LINEAR SYSTEMS AND RAMIFICATION POINTS ON REDUCIBLE NODAL CURVES

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Abstract

In the 80's D. Eisenbud and J. Harris (Invent. math. 85) developed a general theory in order to understand what happens to a linear system and its ramification points on a smooth curve when the curve degenerates to a curve C of compact type. Eisenbud and Harris were able to obtain remarkable results from their theory: refer to loc. cit. for a partial list of the articles where these results are proved. In one of these articles Eisenbud and Harris asked: "What are the limits of Weierstrass points in families of curves degenerating to stable curves C *not* of compact type?" (Invent. math. 87, p. 499). In the present note I hope to have found a satisfactory general answer to their question.

Resumo

Na década de 80 D. Eisenbud e H. Harris (Invent. math. 85) desenvolveram uma teoria geral para compreender o que acontece a um sistema linear e seus pontos de ramificação sobre uma curva não-singular quando esta curva degenera a uma curva C de tipo compacto. Eisenbud e Harris puderam obter resultados extraordinários de tal teoria: pode-se obter no artigo acima citado uma lista parcial dos demais artigos onde tais resultados foram provados. Em um destes artigos Eisenbud e Harris fizeram a seguinte pergunta: "Quais são os limites de pontos de Weierstrass em famílias de curvas degenerando para curvas estáveis C que *não* sejam de tipo compacto?" (traduzido de Invent. math. 87, p. 499). No artigo acima entitulado apresentamos uma resposta geral, que será satisfatória, para a pergunta acima.

1. Introduction

In [1] Eisenbud and Harris developed a general theory in order to understand what happens to a linear system and its ramification points on a smooth curve

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when the curve degenerates to a curve E of compact type. Eisenbud and Harris were able to obtain remarkable results from their theory, and we refer to [1] for a description of some of these results and a partial list of the articles where these results are proved. In one of these articles Eisenbud and Harris asked: *What are the limits of Weierstrass points in families of curves degenerating to stable curves not of compact type?* [2 , p. 499]. In the present note we hope to have found a satisfactory answer to the latter question (Theorem 7).

Actually, in the present note we deal with the more general situation of nodal curves E, not necessarily stable. We also deal with the degeneration of any linear system, not only the canonical system. Moreover, in contrast with the theory developed by Eisenbud and Harris, we do not need to blow up our degeneration family to swerve the degenerating ramification points away from the nodes of E. Indeed, we can assign the appropriate ramification weight to any node of E (Theorem 7, item 2). Therefore, our note is a conceptual addition to the theory of limit linear systems even when E is of compact type.

2. Degenerating linear systems

Set-up: Let S be the spectrum of a discrete valuation ring R. Let π be a parameter of R. Let s (resp. η) denote the special (resp. generic) point of S, and let $\pi \in R$ be a parameter. Let $f : C \longrightarrow S$ be a flat, projective morphism. Suppose that the generic fibre $C(\eta)$ is a geometrically integral curve and the special fibre $C(s)$ is a nodal reduced curve. Assume that C is a regular scheme. Let $C_1, \dots, C_t$ denote the irreducible components of $C(s)$. For each C_i we let

$$C_i^* := C \setminus \bigcup_{j \neq i} C_j.$$

Since C is regular, then $C_1, \dots, C_t$ are Cartier divisors, and any Cartier divisor on C supported in $C(s)$ is a linear combination of $C_1, \dots, C_t$. For every pair of integers (i, j), with $1 \leq i, j \leq t$, we let δ_{ij} denote the intersection number

$C_i \cdot C_j$. It is clear that δ_{ij} is the number of points in $C_i \cap C_j$ if $i \neq j$. Since

$$\mathcal{O}_C \cong \mathcal{O}_C(C_1 + \ldots + C_t), \tag{0.1}$$

as $C_1 + \ldots + C_t$ is the Cartier divisor on C cut out by π, then

$$\delta_{ii} = -\sum_{j \neq i} \delta_{ij}$$

for every $i = 1, \ldots, t$. For each $i = 1, \ldots, t$, we will say that an effective Cartier divisor on C supported in $C(s)$ is *C_i-free* if its support does not contain C_i.

Since C is regular, for every invertible sheaf L_η on $C(\eta)$ there is an invertible sheaf $\mathcal{L}$ on C such that $\mathcal{L}(\eta) \cong L_\eta$. We call such an $\mathcal{L}$ an *extension of L_η to C*. It is easy to see that the sheaves $\mathcal{L} \otimes \mathcal{O}_C(n_1 C_1 + \ldots + n_t C_t)$ are all the extensions of L_η to C.

Set-up: Fix an invertible sheaf L_η on $C(\eta)$ of degree d, and a non-zero subvectorspace $V_\eta \subseteq H^0(C(\eta), L_\eta)$ of rank $r + 1$.

If $\mathcal{L}$ is an extension of L_η to C, then we put:

$$V_{\mathcal{L}} := V_\eta \cap H^0(C, \mathcal{L}),$$

where the above intersection is taken inside $H^0(C(\eta), L_\eta)$. It is clear that $V_{\mathcal{L}}$ is a free R-module of rank $r + 1$ with $V_{\mathcal{L}}(\eta) = V_\eta$. In addition, the induced homomorphism

$$V_{\mathcal{L}}(s) \longrightarrow (f_* \mathcal{L})(s) \longrightarrow H^0(C(s), \mathcal{L}(s))$$

is injective. To summarize, given an extension $\mathcal{L}$ of L_η to C, the linear system (V_η, L_η) extends to the linear system $(V_{\mathcal{L}}, \mathcal{L})$ on C, whose restriction $(V_{\mathcal{L}}(s), \mathcal{L}(s))$ to $C(s)$ is a linear system.

