mathbf{A})$ is *factorizable*, i.e. it can be written as a (restricted) tensor product $\pi = \otimes_p \pi_p$ where π_p is an irreducible admissible representation of G_p, unramified for almost all p.

There are two important results in the theory of cuspidal representations of $GL_2(\mathbf{A})$:

(1) *Multiplicity One*: Let (π, V), (π', V') be two irreducible admissible subrepresentations of $\mathcal{A}_0$. If $\pi \cong \pi'$, then $V = V'$.

(2) *Strong Multiplicity One*: Let π and π' be as in (**1**), and assume that $\pi = \otimes_p \pi_p$, $\pi' = \otimes_p \pi'_p$ and that $\pi_p \cong \pi'_p$ for almost all p. Then $V = V'$.

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We point out that the results described above are actually valid for $GL_n(\mathbf{A})$, for any positive integer n and $\mathbb{Q}$ replaced by any number field.

The main goal of this work is to establish multiplicity one results for a global unitary group G of arbitrary dimension n associated to a quadratic extension of global fields E/F. If v is a place of F then G_v is known to be isomorphic to either $GL_n(F_v)$, in case v splits in E, or to the group $U(\Phi, E_v)$ (to be defined below), if v remains prime in E. Thus locally we always have to consider these two cases although in what follows we concentrate on $U(\Phi, E_v)$.

There are three main ingredients in the proofs of the theorems above. The first is the existence and uniqueness of the *Whittaker Model*, the second is the so-called "small lemma" and the third is the Jacquet-Langlands formula which makes possible the recovery of an automorphic form from its Fourier coefficients and a good portion of this work is understanding and recovering these ingredients in the unitary case.

The two central results of this paper are: multiplicity one for special automorphic representations of $R(\mathbf{A})$ (to be defined below), and a strong multiplicity one type of result for automorphic representations of $G(\mathbf{A})$, namely, if an irreducible discrete representation $\pi = \otimes\pi_v$ is locally isomorphic to a Weil representation at all but a finite number of finite places of F then π in fact equals a global Weil representation.

2. Local set up

Let F be a local, non-archimedean field, and E a quadratic extension of F. Conjugation with respect to E/F will be denoted by a bar. Let $\mathcal{O}_F$ (resp. $\mathcal{O}_E$) be the ring of integers in F (resp. E) and $p \neq 2$ a prime in F. Let $\omega_{E/F}$ be the character of F^* of order two associated to E/F by class field theory. Let ξ be an element of E^* such that $Tr_{E/F}(\xi) = 0$.

For $n = 2m + 1$ we define

$$\Phi_n = \begin{pmatrix} & & & & & & 1 \\ & & & & & \cdot & \\ & & & & 1 & & \\ & & & \xi & & & \\ & & -1 & & & & \\ & \cdot & & & & & \\ -1 & & & & & & \end{pmatrix}$$

(with ξ in the center) and for $n = 2m$, we define Φ_n similarly by deleting ξ. Let $U(\Phi_n)$ be the unitary group associated to the skew-Hermitian form Φ_n, i.e., $U(\Phi_n) = \{g \in GL_n(E) : {}^t\bar{g}\Phi_n g = \Phi_n\}$.

Throughout this section, G denotes either $U(\Phi_n)$ or $GL_n(F)$, Z denotes the center of G and P denotes the maximal parabolic subgroup of G defined by $P = \{g = (g_{i,j}) \in G : g_{i,1} = g_{n,j} = 0,\ i = 2, ..., n,\ j = 1, ..., n-1\}$. If (W, A) is a symplectic space over F, A non-degenerate, then the *Heisenberg group* $H(W)$ associated to (W, A) is the central extension of W consisting of pairs $[w, t]$, $w \in W$, $t \in F$, with group law defined by $[w, t][w', t'] = [w + w', t + t' + \frac{1}{2}A(w, w')]$. The center of $H(W)$ is $\{[0, t],\ t \in F\}$ and is isomorphic to F.

Let N be the unipotent radical of G. If $G = U(\Phi_n)$, N consists of matrices of the form

$$[w, t] = \begin{pmatrix} 1 & {}^t\bar{w}\Phi_{n-2} & d(w, t) \\ & I_{n-2} & w \\ & & 1 \end{pmatrix}$$

where $w \in E^{n-2}$, $t \in F$, $d(w, t) = \frac{1}{2}{}^t\bar{w}\Phi_{n-2}w + t$, and I_{n-2} denotes, as usual, the $(n-2)\times(n-2)$ identity matrix. In this case N is isomorphic to $H(W)$ with $W = E^{n-2}$ (as a vector space over F) and $A(v, w) = Tr_{E/F}({}^t\bar{v}\Phi_{n-2}w)$ (we may also denote (W, A) by (W_{n-2}, A_{n-2}) if we need to be explicit). If $G = GL_n(F)$, then N consists of matrices of the form

$$[u_1 \oplus u_2, t] = \begin{pmatrix} 1 & {}^tu_2 & d(u_1 \oplus u_2, t) \\ & I_{n-2} & u_1 \\ & & 1 \end{pmatrix}$$

where $u_1, u_2 \in F^{n-2}$, $t \in F$, $d(u_1 \oplus u_2, t) = \frac{1}{2}{}^tu_2u_1 + t$, and N is now isomorphic to $H(W)$ with $W = F^{n-2} \oplus F^{n-2}$ and $A(u, v) = {}^tvu - {}^tuv$.

In both cases, let U denote the center of N, and let R be the centralizer of U in P. Let $d(a_1, ..., a_n)$ denote a diagonal matrix with entries $a_1, ..., a_n$.

Let T denote the torus of elements of the form $d(a,1,...,1,\bar{a}^{-1})$, with $a \in E^*$ for the $U(\Phi_n)$ case, and $d(a,1,...,1,b)$, with $a,b \in F^*$ for the $GL_n(F)$ case. Let S' be the subgroup of matrices of the form $d(1,u,1) = \operatorname{diag}(1,u,1)$ with $u \in U(\Phi_{n-2})$ or $GL_{n-2}(F)$, in each case. Let $S = ZS'$ and note that $R = SN = ZS'N$.

Fix a non-trivial additive character ψ of F. Given a Heisenberg group $H(W)$ by the Stone-von Neumann theorem ([MVW], p.28), there is a unique isomorphism class of irreducible admissible representations of $H(W)$ on which the center acts by ψ. This representation is called the *ψ-representation* of $H(W)$ and we will denote by τ_ψ either the isomorphism class or any particular realization of it.

It is well known (see for instance [GR1]) that the choice of a polarization, i.e., a direct sum decomposition $W = X \oplus X^*$ where X and X^* are maximal isotropic subspaces of W, yields a realization of τ_ψ which is called the *Schrödinger model.* The space of τ_ψ is the Schwartz space $\mathcal{S}(X^*)$, of smooth, compactly supported functions on X^*, and for $x \in X$, $x^*, y^* \in X^*$, $t \in F$ and $f \in \mathcal{S}(X^*)$ we have

$$(\tau_\psi([x \oplus x^*, t])f)(y^*) = \psi(\frac{1}{2}A(x^*,x) + A(y^*,x) + t)f(x^* + y^*).$$

Let $S_p(W)$ be the symplectic group of (W,A). An element $g \in S_p(W)$ acts on $H(W)$ by the rule $g.[w,t] = [g(w),t]$, which clearly preserves the center $\{[0,t] : t \in F\}$ pointwise. By the Stone-von Neumann theorem, this action fixes the isomorphism class of τ_ψ and hence gives rise to a projective representation of $S_p(W)$ on the space V_ψ of τ_ψ. We define the *metaplectic group* $M_p(W)$ as the group of pairs (g, M_g) where M_g is an operator on V_ψ such that $M_g \tau_\psi(h) {M_g}^{-1} = \tau_\psi(g(h))$ for $h \in H(W)$. By definition, $M_p(W)$ is a central extension of $S_p(W)$ by $\mathbb{C}^*$ and the projective representation of $S_p(W)$ lifts to an ordinary representation ω_ψ of $M_p(W)$ on V_ψ, defined by $(g, M_g) \mapsto M_g$. called the *metaplectic representation associated to ψ.*

There is a standard way of imbedding an unitary group into the symplectic group $S_p(W)$ for an appropriate W, called the *Howe pairing for unitary groups.* Furthermore, if we consider the Howe pairing for $(U(\Phi_n), U(1))$ (where $U(1)$

denotes the unitary group in one variable, which is isomorphic to E^1, the elements of E of norm 1) then the covering $M_p(W) \to S_p(W)$ is known to split over the image of $U(\Phi_n)$ in $S_p(W)$. The choice of lifting $U(\Phi_n) \to M_p(W)$ is unique up to a character of $U(\Phi_n)$. The choice of such character can actually be shown (Cf. [GR2], p. 457) to be equivalent to a choice of a character γ of E^* whose restriction to F^* equals $\omega_{E/F}$, the character of F^* of order two associated to E/F by class field theory. The restriction of a metaplectic representation of $M_p(W)$ to $U(\Phi_n)$, for some choice of lifting, will be called an *oscillator representation* of $U(\Phi_n)$ denoted by $\omega(\gamma, \psi)$, ψ the fixed additive character of F, γ the character of E^* associated to the choice of lifting $U(\Phi_n) \to Mp(W)$. Any two oscillator representations of $U(\Phi_n)$ differ by a twist by a character of this group.

Finally, for χ a character of Z, let $\omega(\gamma, \psi, \chi)$ be the subrepresentation of $\omega(\gamma, \psi)$ on which Z acts by χ. As a special case of *Howe's conjecture* (Cf. "Theoreme principal", p. 69 of [MVW]) we get that $\omega(\gamma, \psi, \chi)$ is a non-zero, irreducible representation of $U(\Phi_n)$. It is called a *Weil representation* of $U(\Phi_n)$.

The construction of Weil representations for the case $G = GL_n(F)$ is more standard (Cf. for instance [GR1], p. 12), and we again parametrize these representations in the form $\omega(\gamma, \psi, \chi)$, where γ and χ are now characters of F^*. By Theorem(4.2) on [Z] (p. 184) we have that these are irreducible representations of $GL_n(F)$.

3. Models

Suppose that $W = W_1 \oplus W_2$, $A = A_1 \oplus A_2$, where A_i is symplectic on W_i and $W_1 = X_1 \oplus X_1^*$ is a polarization for W_1. Then the Schrödinger model for τ_ψ^1 (the ψ-representation of $H(W_1)$) is realized on the Schwartz space $\mathcal{S}(X_1^*)$, of smooth, compactly supported functions on X_1^*.

If $\mathcal{F}$ is any model for τ_ψ^2 (the ψ-representation of $H(W_2)$) then the ψ-representation of $H(W)$ can be realized on the space $\mathcal{S}(X_1^*) \otimes \mathcal{F}$, which we can identify with $\mathcal{S}(X_1^*, \mathcal{F})$, the space of Schwartz functions on X_1^* with values in $\mathcal{F}$. This realization is called a *mixed model* ([MVW], p.40-42).

3.1. Mixed model

Let (W, A) be the F-vector space underlying E^n, and let A be the alternating form defined by $A(v,w) = Tr_{E/F}({}^t\bar{v}\Phi_n w)$. We consider the following decomposition of W: let $W_{1,n}$ be the F-vector space underlying $V_{1,n} = \{{}^t(x_1, 0, \ldots, 0, x_n) : x_1, x_n \in E\}$. On $W_{1,n}$ we put the symplectic form $A_{1,n}$, the restriction of A to $W_{1,n}$, i.e., $A_{1,n}({}^t(x_1, 0, \ldots, 0, x_n), {}^t(y_1, 0, \ldots, 0, y_n)) = Tr_{E/F}(-\bar{x}_n y_1 + \bar{x}_1 y_n)$.

Let now $V_0 = \{{}^t(0, x_2, \ldots, x_{n-1}, 0) : x_j \in E\}$. and let W_0 be the F-vector space underlying V_0. Let A_0 denote the restriction of the form A to W_0. Denote the ψ-representation of $H(W_{1,n})$ by $\tau_\psi^{1,n}$ and the ψ-representation of $H(W_0)$ by τ_ψ^0.

For a polarization of $W_{1,n}$ we take $W_{1,n} = W_1 \oplus W_n$ where W_1 and W_n are the F-vector spaces underlying $V_1 = \{{}^t(x, 0, ..., 0) : x \in E\}$ and $V_n = \{{}^t(0, ..., 0, z) : z \in E\}$, respectively.

This is a complete polarization of $W_{1,n}$ and we can realize a Schrödinger model for $\tau_\psi^{1,n}$ on $\mathcal{S}(W_n) \cong \mathcal{S}(E)$. Therefore, if $\mathcal{F}$ is any model for τ_ψ^0, we get a realization of τ_ψ on $\mathcal{S}(E) \otimes \mathcal{F} \cong \mathcal{S}(E, \mathcal{F})$ by defining

$$\tau_\psi([{}^t(x, \mathbf{y}, z), t])(f \otimes v) = \tau_\psi^{1,n}([{}^t(x, \mathbf{0}, z), t])f \otimes \tau_\psi^0([{}^t(0, \mathbf{y}, 0), 0])v$$

for all $f \in \mathcal{S}(E)$, for all $v \in \mathcal{F}$.

Since $\tau_\psi^{1,n}$ was realized on the Schrödinger model $\mathcal{S}(E)$, we have that

$$(\tau_\psi^{1,n}([{}^t(x, \mathbf{0}, z), t])f)(e) = \psi(1/2Tr_{E/F}(-\bar{z}x) + Tr_{E/F}(-\bar{e}x) + t)f(z+e)$$

(note the identification $\mathcal{S}(W_n) \equiv \mathcal{S}(E)$).

Finally, identifying $\mathcal{S}(E) \otimes \mathcal{F}$ with $\mathcal{S}(E, \mathcal{F})$ for any $\varphi \in \mathcal{S}(E, \mathcal{F})$ we get that

$$\begin{aligned}(\tau_\psi([{}^t(x, \mathbf{y}, z), t])\varphi)(e) &= \psi(1/2Tr_{E/F}(-\bar{z}x) + Tr_{E/F}(-\bar{e}x) + t) \\ &\quad \tau_\psi^0([{}^t(0, \mathbf{y}, 0), 0])(\varphi(z+e)) \ .\end{aligned}$$

It is easy to check that if $g \in U(\Phi_n)$, then $A(gv, gw) = A(v, w)$, for all $v, w \in W$ (where gv denotes usual matrix multiplication) i.e., $U(\Phi_n) \subset Sp(W)$.

Hence, if $\omega^n(\gamma,\psi)$ denotes a realization of an oscillator representation of $U(\Phi_n)$ on $\mathcal{S}(E,\mathcal{F})$, we have the following formulas for the action of the operators $\omega^n(\gamma,\psi)(g)$, for $g \in U(\Phi_n)$:

Proposition 3.1.1. *For $d(1,u,1) \in S'$ one has*
(1) $(\omega^n(\gamma,\psi)(d(1,u,1)\varphi)(e) = \omega^{n-2}(\gamma,\psi)(u)(\varphi(e))$ *for all* $\varphi \in \mathcal{S}(E,\mathcal{F})$. *The following formulas also hold:*
(2) $(\omega^n(\gamma,\psi)(d(a,1,...,1,\bar{a}^{-1}))\varphi)(e) = \gamma(a)|N_{E/F}(a)|^{1/2}\varphi(\bar{a}e)$ *for all* $a \in E^*$, *where γ is the character of E^* showing up in $\omega^n(\gamma,\psi)$ (associated to the choice of lifting used to realize $\omega^n(\gamma,\psi)$);*
(3) $(\omega^n(\gamma,\psi)([0,t])\varphi)(e) = \psi(tN_{E/F}(e))\varphi(e)$ *for all* $t \in F$;
(4) $(\omega^n(\gamma,\psi)([\mathbf{c},0])\varphi)(e) = \tau^0_\psi([\begin{pmatrix} 0 \\ -e\mathbf{c} \\ 0 \end{pmatrix},0])(\varphi(e))$ *for all* $c \in E^{n-2}$.

Proof. It is necessary to check the commutation relation

$$\omega^n(\gamma,\psi)(g)\tau_\psi(h)\omega^n(\gamma,\psi)(g)^{-1} = \tau_\psi(g.h)$$

for all $h \in H(W)$, for each of the formulas above, but this is accomplished with some simple computations. Observe that this already establishes formulas **(3)** and **(4)**, since the restriction of an oscillator representation to N is unique. Formula(**1**) in fact shows the inductive way that the mixed model is constructed, passing from dimension $n-2$ to dimension n. We have:

$$(\omega^n(\gamma,\psi)(d(1,u,1)(\tau_\psi([\begin{pmatrix} x \\ \mathbf{y} \\ z \end{pmatrix},t])\varphi)(e) = (\omega^{n-2}(\gamma,\psi)(u)(\tau_\psi([\begin{pmatrix} x \\ \mathbf{y} \\ z \end{pmatrix},t])\varphi)(e) =$$

$$\omega^{n-2}(\gamma,\psi)(u)[\psi(\frac{1}{2}Tr(-\bar{z}x) + Tr(-\bar{e}x) + t)(\tau^0_\psi([\begin{pmatrix} 0 \\ \mathbf{y} \\ 0 \end{pmatrix},0])(\varphi(z+e))] =$$

$$\psi(\frac{1}{2}Tr(-\bar{z}x) + Tr(-\bar{e}x) + t)\omega^{n-2}(\gamma,\psi)(u)(\tau^0_\psi([\begin{pmatrix} 0 \\ \mathbf{y} \\ 0 \end{pmatrix},0])(\varphi(z+e)) .$$

On the other hand,

$$(\tau_\psi([d(1,u,1)\begin{pmatrix} x \\ \mathbf{y} \\ z \end{pmatrix},t])\omega^n(\gamma,\psi)(d(1,u,1))\varphi)(e) =$$

$$(\tau_\psi([\begin{pmatrix} x \\ u\mathbf{y} \\ z \end{pmatrix},t])\omega^n(\gamma,\psi)(d(1,u,1))\varphi)(e) =$$

$$\psi(\frac{1}{2}Tr(-\bar{z}x) + Tr(-\bar{e}x) + t)(\tau_\psi^0([\begin{pmatrix} 0 \\ \mathbf{uy} \\ 0 \end{pmatrix}, 0])\omega^{n-2}(\gamma, \psi)(u)(\varphi(z+e)) \ .$$

But (for $n-2$) we have that

$$\omega^{n-2}(\gamma, \psi)(u)\tau_\psi^0([y,t])\omega^{n-2}(\gamma, \psi)(u)^{-1} = \tau_\psi^0([uy,t])$$

for all $u \in U(\Phi_{n-2})$, for all $[y,t] \in H(E^{n-2}) \equiv H(W_0)$, hence we get equality of the two sides of the commutation relation. So we may define $\omega^n(\gamma, \psi)(d(1,u,1))$ by formula(**1**) above. Any other oscillator representation $\tilde{\omega}^n(\gamma, \psi)(d(1,u,1))$ of $U(\Phi_n)$ would differ from $\omega^n(\gamma, \psi)(d(1,u,1))$ by a character of $U(\Phi_n)$, and such a character has to factor through the determinant, hence it is determined by its restriction to S'. Therefore formula(**1**) defines a unique oscillator representation of $U(\Phi_n)$. We establish formula(**2**) with the same type of calculations.

□

Remark. Note that $\varphi(e)$ above belongs to the space $\mathcal{F}$ of the representation τ_ψ^0 of $H(W_0)$. But (W_0, A_0) is naturally identifyed with the F-vector space underlying (E^{n-2}, A_{n-2}) and hence we can realize the oscillator representation $\omega^{n-2}(\gamma, \psi)$ of $U(\Phi_{n-2})$ on $\mathcal{F}$, and formula **(1)** makes sense.

Similarly we can also get a mixed model realization for an oscillator representation of $GL_n(F)$.

3.2. Heisenberg Models, special representations, exceptionality

Even for the $U(\Phi_3)$ case it can be shown (Cf. [GR1], p. 22) that Weil representations do not have Whittaker models. The substitute for those are the Heisenberg models (to be defined below) whose notion is due to Piatetski-Shapiro.

For each non-trivial additive character ψ' of F, let $\hat{R}(\psi')$ be the set of all infinite-dimensional, irreducible, unitary representations of R on which U acts by ψ'. Let $\hat{R}$ denote the union of all the $\hat{R}(\psi')$, with ψ' varying among all the non-trivial additive characters of F.

If (π, V) is an irreducible representation of G (G here denotes either $U(\Phi_n)$ or $GL_n(F)$) and $(\tau, X) \in \hat{R}$, we say that π has a non-trivial *τ-Heisenberg model* if there is a non-zero map $L : V \to X$ such that $L(\pi(r)v) = \tau(r)L(v)$ for all $r \in R$, $v \in V$. The map L is then called a *τ-Heisenberg functional.* The existence and uniqueness of Heisenberg models is not known in general. In chapter (5), we establish this result for Weil representations.

There are some elements of $\hat{R}(\psi)$ that are obtained in a very special way. First identify N with a Heisenberg group $H(W)$. Let (τ_ψ, X) be a model for the ψ-representation of N. The adjoint action of S' on N fixes U pointwise, and this defines an embedding of S' into the symplectic group $Sp(W)$. We identify S' with its image. Now let $\omega(\gamma, \psi)$ denote an oscillator representation of $S' \cong G_{n-2}$ on X. Extend τ_ψ to a representation $\tilde{\tau_\psi}$ of $S'N$ by letting S' act on X by $\omega(\gamma, \psi)$. Let now χ be a character of Z, and extend $\tilde{\tau_\psi}$ to a representation of R by letting Z act on X by χ. We denote the resulting representation by $\tau(\gamma, \psi, \chi)$, and we call it a *special representation* of R. We will use the following notation:

$$\hat{R}(\psi, \chi)^{\circ} = \{\tau(\gamma, \psi, \chi) : \gamma \text{ as above}\}$$

$$\hat{R}(\psi, \chi)^{\circ} = \bigcup\nolimits_{\chi} \hat{R}(\psi, \chi)^{\circ}.$$

Notice that, just like Weil representations, the special elements of $\hat{R}(\psi)$ are also parametrized by the characters γ, ψ and χ. We have the following important result about special representations of R:

Proposition 3.2.1. *Let (ρ, V_ρ) be an irreducible admissible representation of R on which the center ZU acts by $\chi \otimes \psi$. Fix $\tau \in \hat{R}(\psi, \chi)^{\circ}$. Then there exists an irreducible admissible representation σ of $R/N = S$ such that ρ is isomorphic to $\sigma \otimes \tau$.*

Proof. Set

$$\mathcal{H} = \mathrm{Hom}_N(\tau, \rho)$$

By Lemma(I.8), p. 33 of [MVW], every smooth representation of N on which U acts by ψ is isomorphic to a direct sum of copies of τ_ψ. It follows that $\mathcal{H}$ is non-zero. The group S acts on $\mathcal{H}$ through its action (that we will denote by σ) on τ and ρ (i.e., for $\phi \in \mathcal{H}$, $\sigma(s)\phi = \rho(s)\phi\tau(s)^{-1}$). We now show that σ is a smooth representation of S. Let $\phi \in \mathcal{H}$ and let $V \subset V_\rho$ be the image of ϕ. Fix any non-zero element $v \in V$ and let B be the stabilizer of v in R. Then $v \in \rho(b)V \cap V$ for all $b \in B$. The spaces $\rho(b)V$ and V are irreducible as N-modules, and hence $\rho(b)V = V$ for all $b \in B$. Let $\mathcal{F}$ be the space of τ. Since $\phi : \mathcal{F} \to V$ intertwines the N action, it must also intertwine the B action up to twisting by a character. In other words, $\rho(b)\phi\tau(b)^{-1} = \mu(b)\phi$ for some character μ of B. Thus ϕ is fixed by $\ker(\mu)$, which is open in S. This shows that σ is smooth. Evaluation defines a natural map $\sigma \otimes \tau \to \rho$. It is injective since τ is irreducible and hence must be an isomorphism. It follows that σ is irreducible and hence also admissible.

□

Now, the group P acts by conjugation on R and also on $\hat{R}$. For (π, V) a representation of G, let $\Lambda_0(\pi)$ be the set of all $\tau \in \hat{R}$ such that π has a τ-Heisenberg model. Observe that $\Lambda_0(\pi)$ is stable under the adjoint action of P. The following notion is also due to Piatetski-Shapiro:

Definition 3.2.2. *We say that (π, V) is exceptional if $\Lambda_0(\pi)$ contains a special element of $\hat{R}$, and consists of a single P-orbit.*

4. The representations α_τ

4.1. The unitary case.

In this section G denotes $U(\Phi_n)$. For each $(\tau, X) \in \hat{R} = \cup_{\{\psi' \neq 1\}} \hat{R}(\psi')$, let $\mathcal{S}(T, X)$ be the space of smooth, compactly supported, X-valued functions φ of T such that $\varphi(rt) = \tau(r)\varphi(t)$ for all $r \in R \cap T \cong E^1$, for all $t \in T$. Define the following representation α_τ of P on $\mathcal{S}(T, X)$:

(a) $(\alpha_\tau(t)\varphi)(t_0) = \varphi(t_0 t)$ for all $t \in T$;

(b) $(\alpha_\tau(r)\varphi)(t_0) = \tau(t_0 r t_0{}^{-1})(\varphi(t_0))$, for all $r \in R$.

Observe that α_τ is a realization of the representation of P on the space $\mathrm{ind}_R^P(\tau)$ of compactly supported functions in $\mathrm{Ind}_R^P(\tau)$.

Proposition 4.1.1. *(a) For any $\tau \in \hat{R}$, α_τ is an irreducible representation of P. (b) Let $\tau_1, \tau_2 \in \hat{R}$. If $\alpha_{\tau_1} \cong \alpha_{\tau_2}$, then there exists $t \in T$ such that $\tau_2 = {\tau_1}^t$ (where ${\tau_1}^t(r) = \tau_1(trt^{-1})$, for all $t \in T,\ r \in R$).*

Proof. With a little abuse of notation, we identify $a \in E^*$ with $d(a, 1, ..., 1, \bar{a}^{-1}) \in T$.

(a) Without loss of generality, we may assume that $\tau \in \hat{R}(\psi)$ (the calculations that follow work for any non-trivial character ψ' of F, and we may assume that the conductor of ψ' is $\mathcal{O}_F$).

Let φ be a non-zero element of $\mathcal{S}(T, X)$ and let $\mathcal{S}(\varphi)$ be the space generated by φ under the action of P. Translating if necessary, we may assume that $\varphi(1) \neq 0$. Let C be an open subgroup of U and for $x \in T \equiv E^*$ set

$$\varphi_C(x) = [\mathrm{meas}(C)^{-1}\int_C \overline{\psi(u)}\alpha_\tau(u)\varphi\, du](x) = \mathrm{meas}(C)^{-1}[\int_C \psi(u(x\bar{x}-1))\, du]\varphi(x).$$

Note that $\varphi_C \in \mathcal{S}(\varphi)$ and, for $N > 0$, if $C_N = p^{-N}\mathcal{O}_F$, and $\Omega_N = \{x \in E^* : x\bar{x} - 1 \in p^N\mathcal{O}_F\}$, we see that $\varphi_{C_N} = \varphi(x)$ for $x \in \Omega_N$, and $\varphi_{C_N} = 0$ for $x \notin \Omega_N$, i.e., $\varphi_{C_N} = \varphi\mathfrak{C}_{\Omega_N}$ where for any set Ω we denote by $\mathfrak{C}_\Omega$ the characteristic function of Ω. Take any non-zero $f \in \mathcal{S}(T, X)$, and let us show that $f \in \mathcal{S}(\varphi)$. Let $t_1 \in T$ be such that $f(t_1) \neq 0$. Since τ is irreducible and $\varphi(1) \neq 0$, we know that we can find $r_j \in R$, $j = 1, ..., J$, such that $\sum_{j=1}^J \tau(r_j)(\varphi(1)) = f(t_1)$, i.e., $(\sum_{j=1}^J \alpha_\tau(r_j)\varphi)(1) = f(t_1)$. Writing $\varphi_1 = \sum_{j=1}^J \alpha_\tau(r_j)\varphi$, and using the fact that $\tau(\beta)\phi(t) = \phi(\beta t)$, for all $\beta \in E^1 \cong R \cap T$, and for any $\phi \in \mathcal{S}(T, X)$ (in particular for φ_1 and f), we get that $\varphi_1(\beta) = f(t_1\beta)$ for all $\beta \in E^1$. By the local constancy of φ_1 and f and the fact that E^1 is compact, we can get a small neighborhood $\mathcal{N}$ of E^1 such that the restriction of φ_1 to $\mathcal{N}$ equals the restriction of f to $t_1\mathcal{N}$. We can assume that $\mathcal{N}$ is of the form $\{x + p^k\mathcal{O}_E,\ x \in E^1\}$, and it is not hard to prove that we can find $N \gg 0$ such that $\Omega_N \subset \mathcal{N}$, and hence we get that the restriction of φ_1 to Ω_N equals the restriction of f to $t_1\Omega_N$, i.e., $\alpha_\tau(t_1^{-1})\varphi_1\mathfrak{C}_{\Omega_N} = f\mathfrak{C}_{t_1\Omega_N}$ (and recall that $\varphi_1\mathfrak{C}_{\Omega_N} \in \mathcal{S}(\varphi)$). It is easy to check that for any $t, t' \in T$, $N > N' > 0$, we either have $t\Omega_N \cap t'\Omega_{N'} = \emptyset$ or $t\Omega_N \subset t'\Omega_{N'}$. The proof now follows using the compactness of the support of f.

(b) Let us assume that $\tau_1 \in \hat{R}(\psi)$, $\tau_2 \in \hat{R}(\psi_2)$, where ψ_2 is a non-trivial character of F. For any additive character ψ' of F, define $Y(\psi') = \mathrm{span}\{\alpha_\tau(u)\varphi - \psi'(u)\varphi : u \in U, \ \varphi \in \mathcal{S}(T,X)\}$ and $Y_{\psi'} = \mathcal{S}(T,X)/Y(\psi')$. For each $t = d(a,1,...,1,\bar{a}^{-1}) \in T$, denote by ψ^t the character $\psi^t(u) = \psi(tut^{-1}) = \psi(a\bar{a}u)$. The proof of (b) is based on the following two steps: for any $\tau \in \hat{R}(\psi)$,

(i) If $\psi' = \psi^{t_0}$ for some $t_0 \in T$, then $Y_{\psi'} \cong \tau^{t_0}$.

(ii) If $\psi' \neq \psi^{t_0}$ for all $t_0 \in T$, then $Y_{\psi'} = \{0\}$.

To prove (i), we consider the map $ev_{t_0} : (\alpha_\tau, \mathcal{S}(T,X)) \to (\tau^{t_0}, X)$ defined by $\varphi \mapsto \varphi(t_0)$. It is easy to check that ev_{t_0} is an R-map and that $Y(\psi^{t_0}) \subset \ker(ev_{t_0})$. To prove the other inclusion, we use the following simple lemma

Lemma 4.1.2. *Let η be any non-trivial additive character of F. For each $x_0 \in F^*$, we can find a small compact subset C of F^* containing x_0, and an element $t \in F^*$ such that $\eta(tx) \neq 1$ for all $x \in C$.*

To complete the proof of (i), take any $\varphi \in \ker(ev_{t_0})$. The idea is to write $\mathrm{supp}(\varphi) = \Omega$ as a disjoint union of neighbourhoods $\mathcal{C}_j = (t_jE^1 + p^{N_j}\mathcal{O}_E) \cap \Omega$ such that $C_j = \{t\bar{t} - t_0\bar{t}_0, \ t \in \mathcal{C}_j\}$ is "small enough". Then we use lemma (4.1.2) to find $u_j \in F^*$ such that $\varphi_j = \varphi\mathfrak{C}_{\mathcal{C}_j}$ can be written as $\varphi_j = \alpha_\tau(u_j)\phi_j - \psi^{t_0}(u_j)\phi_j$, for some $\phi_j \in \mathcal{S}(T,X)$. This can be accomplished with $\phi_j(t) = [\psi(t_0\bar{t}_0u_j)(\psi((t\bar{t} - t_0\bar{t}_0)u_j) - 1)]^{-1}\varphi_j(t)$. The proof of (ii) is similar.

Finally, to conclude the proof of (b), note that (i) and (ii) show that the T-orbit of $\tau \in \hat{R}$ can be recovered from the family of spaces $Y_{\psi'}$ with ψ' varying among all the non-trivial additive characters of F. If $\alpha_{\tau_1} \cong \alpha_{\tau_2}$, the two families of spaces $Y_{\psi'}$ must be the same, hence τ_1 and τ_2 must have the same T-orbit, and (b) follows.

$\square$

4.2. The $GL_n(F)$ case.

In this section G denotes $GL_n(F)$. For each $(\tau, X) \in \hat{R}$, let now $\mathcal{S}(T, X)$ be the space of smooth, X-valued functions on T which are compactly supported modulo $S \cap T \cong F^*$, and such that $\varphi(rt) = \tau(r)\varphi(t)$ for all $r \in S \cap T$, for all $t \in T$. We define a representation α_τ of P on $\mathcal{S}(T, X)$ just as before, i.e., T acts on $\mathcal{S}(T, X)$ by the right regular representation, and $r \in R$ acts by $(\alpha_\tau(r)\varphi)(t) = \tau(trt^{-1})(\varphi(t))$. As in the previous case, α_τ is a realization of the representation of P on $\mathrm{ind}_R^P(\tau)$. Let D be the subgroup of G of diagonal matrices of the form $d(a, 1, ..., 1, 1)$, with $a \in F^*$. Of course we can identify D with F^*, and let $\mathcal{S}(D, X) \cong \mathcal{S}(F^*, X)$ be the space of smooth, compactly supported, X-valued functions on D. Define the following representation β_τ of P on $\mathcal{S}(D, X)$:

(a') $(\beta_\tau(d_0)\varphi)(d) = \varphi(dd_0)$, for all $d_0 \in D$;

(b') $(\beta_\tau(r)\varphi)(d) = \tau(drd^{-1})(\varphi(d))$, for all $r \in R$.

Note that every $t \in T$ can be uniquely written as $t = ds$ with $d \in D$ and $s \in S \cap T$. Also, one can easily check that restriction to D defines a P-isomorphism between $(\alpha_\tau, \mathcal{S}(T, X))$ and $(\beta_\tau, \mathcal{S}(D, X))$. Similarly to the unitary case (including its proof), we have:

Proposition 4.2.1. *(a) For each $\tau \in \hat{R}$, α_τ is an irreducible representation of P. (b) Let $\tau_1, \tau_2 \in \hat{R}$. If $\alpha_{\tau_1} \cong \alpha_{\tau_2}$, then there exists $t \in T$ such that $\tau_2 = {\tau_1}^t$.*

5. Local results

We now establish the exceptionality of the Weil representations of $U(\Phi_n)$ and $GL_n(F)$, for $n \geq 3$. Although the proof of this result is not the same for the unitary and $GL_n(F)$ cases, both proofs follow a general pattern that we now describe. Denote by (π, V) the Weil representation $(\omega(\gamma, \psi, \chi), V)$ of G (either equal to $U(\Phi_n)$ or $GL_n(F)$) and let $(\tau, X) = (\tau(\gamma, \psi, \chi), X)$. Let $V_0 = \text{span}\{\pi(u)v - v : u \in U,\ v \in V\}$. The exceptionality in both cases is obtained once we establish the steps:

(I) $(\pi|_P, V_0) \cong (\alpha_\tau, \mathcal{S}(T, X))$, and

(II) $\tau_1, \tau_2 \in \Lambda_0(\pi) \Rightarrow \alpha_{\tau_1} \cong \alpha_{\tau_2}$.

Next we give the proof of the exceptionality of Weil representations for the unitary case.

5.1. Exceptionality of Weil representations of $U(\Phi_n)$, $n \geq 3$.

In this subsection G denotes $U(\Phi_n)$. Let (π, V) and (τ, X) be as above. We may assume that V is a mixed model realization of $\omega(\gamma, \psi)$ on $\mathcal{S}(E, X)$, as in section (3). Let us denote by $\mathcal{S}(E, X)_\chi$ the subspace of $\mathcal{S}(E, X)$ on which Z acts by χ (so that $\mathcal{S}(E, X)_\chi$ is the space of $\pi = \omega(\gamma, \psi, \chi)$). The following result comes directly from proposition (3.1.1):

Lemma 5.1.1. *$L : \mathcal{S}(E, X)_\chi \to X$ given by $\varphi \mapsto \varphi(-1)$ is a non-zero Heisenberg functional for π.*

Now we want to prove step(**I**) above, namely

Lemma 5.1.2. *Let $V_0 = \{\pi(u)v - v : u \in U,\ v \in \mathcal{S}(E, X)_\chi\}$. The map $\mathcal{L} : (\pi|_P, V_0) \to (\alpha_\tau, \mathcal{S}(T, X))$ defined by $v \mapsto \phi_v$ where $\phi_v(t) = L(\pi(t)v)$, is a P-isomorphism.*

Proof. Let us first show that $\mathcal{L}$ is well-defined, i.e., $\phi_v \in \mathcal{S}(T, X)$ for all $v \in V_0$. Let us denote the matrix $d(a, 1, ..., 1, \bar{a}^{-1}) \in T$ simply by $d(a)$. For any $d(\beta) \in R \cap T \cong E^1$, we have $\phi_v(d(\beta)t) = L(\pi(d(\beta))\pi(t)v) = \tau(d(\beta))\phi_v(t)$ so that ϕ_v satisfies the transformation law reqired to the elements in $\mathcal{S}(T, X)$. It remains to show that ϕ_v is compactly supported. Without loss of generality, we may assume that $v = \pi(u)w - w$, for some $u \in U$, $w \in V$. Then one can easily check that $\phi_v(d(a)) = (\psi(a\bar{a}u) - 1)L(\pi(d(a))w)$ from where we see that $\phi_v(d(a)) = 0$ if $\mathrm{val}_p(a) \gg 0$. On the other hand, by smoothness of π we can find a large integer $M > 0$ such that $\pi([0,t])v = v$ for all $t \in p^M\mathcal{O}_F$, and hence $\phi_v(d(a)) = \psi(a\bar{a}t)\phi_v(d(a))$ If $\mathrm{val}(a) \ll 0$, we can find $t \in p^M\mathcal{O}_F$ such that $\psi(a\bar{a}t) \neq 1$, and consequently $\phi_v(d(a)) = 0$. We conclude that ϕ_v is compactly supported and hence belongs to $\mathcal{S}(T,X)$. To prove that $\mathcal{L}$ is injective, we use proposition (3.1.1). If $\varphi \in V = \mathcal{S}(E,X)_\chi$ is sucht that $\mathcal{L}(\varphi) = 0$, then $\phi_\varphi(t) = 0$ for all $t \in T$, i.e., $L(\pi(t)\varphi) = 0$ for all $t \in T$, or equivalently, $(\pi(d(a))\varphi)(-1) = 0$ for all $a \in E^*$, and formula(2) in proposition (3.1.1) shows that we then must have $\varphi = 0$. Finally, $\mathcal{L}$ is clearly a P-map ($\mathcal{L}$ is the map associated to L by Frobenius reciprocity) and since from proposition (4.1.1) we know that α_τ is irreducible, $\mathcal{L}$ must be surjective, and this completes the proof of lemma (5.1.2).

□

Hence we have shown that $\tau = \tau(\gamma,\psi,\chi)$ belongs to $\Lambda_0(\pi)$, for $\pi = \omega(\gamma,\psi,\chi)$.

Lemma 5.1.3. *If τ' is an element of $\Lambda_0(\pi)$, then $\alpha_{\tau'} \cong \alpha_\tau$.*

Proof. Let $\tau' = (\tau', X')$, and assume that $\tau' \in \hat{R}(\psi')$, for some non-trivial additive character ψ' of F. Let $L' : V \to X'$ be a non-zero τ'-Heisenberg functional for $\pi = (\pi, \mathcal{S}(E,X)_\chi)$. If $L'(v_0) = 0$ for all $v_0 \in V_0$, then for any $v \in V$, choose $u \in U$ such that $\psi'(u) \neq 1$, and we get $0 = L'(\pi(u)v - v) = (\psi'(u) - 1)L'(v)$ from where we conclude that $L'(v) = 0$ for all $v \in V$, which is impossible. Define $\mathcal{L}' : (\pi|_P, V_0) \to (\alpha_{\tau'}, \mathcal{S}(T,X'))$ by $v \mapsto \phi'_v$ where $\phi'_v(t) = L'(\pi(t)v)$ (as before). Note that clearly $\phi'_v(d(\beta)t) = \tau'(d(\beta))\phi'_v(t)$ for all $d(\beta) \in R \cap T$, and the same calculations done for ϕ_v (with ψ replaced by ψ') show that ϕ'_v is compactly supported, hence $\mathcal{L}'$ is well-defined and is a P-map. Also $\mathcal{L}' \equiv 0$ implies $\phi'_{v_0} \equiv 0$ for all $v_0 \in V_0$, and thus $\phi'_{v_0}(1) = L'(v_0) = 0$ for all $v_0 \in V_0$, which can not happen (as we have seen before). Hence $\mathcal{L}' \neq 0$ and by irreducibility we get $(\pi|_P, V_0) \cong (\alpha_{\tau'}, \mathcal{S}(T,X'))$ (note $(\pi|_P, V_0)$ is irreducible because it is isomorphic to $(\alpha_\tau, \mathcal{S}(T,X))$, and both α_τ and $\alpha_{\tau'}$ are irreducible by proposition (4.1.1)). By the previous lemma we get $(\alpha_\tau, \mathcal{S}(T,X)) \cong (\alpha_{\tau'}, \mathcal{S}(T,X'))$.

□

Using the lemmas in this section and proposition (4.1.1) (part (b)) we obtain:

Theorem 5.1.4. *The representation $\pi = \omega(\gamma, \psi, \chi)$ is an exceptional representation of $U(\Phi_n)$.*

5.2. Existence and uniqueness of Heisenberg models for Weil representations.

In this subsection G denotes either $U(\Phi_n)$ or $GL_n(F)$. We have already shown the existence of Heisenberg models for Weil representations of G, and the next result establishes the uniqueness of these models.

Theorem 5.2.1. *Let $(\pi, V) = (\omega(\gamma, \psi, \chi), \mathcal{S}(T,X))$ be a Weil representation of G. For any $\tau' \in \hat{R}$ we have dim(Hom$_R(\pi, \tau')) \leq 1$.*

Proof. We know that for $\tau = (\tau(\gamma, \psi, \chi), X)$ we have $\dim(\mathrm{Hom}_R(\pi, \tau)) \geq 1$ (by the existence part already proved). From the exceptionality of π, we also know that $\dim(\mathrm{Hom}_R(\pi, \tau')) = 0$ if τ' does not belong to the T-orbit of τ. On the other hand, if $\tau_1 = \tau^t$ for some $t \in T$, given $L \in \mathrm{Hom}_R(\pi, \tau)$ one can easily check that $L_t = L \circ \pi(t)$ belongs to $\mathrm{Hom}_R(\pi, \tau^t)$, hence $\dim(\mathrm{Hom}_R(\pi, \tau^t)) = \dim(\mathrm{Hom}_R(\pi, \tau))$ for all $t \in T$, and all we have to show is that $\dim(\mathrm{Hom}_R(\pi, \tau)) = 1$. For that we use **(I)** once again. First, by Schur's lemma, we have that $\dim(\mathrm{Hom}_P((\pi|_P, V_0), (\alpha_\tau, \mathcal{S}(T,X)) = 1$ (note this dimension is at least 1 by **(I)**). Now, if $L_1, L_2 : (\pi, V) \to (\tau, X)$ are non-zero elements of $\mathrm{Hom}_R(\pi, \tau)$ then, as we have seen before, the maps $\mathcal{L}_i : V_0 \to \mathcal{S}(T,X)$ given by $v \mapsto \phi_v^i$ (where $\phi_v^i(t) = L_i(\pi(t)v)$) are non-zero elements of $\mathrm{Hom}_P((\pi|_P, V_0), (\alpha_\tau, \mathcal{S}(T,X))$, hence there exists $\lambda \in \mathbb{C}^*$ such that $\mathcal{L}_1 = \lambda \mathcal{L}_2$. This easily implies $L_1|_{V_0} = \lambda L_2|_{V_0}$. But then the element $L = L_1 - \lambda L_2$ of $\mathrm{Hom}_R(\pi, \tau)$ is such that $L|_{V_0} = 0$, and this forces $L \equiv 0$, which gives $L_1 = \lambda L_2$. Therefore $\dim(\mathrm{Hom}_R(\pi, \tau)) = 1$, as we wanted.

□

6. Global set up

Let now F be a number field, and E a quadratic extension of F (with conjugation again denoted by a bar). Let $\mathbf{A}$ denote the adele ring of F. For each place v of F let $E_v = E \otimes_F F_v$ (note that E_v is a quadratic extension of F_v if v remains prime in E, and $E_v \cong E_w \oplus E_{w'}$, with $E_w \cong E_{w'} \cong F_v$, if v splits into w and w' in E). Roughly speaking (see [GR2] for more details on the constructions of this section), we now "adelize" section (2). Let Φ_n be as in section (2), let V ve an n-dimensional vector space over E, and let $G = G_n$ be the unitary group attached to (V, Φ_n). For all v, let K_v be the group of integral matrices in G_v. For almost all v, K_v is a maximal compact subgroup of G_v. We will indicate by $N(\mathbf{A})$, $R(\mathbf{A})$, $P(\mathbf{A})$, etc, the global counterparts of the groups locally denoted by the letters N, R, P, etc. Taking the necessary precautions, essentially all the notions we defined locally are carryed out for the global setting, and global Weil representations as well as global special representations are still parametrized in the form $\omega(\gamma, \psi, \chi)$ and $\tau(\gamma, \psi, \chi)$ (respectively) where γ and χ are Hecke characters and ψ is a non-trivial additive character of $F\backslash\mathbf{A}$.

To fix the notation, let

$$L^2_{\psi,\chi}(R(F)\backslash R(\mathbf{A}))$$

be the Hilbert space of square-integrable functions ϕ on $R(F)\backslash R(\mathbf{A})$ such that $\phi(z[0,t]r) = \chi(z)\psi(t)\phi(r)$ for all $z \in Z$, for all $t \in \mathbf{A}$. Let $\mathcal{Z}(\psi, \chi)$ be the set of irreducible subrepresentations of $R(\mathbf{A})$ with central character $\chi \otimes \psi$, and let $\hat{R}(\psi, \chi)^{\circ}$ be the subset of $\mathcal{Z}(\psi, \chi)$ consisting of the special elements.

7. Global results

In this section we establish the two central results of the global part of this paper: multiplicity one for special automorphic representations of $R(\mathbf{A})$, and a strong multiplicity one type of result for automorphic representations of $G(\mathbf{A})$, namely, if an irreducible discrete representation $\pi = \otimes\pi_v$ is locally isomorphic to a Weil representation at all but a finite number of finite places of F then π in fact equals a global Weil representation.

7.1. Multiplicity one for special representations of $R(\mathbf{A})$.

We start with the following consequence of strong approximation (Cf. [K]):

Proposition 7.1.1. *Let $\sigma = \otimes\sigma_v$ be an irreducible automorphic representation of $G(\mathbf{A})$. Suppose that there exists a finite place v of F such that σ_v equals the trivial character on G_v. Then σ is one dimensional.*

Proof. It suffices to prove that σ is trivial on $H = SG(\mathbf{A}) = \{g \in G(\mathbf{A}) : det(g) = 1\}$ (since $H\backslash G$ is compact Abelian). For v finite H_v is non-compact and for the v in the statement of the proposition we have that $\sigma_v|_{H_v} \equiv 1$. By strong approximation (Cf. [K]), $H(F).H_v$ is dense in H. Let f be any element of σ. Then for any $\gamma \in H(F)$, $h_v \in H_v$ we get that

$$f(\gamma h_v) = f(h_v) = (\sigma(h_v)f)(1) = f(1)$$

hence f is constant on $H(F) \times H_v$, so f is constant on H. Thus, for any $h \in H$ and any $f \in \sigma$,

$$(\sigma(h)f)(g) = f(gh) = f(ghg^{-1}g) = (\sigma(g)f)(ghg^{-1}) = (\sigma(g)f)(1) = f(g)$$

so, $\sigma(h)f = f$ for any $h \in H$ and any $f \in \sigma$, i.e., $\sigma|_H \equiv 1$.

□

Lemma 7.1.2. *Let $\tau \in \mathcal{Z}(\psi, \chi)$, $\tau = \otimes\tau_v$. Suppose that $\tau_{v_0} \in \hat{R}(\psi_{v_0}, \chi_{v_0})^\circ$ for some finite place v_0 of F. Then $\tau \in \hat{R}(\psi, \chi)^\circ$.*

Proof. It is not hard to prove that we can write $\tau \cong \sigma \otimes \tau^1$ with σ an irreducible automorphic representation of $S'(\mathbf{A}) \cong G_{n-2}(\mathbf{A})$ and $\tau^1 \in \hat{R}(\psi, \chi)^\circ$. For all v we have that $\tau_v \cong \sigma_v \otimes \tau_v^1$, in particular for $v = v_0$. Since both τ_{v_0} and $\tau_{v_0}^1$ belong to $\hat{R}(\psi_{v_0}, \chi_{v_0})^\circ$ (note that since σ_{v_0} is a representation of $S'(F_{v_0})$, τ_{v_0} and $\tau_{v_0}^1$

have the same central character $\chi_{v_0} \otimes \psi_{v_0}$) there exists a character ν_{v_0} of $S^{'}(F_{v_0})$ such that $\tau^1_{v_0} \cong \nu_{v_0} \otimes \tau_{v_0}$, and hence $\tau_{v_0} \cong (\sigma_{v_0} \otimes \nu_{v_0}) \otimes \tau_{v_0}$. This forces σ_{v_0} to be one-dimensional (otherwise the restriction of $(\sigma_{v_0} \otimes \nu_{v_0}) \otimes \tau_{v_0}$ to N would not be irreducible, and could not be isomorphic to τ_{v_0} restricted to N) and by the same argument given in the proof of theorem (7.1.3) below, we get that $\sigma_{v_0} \otimes \nu_{v_0} \equiv 1$. By proposition (7.1.1) we get that σ is one dimensional, i.e., a character of $S^{'}(\mathbf{A})$. Hence $\sigma \otimes \tau^1$ still belongs to $\hat{R}(\psi, \chi)^{\circ}$ (if $\tau^1(zsn) = \chi(z)\omega_{\psi}^{s^{'}}(s)\tau_{\psi}(n)$, then $\sigma \otimes \tau^1(zsn) = \chi(z)(\sigma \otimes \omega_{\psi}^{s^{'}})(s)\tau_{\psi}(n))$.

□

We now come to the main result of this paragraph.

Theorem 7.1.3. *Let* $L^2_{\psi}(R(F)\backslash R(\mathbf{A})) = \{\varphi \in L^2(R(F)\backslash R(\mathbf{A})) : \rho(u)\varphi = \psi(u)\varphi,\ \forall u \in U(\mathbf{A})\}$ *(where* ρ *denotes right translation). Each special automorphic representation of* $R(\mathbf{A})$ *occurs as an irreducible constituent of* $L^2_{\psi}(R(F)\backslash R(\mathbf{A}))$ *with multiplicity one.*

Proof. Let $\tau^0 = \otimes \tau^0_v$ and $\tau^1 = \otimes \tau^1_v$ be special automorphic representations of $R(\mathbf{A})$. We need to show that $\tau^0 \cong \tau^1$ implies $\tau^0 = \tau^1$. The proof is divided in two steps:

(1): If ν is a character of $R(F)N(\mathbf{A})\backslash R(\mathbf{A})$ and $\tau^0 \cong \tau^0 \otimes \nu$, then $\nu \equiv 1$.

Proof. Indeed, realizing both τ^0 and $\tau^0 \otimes \nu$ on the same space $\mathcal{F}$ (they both can be realized on the Schrödinger model for instance), let

$$\phi : (\tau^0, \mathcal{F}) \to (\tau^0 \otimes \nu, \mathcal{F})$$

be an isomorphism. Since both τ^0 and $\tau^0 \otimes \nu$ equal τ_{ψ} (the ψ-representation of $N(\mathbf{A})$) when restricted to $N(\mathbf{A})$, and τ_{ψ} is irreducible, by Schur's lemma ϕ must be of the form $\lambda I_{\mathcal{F}}$ ($I_{\mathcal{F}}$ the identity map of $\mathcal{F}$) for some $\lambda \in \mathbb{C}^*$. It is now easy to conclude that $\nu \equiv 1$.

(2): If $(\tau^0, \mathcal{F}_0)$ and $(\tau^1, \mathcal{F}_1)$ are special automorphic representations of $R(\mathbf{A})$, then there exists a character ν of $R(F)N(\mathbf{A})\backslash R(\mathbf{A})$ such that $\mathcal{F}_1 = \{\nu\varphi : \varphi \in \mathcal{F}_0\}$.

Proof. For $r_0 \in R(\mathbf{A})$ consider the maps ($j = 0, 1$):

$$\begin{array}{rcccl} T_j(r_0) & : & \mathcal{F}_j & \to & L^2_\psi(N(F)\backslash N(\mathbf{A})) \\ & & \varphi & \mapsto & (\tau^j(r_0)\varphi)|_{N(\mathbf{A})} \end{array} .$$

Clearly $T_j(r_0)$ is a non-zero map for all $r_0 \in R(\mathbf{A})$. Furthermore, one easily checks that $T_j(r_0)$ is an intertwining map between $\tau^j|_N$ and $\tau_\psi^{r_0}$ (where $\tau_\psi^{r_0}(n) = \tau_\psi(r_0 n r_0{}^{-1})$). Since $\tau^j|_N = \tau_\psi$ and $\tau_\psi^{r_0}$ are both irreducible, $T_j(r_0)$ is an $N(\mathbf{A})$-isomorphism for all $r_0 \in R(\mathbf{A})$. Hence $T_1(r_0)^{-1}T_0(r_0) : \mathcal{F}_0 \to \mathcal{F}_1$ is an $N(\mathbf{A})$-isomorphism, and by Schur's lemma (once again!) applyed to the representation τ_ψ of $N(\mathbf{A})$ we get that $T_1(r_0)^{-1}T_0(r_0) = \mu(r_0)T_1(1)^{-1}T_0(1)$ for a non-zero scalar $\mu(r_0)$. Noticing that $T_j(r_1r_2) = T_j(r_1)\tau^j(r_2)$, it is simple to check that $\mu(r_1r_2) = \mu(r_1)\mu(r_2)$. If for each $r_0 \in R(\mathbf{A})$ we write $T(r_0) = T_1(r_0)^{-1}T_0(r_0) : \mathcal{F}_0 \to \mathcal{F}_1$ then for each $\varphi_1 \in \mathcal{F}_1$ there exists a unique $\varphi_0 \in \mathcal{F}_0$ such that $\varphi_1 = T(1)\varphi_0$, and in this case we see that (noticing that $T_j(1)\varphi_j = \varphi_j|_N$) $\varphi_1|_{N(\mathbf{A})} = \varphi_0|_{N(\mathbf{A})}$ and $\varphi_0(r) = \mu(r)\varphi_1(r)$ for all $r \in R(\mathbf{A})$. From this we conclude that μ is a smooth function on $R(F)N(\mathbf{A})\backslash R(\mathbf{A})$ (and since μ is also multiplicative, it is a character of $R(F)N(\mathbf{A})\backslash R(\mathbf{A})$). For $\nu = \mu^{-1}$, the equation $\varphi_1(r) = \nu(r)\varphi_0(r)$ for any $\varphi_1 \in \mathcal{F}_1$ (and $\varphi_0 \in \mathcal{F}_0$ such that $\varphi_1 = T(1)\varphi_0$) says that $\mathcal{F}_1 = \{\nu\varphi_0 : \varphi_0 \in \mathcal{F}_0\}$.

To conclude the proof of theorem (7.1.3), given $(\tau^0, \mathcal{F}_0)$ and $(\tau^1, \mathcal{F}_1)$ as above, consider the map $T_\nu : \mathcal{F}_0 \to \mathcal{F}_1$ defined by $\varphi \mapsto \nu\varphi$ (for ν as in **(2)**). It is easy to check that $T_\nu(\tau^0(r)\varphi) = (\nu^{-1} \otimes \tau^1)(r)T_\nu(\varphi)$ and since both τ^0 and τ^1 are irreducible we get that $\tau^0 \cong \nu^{-1} \otimes \tau^1$. If we had that $\tau^0 \cong \tau^1$, then we would have $\nu^{-1} \otimes \tau^1 \cong \tau^1$, and by step(1) we would conclude that $\nu \equiv 1$, which in turn would imply $\mathcal{F}_0 = \mathcal{F}_1$. The proof is then complete.

$\square$

7.2. Strong multiplicity one for a. e. Weil representations.

The proof of this result follows the same pattern of the GL_2 case (Cf. [PS]). The central idea is to construct "two" non-zero automorphic forms $f^1 \in \pi$ and $f^0 \in \omega$ and use uniqueness of Heisenberg models (proved for Weil representations), formula (2)(below), some archimedean calculations and a "small lemma" - type of argument to see that $f^1 = f^0$ and thus to conclude by irreducibility (of both π and ω) that $\pi = \omega$. We present the details in what follows.

Let us begin with the following important lemma ([GR2], p.453):

Lemma 7.2.1. *Let (π, V), $\pi = \otimes \pi_v$ be an infinite dimensional irreducible discrete representation of $G(\mathbf{A})$ and let φ be a non-zero element of V. Then there exists a non-trivial character ψ of $F \backslash \mathbf{A}$ such that $\varphi_\psi \neq 0$.*

Proof. Let H be the closed subgroup of $G(\mathbf{A})$ generated by $U(\mathbf{A})$ and $SG(F) = \{g \in G(F) : det(g) = 1\}$. If $\varphi_\psi = 0$ for all non-trivial ψ, then $\varphi = \varphi_U$ and therefore φ is left-invariant under H.

We will get a contradiction (to the infinite dimensionality of π) if we show that $H = SG(\mathbf{A}) = \{g \in G(\mathbf{A}) : det(g) = 1\}$. Indeed, if that is the case, then the left-invariance of φ under H implies $\pi(h)\varphi = \varphi$ for all $h \in H$. Since π is irreducible, this equation has to hold for any element of π, and hence π may be viewed as a representation on $H \backslash G$, which is compact abelian, thus π would have to be finite dimensional.

Now, to see that $H = SG(\mathbf{A})$, notice first that for all v, H contains the closed subgroup H_v generated by $\{\gamma u \gamma^{-1} : \gamma \in SG(F),\ u \in U_v\}$. Since $SG(F)$ is dense in $SG(F_v)$, H_v is a closed normal non-central subgroup of $SG(F_v)$. Hence $H_v = SG(F_v)$ and $H = SG(\mathbf{A})$.

□

Lemma 7.2.2. *Let π be as in the previous lemma. Assume that $\pi_v \cong \omega_v^0 = \omega(\gamma_v^0, \psi_v^0, \chi_v^0)$ (for some local Weil representation ω_v^0) for almost all v. Then* *there exists a global Weil representation $\omega = \otimes\omega_v$ such that $\pi_v \cong \omega_v$ for almost all finite v.*

Proof. By lemma (7.2.1), there exists a non-trivial character ψ of $F\backslash\mathbf{A}$ such that $f_\psi \neq 0$ for some $f \in \pi$. Hence there exists $\tau = \otimes\tau_v \in \hat{R}(\psi)$ such that $f_\tau \neq 0$ (recall from the end of chapter (6) that $f_\tau = (f_{\psi,R})_\tau$). For a finite place v where $\pi_v \cong \omega_v^0 = \omega(\gamma_v^0, \psi_v^0, \chi_v^0)$ we get that $\tau_v \cong \tau(\gamma_v^0, (\psi_v^0)^{d(a_v)}, \chi_v^0) = \tau(\gamma_v^0, (\psi_v^0)^{a_v \bar{a}_v}, \chi_v^0)$ for some $a_v \in E_v^*$ and hence, by lemma(**??**) we get that τ is special, so there is global data (γ, ψ, χ) such that $\tau = \tau(\gamma, \psi, \chi)$. Let $\omega = \omega(\gamma, \psi, \chi)$. For all finite v for which $\pi_v \cong \omega_v^0$ we then get that ω_v^0 and ω_v have a τ_v-Heisenberg model. This forces $\omega_v^0 \cong \omega_v$, so for the global Weil representation $\omega = \omega(\gamma, \psi, \chi)$ we have that $\pi_v \cong \omega_v$ for almost all finite v.

□

We now come to the main result of this work, and for that we need to make the technical assumption that the extension $E_{v'}/F_{v'}$ equals $\mathbb{C}/\mathbb{R}$ for all the archimedean places v' of F.

Theorem 7.2.3. *Let $\pi = (\pi, V)$ be as in lemma (7.2.1). Assume that there exists a finite set S_1 of places of F, containing only finite places, such that $\pi_v \cong \omega_v^0 = \omega(\gamma_v^0, \psi_v^0, \chi_v^0)$ (for some local Weil representation ω_v^0) for all $v \notin S_1$. Then, there exists a global Weil representation $\omega = (\omega, W)$ such that $\pi = \omega$.*

Proof. Let χ be the central character of π. Recall (from the end of chapter (6)) that by Fourier analysis on the group $U(\mathbf{A})$ we have that any $f \in V$ can be written as

$$f = f_U + \sum_{\{\psi' \neq 1\}} f_{\psi'}$$

where $\{\psi' \neq 1\}$ is the set of all non-trivial additive characters of $F\backslash\mathbf{A}$. Writting simply $d(a)$ for the matrix $d(a, 1, ..., 1, \bar{a}^{-1})$ with $a \in E^*$, by direct calculation, for any $\psi \in \{\psi' \neq 1\}$, we get that

$$f_{\psi^{a\bar{a}}}(g) = f_\psi(d(a)g)$$

for all $g \in G(\mathbf{A})$, and thus we get the formula

$$f(g) = f_U(g) + \sum_{\{\psi'\}} \sum_{a \in E^*/E^1} f_{\psi'}(d(a)g) \tag{1}$$

where $\{\psi'\}$ denotes a set of representatives for the equivalence relation: $\psi_1,\ \psi_2 \in \{\psi' \neq 1\}$ then $\psi_1 \sim \psi_2$ if, and only if, $\psi_2 = {\psi_1}^{a\bar{a}}$ for some $a \in E^*$ (or equivalently, $\psi_2(u) = {\psi_1}^{d(a)}(u) = \psi_1(d(a)ud(a)^{-1})$, for any $u \in U(\mathbf{A})$, identifying $U(\mathbf{A})$ with $\mathbf{A}$).

Now, by lemma (7.2.1), it is possible to find a non-zero $f \in V$ such that $f_\psi \neq 0$ for some non-trivial ψ. Replacing f by a translate, if necessary, we may assume that $f_{\psi,R} \neq 0$, and hence we can find $\tau = \otimes\tau_v$ belonging to $\mathcal{Z}(\psi,\chi)$ such that $(f_{\psi,R})_\tau \neq 0$. This means that we have a non-zero $R(\mathbf{A})$-map $L : \pi \to \tau$. For any finite place $v \notin S_1$ we then get a non-zero map $\omega_v^0 \xrightarrow{\sim} \pi_v \to \tau_v$, hence τ_v must be $P(F_v)$-conjugate to $\tau(\gamma_v^0, \psi_v^0, \chi_v^0)$ and therefore τ_v must be special. By lemma (7.1.2) τ must be special, and then we can find global data (γ,ψ,χ) such that $\tau = \tau(\gamma,\psi,\chi)$. Let $\omega = \omega(\gamma,\psi,\chi)$, and by lemma (7.2.2) we get that $\pi_v \cong \omega_v = \omega(\gamma_v,\psi_v,\chi_v)$ for all finite v such that $v \notin S_1$. We fix the data (γ,ψ,χ) from now on.

We also have the following important fact:

Claim 7.2.4. *For any $f \in V$, and for each character ψ' in the set $\{\psi'\}$ (of characters modulo norms from E^*) there is at most one $\tau' \in \mathcal{Z}(\psi',\chi)$ such that $(f_{\psi',R})_{\tau'} \neq 0$.*

Proof. Let $\tau^1,\ \tau^2 \in \mathcal{Z}(\psi',\chi)$ be such that $f_{\tau^1},\ f_{\tau^2} \neq 0$ for some $f \in V$. Then both τ^1 and τ^2 have to be special by the argument above. Any two elements of $\hat{R}(\psi',\chi)^\circ$ differ by a twist by a character ν of $R(F)N(\mathbf{A})\backslash R(\mathbf{A})$ (since any two oscillator representations of $S'(\mathbf{A})$ have this property) and hence we get that $\tau^2 \cong \nu \otimes \tau^1$ for such ν. For almost all v, by exceptionality of ω_v^0, we have that $\tau_v^2 \cong (\tau_v^1)^{d(a_v)}$ for some $a_v \in E_v^*$ (and $d(a_v) = d(a_v, 1, ..., 1, \bar{a}_v^{-1})$), but since both $\tau^1,\ \tau^2 \in \hat{R}(\psi',\chi)^\circ$, we have that $\tau_v^1,\ \tau_v^2 \in \hat{R}(\psi_v',\chi_v)^\circ$ so $U(F_v)$ acts by ψ_v' on both τ_v^1 and τ_v^2, and this forces $a_v\bar{a}_v = 1$. Thus $\tau_v^1 \cong \tau_v^2$ for almost all v.

Since we also have $\tau_v^2 \cong \nu_v \otimes \tau_v^1$ for all v, we conclude that $\tau_v^1 \cong \nu_v \otimes \tau_v^1$ for almost all v. By Schur's lemma, we get that $\nu_v \equiv 1$ for almost all v and therefore $\nu \equiv 1$, since ν is a Hecke character. Hence $\tau^1 \cong \tau^2$, and by theorem (7.1.3) we get that $\tau^1 = \tau^2$.

Then, by (1), for any $f \in V$, we can write

$$\begin{aligned} f(g) &= f_U(g) + \textstyle\sum_{\{\psi'\}} \sum_{a \in E^*/E^1} f_{\psi'}(d(a)g) \\ &= f_U(g) + \textstyle\sum_{\{\psi'\}} \sum_{a \in E^*/E^1} (\pi(d(a)g) f_{\psi'})(1) \\ &= f_U(g) + \textstyle\sum_{\{\psi'\}} \sum_{a \in E^*/E^1} (\pi(d(a)g) f)_{\psi'}(1) \\ &= f_U(g) + \textstyle\sum_{\{\psi'\}} \sum_{a \in E^*/E^1} (\pi(d(a)g) f)_{\tau(\psi')}(1) \end{aligned}$$

where $\tau(\psi')$ is the unique element of $\hat{R}(\psi', \chi)^\circ$ (given by claim (7.2.4)) for which $(f_{\psi',R})_{\tau'}$ is not (necessarily) identically zero. By Frobenius reciprocity, we get that $\mathrm{Hom}_R(\pi|_R, \tau') \cong \mathrm{Hom}_G(\pi, \mathrm{Ind}_R^G(\tau'))$ and the formula above becomes

$$f(g) = f_U(g) + \sum_{\{\psi'\}} \sum_{a \in E^*/E^1} [\mathcal{L}_{\psi'}(f)(d(a)g)](1) \tag{2}$$

where $\mathcal{L}_{\psi'}$ is the map associated to $(f \mapsto (f_{\psi',R})_{\tau(\psi')})$ by Frobenius reciprocity.

Another important thing to be noticed is that

Claim 7.2.5. *For any $f \in V$, $f_{\psi'} \neq 0$ only for a finite subset $\{\psi_j\}_{j=1}^J$ of $\{\psi'\}$.*

Proof. Indeed, since the map

$$F^*/N(E^*) \hookrightarrow \bigoplus (F_v^*/N(E_v^*))$$

is injective and any $\psi' \in \{\psi'\}$ is of the form $\psi' = \psi^x$ (where recall that $\psi^x(t) = \psi(xt)$, $t \in \mathbf{A}$) for some $x \in F^*$ and for the non-trivial character ψ of $F\backslash\mathbf{A}$ fixed above, we see that $\psi^x \sim \psi$ if, and only if, x is a local norm everywhere.

Now suppose that for some finite $v_0 \notin S_1$ we have that $\psi_{v_0}^x \not\sim \psi_{v_0}$ but $f_{\psi^x} \neq 0$ for some $f \in \pi$. Then $f_{\psi^x}(g_0) \neq 0$ for some $g_0 \in G(\mathbf{A})$, and since $(\pi(g_0)f)_{\psi^x}(1) = (f_{\psi^x})(g_0)$, replacing f by $\pi(g_0)f$, if necessary, we may assume that $f_{\psi^x}(1) \neq 0$. Hence $f_{\psi^x,R} \neq 0$ and we must have $\tau(\psi^x) \in \hat{R}(\psi^x, \chi)^\circ$ such that $(f_{\psi^x,R})_{\tau(\psi^x)} \neq 0$.

Thus we get a non-zero Heisenberg functional from π to $\tau(\psi^x) = \otimes\tau(\psi^x)_v$. In particular, we get a non-zero map

$$\pi_{v_0} \hookrightarrow \pi = \otimes\pi_v \to \tau(\psi^x)$$

from which we derive a non-zero map $\tau(\psi^x) \to \tau(\psi^x)_{v_0}$, and since $\pi_{v_0} \cong \omega_{v_0} = \omega(\gamma_{v_0}, \psi_{v_0}, \chi_{v_0})$ we get (by exceptionality of ω_{v_0}) that $\tau(\psi^x)_{v_0}$ is a $P(F_{v_0})$ conjugate of $\tau(\gamma_{v_0}, \psi_{v_0}, \chi_{v_0})$, which gives $\psi^x_{v_0} \sim \psi_{v_0}$, a contradiction.

Therefore, for any $f \in \pi$, $f_{\psi^x} \equiv 0$ if $\psi^x_v \not\sim \psi_v$ for some finite $v \notin S_1$. For each place $v \in S_1$ or v archimedean, there are at most two orbits of non-trivial additive characters of F modulo norms from E^*, and the claim follows.

Then, for any $f \in \pi$, formulas (1) and (2) become

$$f(g) = f_U(g) + \sum_{\{\psi_j\}} \sum_{a \in E^*/E^1} f_{\psi_j}(d(a)g) \tag{3}$$

$$f(g) = f_U(g) + \sum_{\{\psi_j\}} \sum_{a \in E^*/E^1} [\mathcal{L}_{\psi_j}(f)(d(a)g)](1) \tag{4}$$

where $\{\psi_j\}_{j=1}^J$ is a set of representatives for characters modulo norms from E^*, $\psi_1 = \psi,\ \psi_j = \psi^{\delta_j}$ where δ_j is not a norm from $E^*_{v_j}$ (hence $\psi_j \not\sim \psi$).

Remark. For each $\psi_j \in \{\psi_j\}_{j=1}^J$ let $\tau_j = \tau(\psi_j) = \otimes\tau^j_v$. Note that for v_0 finite, $v_0 \notin S_1$, we have that $\tau^j_{v_0}$ is a $P(F_{v_0})$-conjugate to $\tau_{v_0} = \tau(\gamma_0, \psi_0, \chi_0)$ and since $U(F_{v_0})$ acts by ψ_{v_0} on both τ_{v_0} and $\tau^j_{v_0}$ (after the representatives in $\{\psi_j\}$ have been fixed) we get that $\tau^j_{v_0} \cong \tau_{v_0}$, and we may assume that they coincide.

For the global Weil representation $\omega = \omega(\gamma, \psi, \chi)$ there is only one orbit of non-zero Fourier coefficients (namely the orbit of ψ) and for any $f^0 \in \omega$ we have

$$f^0(g) = f^0_U(g) + \sum_{a \in E^*/E^1} [\mathcal{L}^0(f^0)(d(a)g)](1) \tag{5}$$

where $\mathcal{L}^0$ is the map associated to $(f^0 \mapsto (f^0_{\psi,R})_\tau)$ (with $\tau = \tau(\gamma, \psi, \chi) = \tau(\psi_1)$) by Frobenius reciprocity.

Our goal is to construct "two" non-zero automorphic forms $f^1 \in \pi$ and $f^0 \in \omega$ and to use formulas (4) and (5) to prove that $f^1 = f^0$ (so by irreducibility of both π and ω we conclude that $\pi = \omega$). Since the restriction to $P(\mathbf{A})$ determines an automorphic form (Cf. [PS]), it is enough to construct such f^1 and f^0 such that $f^1|_{P(\mathbf{A})} = f^0|_{P(\mathbf{A})}$. By (4) and (5), for $p = tr \in P(\mathbf{A})$, we get that:

$$f^1(tr) = f^1_U(tr) + \sum_{\{\psi_j\}} \sum_{a \in E^*/E^1} [\tau(\psi_j)(d(a)trt^{-1}d(a)^{-1})\mathcal{L}_{\psi_j}(f^1)(d(a)t)](1) \quad (6)$$

$$f^0(tr) = f^0_U(tr) + \sum_{a \in E^*/E^1} [\tau(d(a)trt^{-1}d(a)^{-1})\mathcal{L}^0(f^0)(d(a)t)](1). \quad (7)$$

Let us consider the following finite sets of places: first we add to S_1 (and denote the new set again by S_1) all the finite places v of F such that π_v (and ω^0_v) is not unramified. Now let us write

$$\begin{array}{lcl} S_{1,n} & = & \{v_1, ..., v_r : v_j \text{ is non-split}\} \\ S_1 & = & \{v_1, ..., v_r, v_{r+1}, ..., v_m\} \\ S_\infty & = & \{v_1^{'}, ..., v_s^{'} : v_j^{'} \text{ is archimedean}\} \\ S_0 & = & S_1 \cup S_\infty \end{array}$$

If S_* is any of the sets above, we use the following notation (writing τ_j for $\tau(\psi_j)$ in (6)):

$$\pi_{S_*} = \otimes_{v \in S_*} \pi_v, \ \pi^{S_*} = \otimes_{v \notin S_*} \pi_v$$

$$\tau^j{}_{S_*} = \otimes_{v \in S_*} \tau^j_v, \ \tau_j{}^{S_*} = \otimes_{v \notin S_*} \tau^j_v$$

$$G_{S_*} = \prod\nolimits_{v \in S_*} G_v, \ G^{S_*} = \prod\nolimits_{v \notin S_*} G_v, \text{ etc.}$$

There are possibly 2^r elements in $\{\psi_j\}$ in formula (6). For each such ψ_j let us denote the map $(f \mapsto (f_{\psi_j,R})_{\tau(\psi_j)})$ by L_j and let us also write simply $\mathcal{L}_j$ for the map $\mathcal{L}_{\psi_j}$ (associated to L_j by Frobenius reciprocity).

Let us also fix the choices of unramified vectors used to realize the various representations above as tensor products. Let us write $\{\xi^0_v\}_{v \notin S_0}$, $\{y^0_v\}_{v \notin S_0}$ and $\{x^0_v\}_{v \notin S_0}$ for the choices of unramified vectors used to realize $\pi \cong \otimes \pi_v$, $\omega \cong \otimes \omega_v$ and $\tau_j \cong \otimes \tau^j_v$ respectively (recall that we are assuming that all the τ^j_v coincide for $v \notin S_0$, since they are all isomorphic and thus we avoid several choices of unramified vectors and simplify the notation). We should point out here that, for $v \notin S_0$, ω_v unramified forces τ_v to be also unramified.

We need to change directions for a little while, and make some archimedean calculations before proceeding with the proof of theorem (7.2.3).

Just for the moment, let $E = \mathbb{C}$, $F = \mathbb{R}$ and consider all the groups defined in section (2) for this new situation (in the definition of $U(\Phi_{2m+1})$ we take for instance $\xi = i \in \mathbb{C}$). Let $G = G_n = U(\Phi_n)$. Fix ψ as usual.

Let $\mathcal{F}$ be a model for the Stone-von Neumann representation τ_ψ of N with central character ψ. We extend τ_ψ to a representation of $S'N$ by letting $s \in S'$ act on $\mathcal{F}$ by an oscillator representation $\omega^{n-2}(\gamma, \psi)$ of G_{n-2} (as before). For each character ν of the center Z' of S' (which is isomorphic to $\mathbb{C}^1$, the norm one elements in $\mathbb{C}$), let $\mathcal{F}(\nu) \subset \mathcal{F}$ be the subspace on which Z' acts by ν. Then $\mathcal{F} = \hat{\oplus}\mathcal{F}(\nu)$ (Hilbert space direct sum).

Writting $\gamma_1 = \gamma|_{\mathbb{C}^1}$, for each character χ of $\mathbb{C}^1$, we define

$$\mathcal{C}_c^\infty(E^*, \mathcal{F})_\chi = \bigoplus_\nu [\mathcal{C}_c^\infty(E^*)_\nu \otimes \mathcal{F}^0(\gamma_1^{-1}\chi\nu)]$$

where $\mathcal{C}_c^\infty(E^*)_\nu$ is the space of smooth, compactly supported functions $\varphi : E^* \to \mathbb{C}$ such that $\varphi(e^{i\theta}x) = \nu(e^{i\theta})\varphi(x)$ for all $e^{i\theta} \in \mathbb{C}^1$, for all $x \in E^*$ (note that we can identify $\mathcal{C}_c^\infty(E^*)_\nu$ with $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$ - where $\mathbb{R}_+^*$ of course denotes the real positive numbers - by writting φ in the form $\varphi(e^{i\theta}x) = \nu(e^{i\theta})f(x)$ for $x \in \mathbb{R}_+^*$ and $f \in \mathcal{C}_c^\infty(\mathbb{R}_+^*)$) and $\mathcal{F}^0(\gamma_1^{-1}\chi\nu)$ is the space of K'-finite vectors in $\mathcal{F}(\gamma_1^{-1}\chi\nu)$ relative to the action of S' (K' is a maximal compact subgroup of S').

The group $T = \{d(a) = d(a, 1, ..., 1, \bar{a}^{-1}) : a \in E^*\}$ acts on $\varphi_\nu \otimes \xi \in \mathcal{C}_c^\infty(E^*, \mathcal{F})_\chi$ (with $\xi \in \mathcal{F}^0(\gamma_1^{-1}\chi\nu)$) by $d(a).(\varphi_\nu \otimes \xi) = \varphi_\nu^a \otimes \xi$, where φ_ν^a is the function in $\mathcal{C}_c^\infty(E^*)_\nu$ defined by $\varphi_\nu^a(x) = \gamma(a)|a\bar{a}|^{1/2}\varphi_\nu(x\bar{a})$. We also have an action of the pair $(\mathfrak{U}(S'), K')$ on $\mathcal{C}_c^\infty(E^*, \mathcal{F})_\chi$ (where $\mathfrak{U}(S')$ denotes the universal enveloping algebra associated to S') and it is easy to check that the center Z of G acts by χ on $\mathcal{C}_c^\infty(E^*, \mathcal{F})_\chi$.

Let now $\mathcal{A}^\infty(E^*, \mathcal{F})$ be the space of all smooth functions on E^* with values in $\mathcal{F}$. Let P acts on $\mathcal{A}^\infty(E^*, \mathcal{F})$ by the mixed model formulas (Cf. proposition (3.1.1)) and we denote by $\mathcal{A}^\infty(E^*, \mathcal{F})_\chi$ the sbspace of $\mathcal{A}^\infty(E^*, \mathcal{F})$ on which the center Z of P acts by χ. With the same notation as before, let $\tau = \chi \otimes \omega^{n-2}(\gamma,\psi) \otimes \tau_\psi$ be the representation of R on $\mathcal{F}$ such that for $z \in Z$, $s \in S'$, $n \in N$ we define $\tau(zsn) = \chi(z)\omega^{n-2}(\gamma,\psi)(s)\tau_\psi(n)$. It is simple to prove that $\mathrm{Ind}_R^P(\tau)$ is isomorphic to the space of smooth (not necessarily compactly supported) functions

$$\tilde{\mathcal{C}}_\tau(E^*,\mathcal{F}) = \{\phi : E^* \to \mathcal{F} \text{ such that } \phi(e^{i\theta}x) = \tau(d(e^{i\theta}))\phi(x)\}$$

where the action α_τ on $\tilde{\mathcal{C}}_\tau(E^*,\mathcal{F})$ is given by the formulas in section (4.1). Furthermore, the map

$$\begin{array}{ccc} \mathcal{A}^\infty(E^*,\mathcal{F})_\chi & \to & \tilde{\mathcal{C}}_\tau(E^*,\mathcal{F}) \\ \varphi & \mapsto & \tilde{\varphi} \end{array}$$

where $\tilde{\varphi}(x) = \gamma(x)|x\bar{x}|^{1/2}\varphi(\bar{x})$ is a P-isomorphism.

Notice that the map

$$\begin{array}{ccc} \mathcal{C}_c^\infty(E^*,\mathcal{F})_\chi & \to & \mathcal{A}^\infty(E^*,\mathcal{F})_\chi \\ \varphi_\nu \otimes \xi & \mapsto & \varphi_{\nu,\xi} \end{array}$$

where $\varphi_{\nu,\xi}(x) = \varphi_\nu(x)\xi$, defines a natural embedding. On the other hand, global Fourier coefficients $(f \mapsto (f_{\psi,R})_\tau)$ give rise to a linear map (after applying Frobenius reciprocity)

$$\phi : \mathcal{C}_c^\infty(E^*,\mathcal{F})_\chi \to \mathcal{A}^\infty(E^*,\mathcal{F})_\chi$$

and if we write

$$\mathcal{C}^\infty(E^*,\mathcal{F})_\chi = \bigoplus_\nu [\mathcal{C}^\infty(E^*)_\nu \otimes \mathcal{F}^0(\gamma_1^{-1}\chi\nu)]$$

($\mathcal{C}^\infty(E^*)_\nu$ equals $\mathcal{C}_c^\infty(E^*)_\nu$ without the support condition) then we see that the image of ϕ is contained in $\mathcal{C}^\infty(E^*,\mathcal{F})_\chi$, since ϕ intertwines the action of K'. Thus we may view ϕ as a map from $\mathcal{C}_c^\infty(E^*,\mathcal{F})_\chi$ to $\mathcal{C}^\infty(E^*,\mathcal{F})_\chi$ which intertwines the action of the groups T, U, and of the pair $(\mathfrak{U}(S'),K')$. Therefore it induces the maps ϕ_ν below, that make the following diagram commute:

$$\begin{array}{ccc} \bigoplus[\mathcal{C}_c^\infty(E^*)_\nu \otimes \mathcal{F}^0(\gamma_1^{-1}\chi\nu)] & \xrightarrow{\phi} & \bigoplus[\mathcal{C}^\infty(E^*)_\nu \otimes \mathcal{F}^0(\gamma_1^{-1}\chi\nu)] \\ \uparrow & & \downarrow \\ \mathcal{C}_c^\infty(E^*)_\nu \otimes \mathcal{F}^0(\gamma_1^{-1}\chi\nu) & \xrightarrow{\phi_\nu} & \mathcal{C}^\infty(E^*)_\nu \otimes \mathcal{F}^0(\gamma_1^{-1}\chi\nu) \end{array}$$

(where the vertical maps are just the standard inclusion and projection).

The maps ϕ_ν necessarily factor, since $\mathcal{F}^0(\gamma_1^{-1}\chi\nu)$ is a direct sum of finite dimensional representations of K'. In fact, we see that ϕ_ν is determined by maps $\mathcal{C}_c^\infty(E^*)_\nu \to \mathcal{C}^\infty(E^*)_\nu$ and these reduce to maps $\mathcal{C}_c^\infty(\mathbb{R}_+^*) \to \mathcal{C}^\infty(\mathbb{R}_+^*)$ commuting with the action of the groups $\{d(a) : a \in \mathbb{R}_+^*\}$ and $\{[0,t] : t \in \mathbb{R}\}$, which is given by the formulas (coming from the mixed model formulas):

. $(d(a)f) = \gamma(a)|a|f(xa)$

. $([0,t]f) = \psi(tx^2)f(x)$

for all $a \in \mathbb{R}_+^*$, $t \in \mathbb{R}$ and $f \in \mathcal{C}^\infty(\mathbb{R}_+^*)$). Now we apply the technical lemma below to conclude that each ϕ_ν (and therefore ϕ) is the standard inclusion.

Technical Lemma 1. *Let the group of matrices* $H = \{\begin{pmatrix} a & t \\ 0 & a^{-1}\end{pmatrix} : a \in \mathbb{R}_+^*,\ t \in \mathbb{R}\} \subset SL_2(\mathbb{R})$ *act on the spaces* $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$ *and* $\mathcal{C}^\infty(\mathbb{R}_+^*)$ *by the formulas:*

. $(\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}.f)(x) = \gamma(a)|a|f(xa)$

. $(\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}.f)(x) = e^{itx^2}f(x)$.

Let $T : \mathcal{C}_c^\infty(\mathbb{R}_+^*) \to \mathcal{C}^\infty(\mathbb{R}_+^*)$ *be a linear map which is continuous in the topology defined by the semi-norms*

$$P_{K,m}(f) = \sup_{\substack{x\in K\\ 0\le j\le m}} |f^{(j)}(x)|$$

for $K \subset \mathbb{R}_+^*$ *compact. Assume also that* T *intertwines the action of* H*. Then* T *is a multiple of the standard inclusion.*

Proof. For $t \in \mathbb{R}$ let $u_t(x) = e^{itx^2}$. If f is any function in $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$, then $\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}.f = u_t f$ and hence we get that $T(u_t f) = u_t T(f)$. By linearity, $T(\sum u_{t_j} f) = (\sum u_{t_j})T(f)$ for all u_{t_j} and f as above.

Consider the distribution

$$\begin{array}{rccc} \lambda & : & \mathcal{C}_c^\infty(\mathbb{R}_+^*) & \to & \mathbb{C} \\ & & f & \mapsto & T(f)(1)\ . \end{array}$$

We claim that λ is a distribution supported at $\{1\}$. To see this, let $f \in \mathcal{C}_c^\infty(\mathbb{R}_+^*)$ be such that $1 \notin \mathrm{supp}(f)$. Then f vanishes on an open interval around

1. Choose functions

$$u(x) = \sum_{j=1}^{N} a_j e^{it_j x^2}$$

such that $u(1) = 0$ and u approaches the constant 1 uniformly on $\mathrm{supp}(f)$. Then, $T(uf) = uT(f)$ implies that

$$T(uf)(1) = u(1)T(f)(1) = 0 \ .$$

By continuity of T, we get $T(f)(1) = 0$, i.e., $\lambda(f) = 0$, hence we conclude that $\mathrm{supp}(\lambda) = \{1\}$.

Now, by [Y], Theorem(3), p. 64 we get that

$$\lambda(f) = \sum_{j=0}^{M} a_j f^{(j)}(1) \ .$$

But for any u as above we have

$$\lambda(uf) = T(uf)(1) = u(1)T(f)(1) = u(1)\lambda(f) = u(1)\sum_{j=0}^{M} a_j f^{(j)}(1) \ ,$$

thus,

$$\sum_{j=0}^{M} a_j (uf)^{(j)}(1) = u(1)\sum_{j=0}^{M} a_j f^{(j)}(1)$$

and this implies that $\lambda(f) = a_0 f(1)$ for all $f \in \mathcal{C}_c^\infty(\mathbb{R}_+^*)$. In other words, $T(f)(1) = a_0 f(1)$ for all $f \in \mathcal{C}_c^\infty(\mathbb{R}_+^*)$, hence by dilation invariance we get $T(f)(x) = a_0 f(x)$, for all $x \in \mathbb{R}_+^*$ and for all $f \in \mathcal{C}_c^\infty(\mathbb{R}_+^*)$, which proves the lemma.

□

Now we come back to the proof of theorem (7.2.3). Using the local uniqueness of Heisenberg models for Weil representations established in section (5.2), one employs a standard procedure (Cf. [G]) to prove that $L_j : \pi_{S_0} \otimes \pi^{S_0} \to \tau_{S_0}^j \otimes \tau_j^{S_0}$ can be written as $L_j = L_{S_0}^j \otimes L_j^{S_0} = L_{S_0}^j \otimes (\prod_{v \notin S_0} L_v^1)$, where $L_v^1 : \pi_v \to \tau_v$ is the unique element L_v of $\mathrm{Hom}_{R_v}(\pi_v, \tau_v)$ such that $L_v(\xi_v^0) = x_v^0$ (note that since $\pi_v \cong \omega_v$ for all $v \notin S_0$, we have $\dim(\mathrm{Hom}_{R_v}(\pi_v, \tau_v)) = 1$).

From this factorization we also get

$$\mathcal{L}_j = \mathcal{L}_{S_0}^j \otimes (\otimes_{v \notin S_0} \mathcal{L}_v^1) \ ,$$

where $\mathcal{L}^j_{S_0}$ (respectively $\mathcal{L}^1_v$) is the map associated to $L^j_{S_0}$ (respectively L^1_v) by Frobenius reciprocity.

In formula (6) we are only concerned with $\mathcal{L}_j|_{T(\mathbf{A})}$, and for any $f \in \pi$ we then have the formula:

$$\mathcal{L}_j(f)|_{T(\mathbf{A})} = \mathcal{L}^j_{S_0}(f_{S_0})|_{T_{S_0}} \otimes (\otimes_{v \notin S_0} \mathcal{L}^1_v(f_v)|_{T_v}) .$$

If for each $v' \in S_\infty$ we choose $f_{v'}$ in $\mathcal{C}^\infty_c(E^*_{v'}, \mathcal{F}_{v'})_{\chi_{v'}}$, then by our archimedean calculations above the map $\mathcal{L}^j_{S_0}(f_{S_0})|_{T_{S_0}}$ has to factor at v', and actually $\mathcal{L}^j_{v'}(f_{v'})|_{T_{v'}}$ has to be the canonical inclusion from $\mathcal{C}^\infty_c(E^*_{v'}, \mathcal{F}_{v'})_{\chi_{v'}}$ to $\mathcal{C}^\infty(E^*_{v'}, \mathcal{F}_{v'})_{\chi_{v'}}$. Picking a non-zero $f^0_{v'}$ for each archimedean place v', for any $f = (\otimes_{v' \in S_\infty} f^0_{v'}) \otimes f^{S_\infty}$, we get the factorization

$$\mathcal{L}_j(f)|_{T(\mathbf{A})} = \mathcal{L}^j_{S_1}(f_{S_1})|_{T_{S_1}} \otimes (\otimes_{v' \in S_\infty} \mathcal{L}^j_{v'}(f^0_{v'})|_{T_{v'}}) \otimes (\otimes_{v \notin S_0} \mathcal{L}^1_v(f_v)|_{T_v}). \qquad (8)$$

Let us divide the rest of the proof into steps:

Step(1): For $v_k \in S_1 = \{v_1, ..., v_m\}$, let us write $\tau_{v_k} = (\tau_{v_K}, \mathcal{F}_{v_k})$ and denote (π_{v_k}, V_{v_k}) simply by (π_k, V_k). For $f_1 \otimes ... \otimes f_m \in V_1 \otimes ... \otimes V_m$ we will write $\varphi_{f_1 \otimes ... \otimes f_m}(\,\cdot\,, \tau^j_{S_1})$ for the function $\mathcal{L}^j_{S_1}(f_1 \otimes ... \otimes f_m)|_{T^{S_1}} \in \mathcal{S}(T_{v_1} \times ... \times T_{v_m}, \mathcal{F}_{v_1} \otimes ... \otimes \mathcal{F}_{v_m}) \cong \otimes^m_{k=1} \mathcal{S}(T_{v_k}, \mathcal{F}_{v_k})$.

Then there exists $\mathfrak{f}^0_1 \otimes ... \otimes \mathfrak{f}^0_m \in V^0_1 \otimes ... \otimes V^0_m$ (recall $V^0_k = V_k(U(F_{v_k})) = \mathrm{span}\{\pi_k(u)\xi - \xi : u \in U(F_{v_k}),\ \xi \in V_k\}$) such that $\varphi_{\mathfrak{f}^0_1 \otimes ... \otimes \mathfrak{f}^0_m}(\,\cdot\,, \tau^1_{S_1}) \neq 0$ and $\varphi_{\mathfrak{f}^0_1 \otimes ... \otimes \mathfrak{f}^0_m}(\,\cdot\,, \tau^j_{S_1}) = 0$ for all $j = 2, ..., J$.

Proof. The proof is divided in two parts. First we show that it is possible to get $f^0_1 \otimes ... \otimes f^0_m \in V^0_1 \otimes ... \otimes V^0_m$ such that $\varphi_{f^0_1 \otimes ... \otimes f^0_m}(\,\cdot\,, \tau^1_{S_1}) \neq 0$ and then we work on $f^0_1 \otimes ... \otimes f^0_m$ a bit more to get $\mathfrak{f}^0_1 \otimes ... \otimes \mathfrak{f}^0_m$ satisfying step(1).

Recall that we know that $L^1_{S_1} : \pi_{S_1} \to \tau^1_{S_1}$ is a non-zero map. Suppose that we had $L^1_{S_1}|_{V^0_1 \otimes ... \otimes V^0_m} \equiv 0$. Choose $u \in U(F_{v_1})$ such that $\psi_{v_1}(u) \neq 1$. Then, for any $f_1 \otimes f^0_2 \otimes ... \otimes f^0_m \in V_1 \otimes V_2 ... \otimes V_m$ we have

$$\begin{aligned}
& L^1_{S_1}(\pi_1 \otimes ... \otimes \pi_m(u, 1, ..., 1) f_1 \otimes f^0_2 \otimes ... \otimes f^0_m) \\
= \ & L^1_{S_1}(\pi_1(u) f_1 \otimes f^0_2 \otimes ... \otimes f^0_m) \\
= \ & L^1_{S_1}((\pi_1(u) f_1 - f_1) \otimes f^0_2 \otimes ... \otimes f^0_m + f_1 \otimes f^0_2 \otimes ... \otimes f^0_m) \\
= \ & L^1_{S_1}(f_1 \otimes f^0_2 \otimes ... \otimes f^0_m)
\end{aligned}$$

since $(\pi_1(u)f_1 - f_1) \in V_1^0$. On the other hand,

$$\begin{aligned} & L^1_{S_1}(\pi_1 \otimes ... \otimes \pi_m(u,1,...,1)f_1 \otimes f_2^0 \otimes ... \otimes f_m^0) \\ = \; & \tau^1_{S_1}(u,1,...,1)L^1_{S_1}(f_1 \otimes f_2^0 \otimes ... \otimes f_m^0) \\ = \; & \tau^1_{v_1} \otimes \tau^1_{v_2} \otimes ... \otimes \tau^1_{v_m}(u,1,...,1)L^1_{S_1}(f_1 \otimes f_2^0 \otimes ... \otimes f_m^0) \\ = \; & \psi_{v_1}(u)L^1_{S_1}(f_1 \otimes f_2^0 \otimes ... \otimes f_m^0) \end{aligned}$$

and hence $L^1_{S_1}(f_1 \otimes f_2^0 \otimes ... \otimes f_m^0) = 0$, or yet, $L^1_{S_1}|_{V_1 \otimes V_2 ... \otimes V_m} \equiv 0$.

The same calculations now done for V_2 would give us that $L^1_{S_1}|_{V_1 \otimes V_2 \otimes V_3^0 \otimes ... \otimes V_m^0} \equiv 0$, and finally after m steps we would get $L^1_{S_1} \equiv 0$, a contradiction.

Hence $L^1_{S_1}|_{V_1^0 \otimes ... \otimes V_m^0} \not\equiv 0$ and choosing $f_1^0 \otimes ... \otimes f_m^0$ in $V_1^0 \otimes ... \otimes V_m^0$ such that $L^1_{S_1}(f_1^0 \otimes ... \otimes f_m^0) \neq 0$ (note that clearly $L^1_{S_1}|_{V_1^0 \otimes ... \otimes V_m^0} \not\equiv 0$ implies that we can find $f_1^0 \otimes ... \otimes f_m^0$ - a single tensor - such that $L^1_{S_1}(f_1^0 \otimes ... \otimes f_m^0) \neq 0$) we have that $\varphi_{f_1^0 \otimes ... \otimes f_m^0}(\,\cdot\,, \tau^1_{S_1}) \not\equiv 0$.

Let us write each $(\tau^j_{S_1}, \mathcal{F}_j)$ as $(\tau^j_{v_1} \otimes ... \otimes \tau^j_{v_m}, \mathcal{F}^j_{v_1} \otimes ... \otimes \mathcal{F}^j_{v_m})$. Note that for each $f_1 \otimes ... \otimes f_m \in V_1^0 \otimes ... \otimes V_m^0$, $\varphi_{f_1 \otimes ... \otimes f_m}(\,\cdot\,, \tau^j_{S_1})$ belongs to $\mathcal{S}(T_1 \times ... \times T_m, \mathcal{F}^j_{v_1} \otimes ... \otimes \mathcal{F}^j_{v_m})$ which is isomorphic to $\mathcal{S}(T_1, \mathcal{F}^j_{v_1}) \otimes ... \otimes \mathcal{S}(T_m, \mathcal{F}^j_{v_m})$ (on which $\alpha_{\tau^j_{v_1}} \otimes ... \otimes \alpha_{\tau^j_{v_m}}$ acts).

We claim that $\alpha_{\tau^j_{v_1}} \otimes ... \otimes \alpha_{\tau^j_{v_m}} \cong \alpha_{\tau^l_{v_1}} \otimes ... \otimes \alpha_{\tau^l_{v_m}}$ if, and only if $j = l$.

To see that, first notice that it suffices to prove that $\alpha_{\tau^j_{v_1}} \otimes ... \otimes \alpha_{\tau^j_{v_m}} \cong \alpha_{\tau^l_{v_1}} \otimes ... \otimes \alpha_{\tau^l_{v_m}}$ if, and only if $\alpha_{\tau^j_{v_k}} \cong \alpha_{\tau^l_{v_k}}$ for all $k = 1, ..., m$, since (from our local computations on the representations α_τ) this last condition implies that $\tau^j_{S_1}$ and $\tau^l_{S_1}$ are $T_1 \times ... \times T_m$-conjugates, which can not happen unless $\tau^j_{S_1} = \tau^l_{S_1}$ (i.e., $j = l$).

Now, clearly $\alpha_{\tau^j_{v_k}} \cong \alpha_{\tau^l_{v_k}}$ for all $k = 1, ..., m$ implies $\alpha_{\tau^j_{v_1}} \otimes ... \otimes \alpha_{\tau^j_{v_m}} \cong \alpha_{\tau^l_{v_1}} \otimes ... \otimes \alpha_{\tau^l_{v_m}}$. Conversely, assume that we have an isomorphism $\phi : \alpha_{\tau^j_{v_1}} \otimes ... \otimes \alpha_{\tau^j_{v_m}} \to \alpha_{\tau^l_{v_1}} \otimes ... \otimes \alpha_{\tau^l_{v_m}}$. Fix a non-zero vector $\xi_2 \otimes ... \otimes \xi_m \in \alpha_{\tau^j_{v_2}} \otimes ... \otimes \alpha_{\tau^j_{v_m}}$ and basis $\{y^1_{i_1}\}, ..., \{y^m_{i_m}\}$ of $\alpha_{\tau^l_{v_1}}$,...,$\alpha_{\tau^l_{v_m}}$ respectively (where the i_k belong to some families of indeces). For each $h \in \alpha_{\tau^j_{v_1}}$ we have that

$$\phi(h \otimes \xi_2 \otimes ... \otimes \xi_m) = \sum_{i_1} ... \sum_{i_m} \lambda_{i_1,...,i_m}(h) y^1_{i_1}(h) \otimes ... \otimes y^m_{i_m}(h) \tag{9}$$

(this is a finite sum, listing of course only the non-zero $\lambda_{i_1,...,i_m}(h)$).

The thing to notice here is that $y^2_{i_2}(h) \otimes ... \otimes y^m_{i_m}(h)$ belong to a finite dimensional subspace of $\alpha_{\tau^l_{v_2}} \otimes ... \otimes \alpha_{\tau^l_{v_m}}$, as we vary h.

To see this, let $P^0_{v_2} \times ... \times P^0_{v_m}$ be an open subgroup of $P(F_{v_2}) \times ... \times P(F_{v_m})$ such that $(\alpha_{\tau^j_{v_2}} \otimes ... \otimes \alpha_{\tau^j_{v_m}})(p_2, ..., p_m)\xi_2 \otimes ... \otimes \xi_m = \xi_2 \otimes ... \otimes \xi_m$, for all $(p_2, ..., p_m) \in P^0_{v_2} \times ... \times P^0_{v_m}$. For a fix $k \in \{2, ..., m\}$ let $\hat{p_k} = (1, ..., 1, p_k, 1, ..., 1) \in P^0_{v_2} \times ... \times P^0_{v_k} \times ... \times P^0_{v_m}$. Then we have that

$$\begin{array}{rl} & \phi((\alpha_{\tau^j_{v_1}} \otimes ... \otimes \alpha_{\tau^j_{v_m}})(1, \hat{p_k})h \otimes \xi_2 \otimes ... \otimes \xi_m) \\ = & \phi(h \otimes \xi_2 \otimes ... \otimes \xi_m) \\ = & \sum_{i_1} ... \sum_{i_m} \lambda_{i_1,...,i_m}(h) y^1_{i_1}(h) \otimes ... \otimes y^m_{i_m}(h) \end{array}$$

and on the other hand,

$$\begin{array}{rl} & \phi((\alpha_{\tau^j_{v_1}} \otimes ... \otimes \alpha_{\tau^j_{v_m}})(1, \hat{p_k})h \otimes \xi_2 \otimes ... \otimes \xi_m) \\ = & \alpha_{\tau^l_{v_1}} \otimes ... \otimes \alpha_{\tau^l_{v_m}}(1, \hat{p_k})\phi(h \otimes \xi_2 \otimes ... \otimes \xi_m) \\ = & \sum_{i_1} ... \sum_{i_m} \lambda_{i_1,...,i_m}(h)[y^1_{i_1}(h) \otimes ... \otimes \alpha_{\tau^l_{v_k}}(p_k) y^k_{i_k}(h) \otimes ... \otimes y^m_{i_m}(h)] \end{array}$$

from what we get that

$$\begin{array}{rl} & \lambda_{i_1,...,i_m}(h) y^1_{i_1}(h) \otimes ... \otimes y^k_{i_k}(h) \otimes ... \otimes y^m_{i_m}(h) \\ = & \lambda_{i_1,...,i_m}(h) y^1_{i_1}(h) \otimes ... \otimes \alpha_{\tau^l_{v_k}}(p_k) y^k_{i_k}(h) \otimes ... \otimes y^m_{i_m}(h) \end{array}$$

hence $y^k_{i_k}(h) \in (\alpha_{\tau^l_{v_k}})^{P^0_{v_k}}$, which is finite dimensional (by admissibility). Since $k \in \{2, ..., m\}$ was arbitrary, we get that $y^2_{i_2}(h) \otimes ... \otimes y^m_{i_m}(h)$ belongs to a finite dimensional subspace of $\alpha_{\tau^l_{v_2}} \otimes ... \otimes \alpha_{\tau^l_{v_m}}$, independent of h.

Let us then fix $\{y^2_{i_2}\}^{N_2}_{i_2=1}, ..., \{y^m_{i_m}\}^{N_m}_{i_m=1}$ basis for $(\alpha_{\tau^l_{v_2}})^{P^0_{v_2}}, ..., (\alpha_{\tau^l_{v_m}})^{P^0_{v_m}}$ respectively. Then, for each $h \in \alpha_{\tau^j_{v_1}}$ we can write (let us write $\xi = \xi_2 \otimes ... \otimes \xi_m$)

$$\phi(h \otimes \xi) = \sum_{i_2=1}^{N_2} ... \sum_{i_m=1}^{N_m} z_{i_2,...,i_m}(h) \otimes y^2_{i_2}(h) \otimes ... \otimes y^m_{i_m}(h)$$

with $z_{i_2,...,i_m}(h) \in \alpha_{\tau^l_{v_1}}$. Since $\xi \neq 0$ and ϕ has trivial kernel, $z_{i_2,...,i_m}(h)$ must be a non-zero element of $\alpha_{\tau^l_{v_1}}$ when $h \neq 0$, for some $i_{k_2}, ..., i_{k_m}$. Let us assume for simplicity that for some $h \neq 0$, $z_{1,...,1}(h) \neq 0$. then we a get a non-zero P_{v_1}-map

$$\begin{array}{ccccccc} \alpha_{\tau^j_{v_1}} & \stackrel{i_1}{\hookrightarrow} & \alpha_{\tau^j_{v_1}} \otimes ... \otimes \alpha_{\tau^j_{v_m}} & \stackrel{\phi}{\to} & \alpha_{\tau^l_{v_1}} \otimes ... \otimes \alpha_{\tau^l_{v_m}} & \stackrel{\wp_{1,...,1}}{\to} & \alpha_{\tau^l_{v_1}} \\ h & \mapsto & h \otimes \xi & \mapsto & \phi(h \otimes \xi) & \mapsto & z_{1,...,1}(h)\ . \end{array}$$

By irreducibility of both $\alpha_{\tau^j_{v_1}}$ and $\alpha_{\tau^l_{v_1}}$ we get that $\alpha_{\tau^j_{v_1}} \cong \alpha_{\tau^l_{v_1}}$. Similarly $\alpha_{\tau^j_{v_k}} \cong \alpha_{\tau^l_{v_k}}$ for all $k = 2, ..., m$ and this completes the proof of our claim.

Now let us go back to the proof of Step(1). Recall that we have already selected an element $f^0_1 \otimes ... \otimes f^0_m$ of $V^0_1 \otimes ... \otimes V^0_m$ such that $\varphi_{f^0_1 \otimes ... \otimes f^0_m}(\,\cdot\,, \tau^1_{v_1} \otimes ... \otimes \tau^1_{v_m}) \neq 0$. Therefore the J elements $\varphi_{f^0_1 \otimes ... \otimes f^0_m}(\,\cdot\,, \tau^j_{v_1} \otimes ... \otimes \tau^j_{v_m})$ form a finite collection of vectors belonging to the inequivalent irreducible representations $\alpha_{\tau^j_{v_1}} \otimes ... \otimes \alpha_{\tau^j_{v_m}}$ and then it is possible (see for instance [L], p. 650) to find a smooth compactly supported function $\lambda = \lambda_1 \otimes ... \otimes \lambda_m$ on $P_1 \times ... \times P_m$ such that

$$(\alpha_{\tau^1_{v_1}} \otimes ... \otimes \alpha_{\tau^1_{v_m}})(\lambda)\varphi_{f^0_1 \otimes ... \otimes f^0_m}(\,\cdot\,, \tau^1_{v_1} \otimes ... \otimes \tau^1_{v_m}) = \varphi_{f^0_1 \otimes ... \otimes f^0_m}(\,\cdot\,, \tau^1_{v_1} \otimes ... \otimes \tau^1_{v_m})$$

and

$$(\alpha_{\tau^1_{v_1}} \otimes ... \otimes \alpha_{\tau^1_{v_m}})(\lambda)\varphi_{f^0_1 \otimes ... \otimes f^0_m}(\,\cdot\,, \tau^j_{v_1} \otimes ... \otimes \tau^j_{v_m}) = 0$$

for all $j = 2, ..., J$. Since

$$(\alpha_{\tau^1_{v_1}} \otimes ... \otimes \alpha_{\tau^1_{v_m}})(\lambda)\varphi_{f^0_1 \otimes ... \otimes f^0_m}(\,\cdot\,, \tau^1_{v_1} \otimes ... \otimes \tau^1_{v_m}) =$$

$$\varphi_{\pi_1(\lambda_1) f^0_1 \otimes ... \otimes \pi_m(\lambda_m) f^0_m}(\,\cdot\,, \tau^j_{v_1} \otimes ... \otimes \tau^j_{v_m})$$

for all j and $\pi_1(\lambda_1) f^0_1 \otimes ... \otimes \pi_m(\lambda_m) f^0_m$ still belongs to $V^0_1 \otimes ... \otimes V^0_m$, step(1) follows.

Step(2): There exists non-zero $f^1 \in \pi$, and non-zero $f^0 \in \omega$ such that $\mathcal{L}_1(f^1)|_{T(\mathbf{A})} = \mathcal{L}^0(f^0)|_{T(\mathbf{A})}$ and $\mathcal{L}_j(f^1)|_{T(\mathbf{A})} = 0$ for all $j = 2, ..., J$.

Proof. For each $v' \in S_\infty$ choose $f^0_{v'}$ as above. For $\mathfrak{f}^0_1 \otimes ... \otimes \mathfrak{f}^0_m$ as in step(1), let

$$f^1 = (\mathfrak{f}^0_1 \otimes ... \otimes \mathfrak{f}^0_m) \otimes (\otimes_{v' \in S_\infty} f^0_{v'}) \otimes (\otimes_{v \notin S_0} \xi^0_v) \; .$$

By formula (8) we see that

$$\mathcal{L}_j(f^1)|_{T(\mathbf{A})} = \mathcal{L}^j_{S_1}(\mathfrak{f}^0_1 \otimes ... \otimes \mathfrak{f}^0_m)|_{T_{S_1}} \otimes (\otimes_{v' \in S_\infty} \mathcal{L}^j_{v'}(f^0_{v'})|_{T_{v'}}) \otimes (\otimes_{v \notin S_0} \mathcal{L}^1_v(\xi^0_v)|_{T_v})$$

is non-zero precisely for $j = 1$ (by step(1)).

Let us now look at the ω side. First, for $v \notin S_0$ we have the diagram

$$\begin{tikzcd} \pi_v \arrow[rr, "\overset{\sigma_v}{\sim}"] \arrow[dr, "L_v^1"'] & & \omega_v \arrow[dl, "L_v^0"] \\ & \tau_v & \end{tikzcd}$$

and by uniqueness of Heisenberg models we get that there exists $\lambda_v \in \mathbb{C}^*$ such that $L_v^1 = \lambda_v L_v^0 \circ \sigma_v$. By changing our fixed unramified vectors by multiples if necessary, we may assume that $L_v^1 = L_v^0 \circ \sigma_v$ for all $v \notin S_0$. From this follows that $\mathcal{L}_v^1 = \mathcal{L}_v^0 \circ \sigma_v$ for all $v \notin S_0$.

Finally, let $\mathcal{L}_{S_1}^1(\mathfrak{f}_1^0 \otimes ... \otimes \mathfrak{f}_m^0)|_{T_{S_1}} = \sum_{i=1}^I \phi_1^i \otimes ... \otimes \phi_m^i$ which belongs to $\mathcal{S}(T_{v_1} \times ... \times T_{v_m}, \mathcal{F}_{v_1} \otimes ... \otimes \mathcal{F}_{v_m}) \cong \otimes_{k=1}^m \mathcal{S}(T_{v_k}, \mathcal{F}_{v_k})$. For each $i = 1, ..., I$ choose $h_k^i \in \omega_{v_k}$ such that $\mathcal{L}_{v_k}^0(h_k^i)|_{T_{v_k}} = \phi_k^i$ (recall that locally, for a finite v and a Weil representation (ω_v, W_v) we have that $(\omega_v|_{P_v}, W_v^0) \cong (\alpha_{\tau_v}, \mathcal{S}(T_v, \mathcal{F}_v))$ and the isomorphism is given by $(h \mapsto \mathcal{L}_v^0(h)|_{T_v})$, so we can choose the h_k^i above).

Now let

$$f^0 = [\sum_{i=1}^I h_1^i \otimes ... \otimes h_m^i] \otimes (\otimes_{v' \in S_\infty} f_{v'}^0) \otimes (\otimes_{v \notin S_0} \sigma_v(\xi_v^0)) \ .$$

Then, it is easy to see that $\mathcal{L}^0(f^0)|_{T(\mathbf{A})} = \mathcal{L}^1(f^1)|_{T(\mathbf{A})}$ and step(2) follows (we are assuming for simplincity that for $v' \in S_\infty$, $\mathcal{L}_{v'}^1(.)|_{T_{v'}}$ and $\mathcal{L}_{v'}^0(.)|_{T_{v'}}$ are both equal to the standard inclusion

$$\mathcal{C}_c^\infty(E^*v', \mathcal{F}_{v'})_{\chi_{v'}} \hookrightarrow \mathcal{C}^\infty(E_{v'}^*, \mathcal{F}_{v'})_{\chi_{v'}}$$

rather than non-zero multiples).

Step(3): $f^1 = f^0$.

Proof. Let $\varphi = f^1 - f^0$. Since φ is an automorphic form on $G(\mathbf{A})$, to prove that $\varphi = 0$ it is enough to show that $\varphi|_{P(\mathbf{A})} = 0$. By step(2) above and formulas (6) and (7) we see that for any $p \in P(\mathbf{A})$, $\varphi(p) = \varphi_U(p) = f_U^1(p) - f_U^0(p)$.

We claim that the choices of f^1 and f^0 force $f^1|_{P(\mathbf{A})} = f^0|_{P(\mathbf{A})} = 0$. Indeed, we have $f^1 = (\mathfrak{f}_1^0 \otimes ... \otimes \mathfrak{f}_m^0) \otimes (f^1)^{S_1}$ and $\mathfrak{f}_k^0 \in V_{v_k}^0$ for $k = 1, ..., m$. Say

$\mathfrak{f}_1^0 = \sum_{j=1}^{J_1} \pi_{v_1}(u_j)h_j - h_j$ with $u_j \in U(F_{v_1})$ and $h_j \in V_{v_1}$. Let $\tilde{u}_j = (u_j, 1, 1, ...) \in U(\mathbf{A})$. Then,

$$\begin{aligned} f^1 &= (\textstyle\sum_{j=1}^{J_1} \pi_{v_1}(u_j)h_j - h_j) \otimes (f^1)^{\{v_1\}} \\ &= \textstyle\sum_{j=1}^{J_1} \pi_{v_1}(u_j)h_j \otimes (f^1)^{\{v_1\}} - h_j \otimes (f^1)^{\{v_1\}} \\ &= \textstyle\sum_{j=1}^{J_1} \pi(\tilde{u}_j)h_j \otimes (f^1)^{\{v_1\}} - h_j \otimes (f^1)^{\{v_1\}} \\ &= \textstyle\sum_{j=1}^{J_1} \pi(\tilde{u}_j)\tilde{h}_j - \tilde{h}_j \end{aligned}$$

($\tilde{h}_j = h_j \otimes (f^1)^{\{v_1\}}$, of course). Hence, for any $p \in P(\mathbf{A})$,

$$\begin{aligned} f_U^1(p) &= \textstyle\int_{U(F)\backslash U(\mathbf{A})} f^1(up)\, du \\ &= \textstyle\int_{U(F)\backslash U(\mathbf{A})} (\sum_{j=1}^{J_1} \pi(\tilde{u}_j)\tilde{h}_j - \tilde{h}_j)(up)\, du \\ &= \textstyle\sum_{j=1}^{J_1} (\int_{U(F)\backslash U(\mathbf{A})} \pi(\tilde{u}_j)\tilde{h}_j(up)\, du - \int_{U(F)\backslash U(\mathbf{A})} \tilde{h}_j(up)\, du) \\ &= 0\ . \end{aligned}$$

Also, $f^0 = [\sum_{i=1}^{I} h_1^i \otimes ... \otimes h_m^i] \otimes (\otimes_{v' \notin S_\infty} f_{v'}^0) \otimes (\otimes_{v \notin S_0} \sigma_v(\xi_v^0))$ with each h_k^i in $W_{v_k}^0$, and the same argument shows that $f_U^0|_{P(\mathbf{A})} = 0$. We conclude that $\varphi_U|_{P(\mathbf{A})} = 0$, therefore $\varphi = 0$ and $f^1 = f^0$.

Finally, note that this also concludes the proof of the theorem (7.2.3), since we have constructed a non-zero element $f^1 = f^0$ in $V \cap W$.

□

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