Definition. We say that $(V_{\mathcal{L}}(s), \mathcal{L}(s))$ is a *limit linear system.*

Theorem 1. *For every irreducible component $C_i \subseteq C(s)$, there is a unique extension $\mathcal{L}_i$ of L_η to C with the following properties:*

(1) *the canonically induced homomorphism*

$$V_{\mathcal{L}_i}(s) \longrightarrow H^0(C_i, \mathcal{L}_i(s)|_{C_i})$$

is injective;

(2) *if $\mathcal{I}$ is an extension of L_η to C such that the induced homomorphism*

$$V_{\mathcal{I}}(s) \longrightarrow H^0(C_i, \mathcal{I}(s)|_{C_i}) \tag{1.1}$$

is injective, then there is an effective, C_i-free Cartier divisor D on C supported in $C(s)$ such that $\mathcal{I} \cong \mathcal{L}_i(D)$ and the induced homomorphism $V_{\mathcal{L}_i} \hookrightarrow V_{\mathcal{I}}$ is an isomorphism.

Proof. We first show that there is an extension $\mathcal{I}$ of L_η to C satisfying (1.1). In fact, choose any extension $\mathcal{J}$ of L_η to C. Let $n_1, \dots, n_t$ be integers such that

$$\deg_{C_j} \mathcal{I}(s) < 0$$

for every $j \neq i$, where $\mathcal{I} := \mathcal{J} \otimes \mathcal{O}_C(n_1C_1 + \ldots + n_tC_t)$. Then

$$V_{\mathcal{I}}(s) \subseteq H^0(C(s), \mathcal{I}(s)) \subseteq \bigoplus_{j=1}^{t} H^0(C_j, \mathcal{I}(s)|_{C_j}) = H^0(C_i, \mathcal{I}(s)|_{C_i}).$$

Assume now that there are two extensions $\mathcal{I}_1$ and $\mathcal{I}_2$ of L_η to C such that

$$V_{\mathcal{I}_m}(s) \longrightarrow H^0(C_i, \mathcal{I}_m(s)|_{C_i})$$

is injective for $m = 1, 2$. We claim that there is an extension $\mathcal{N}$ of L_η to C such that $\mathcal{I}_m \cong \mathcal{N}(D_m)$ for an effective, C_i-free Cartier divisor D_m on C supported in $C(s)$, and the induced homomorphism $V_{\mathcal{N}} \longrightarrow V_{\mathcal{I}_m}$ is an isomorphism, for $m = 1, 2$. In fact, since both $\mathcal{I}_1$ and $\mathcal{I}_2$ are extensions of L_η, then there is a Cartier divisor D on C supported in $C(s)$ such that $\mathcal{I}_1 \cong \mathcal{I}_2(D)$. It follows from (0.1) that we may assume that $D = D_1 - D_2$, where D_1 and D_2 are disjoint, effective, C_i-free Cartier divisors on C supported in $C(s)$. Let

$$\mathcal{M} := \mathcal{I}_2(D_1) \quad \text{and} \quad \mathcal{N} := \mathcal{I}_2(-D_2).$$

It is clear that

$$\mathcal{M} \cong \mathcal{I}_1(D_2) \quad \text{and} \quad \mathcal{N} \cong \mathcal{I}_1(-D_1).$$

For $m = 1, 2$, the inclusion $\mathcal{I}_m \longrightarrow \mathcal{M}$ induces the following commutative diagram:

$$\begin{array}{ccc} V_{\mathcal{M}}(s) & \longrightarrow & H^0(C_i, \mathcal{M}(s)|_{C_i}) \\ \uparrow & & \uparrow \\ V_{\mathcal{I}_m}(s) & \longrightarrow & H^0(C_i, \mathcal{I}_m(s)|_{C_i}). \end{array}$$

Since D_1 and D_2 are C_i-free, it follows that the right vertical homomorphism is an embedding for $m = 1, 2$. On the other hand, the bottom horizontal homomorphism is injective by assumption. It follows that the left vertical homomorphism is injective. Since $V_{\mathcal{M}}$ and $V_{\mathcal{I}_m}$ are free R-modules of same rank, then $V_{\mathcal{M}} = V_{\mathcal{I}_m}$ for $m = 1, 2$. On the other hand, since D_1 and D_2 are disjoint, then $V_{\mathcal{N}} = V_{\mathcal{I}_1} \cap V_{\mathcal{I}_2}$ inside $V_{\mathcal{M}}$. Therefore, $V_{\mathcal{N}} = V_{\mathcal{I}_m}$ for $m = 1, 2$. Our claim is proved. Note that $\mathcal{N} \cong \mathcal{I}_m$ if and only if $D_m = 0$.

We finally show that there is a sheaf $\mathcal{L}_i$ as in the statement of the theorem. Let $\mathcal{I}_1$ be an extension of L_η to C such that the induced homomorphism

$$V_{\mathcal{I}_1}(s) \longrightarrow H^0(C_i, \mathcal{I}_1(s)|_{C_i})$$

is injective. If $\mathcal{I}_1$ satisfies the second property in the statement of the theorem as well, then we are done: put $\mathcal{L}_i := \mathcal{I}_1$. If not, by applying the reasoning in the above paragraph, there is an extension $\mathcal{I}_2$ of L_η and a non-zero, effective, C_i-free Cartier divisor D_1 on C supported in $C(s)$ such that $\mathcal{I}_2 \cong \mathcal{I}_1(-D_1)$ and the induced homomorphism $V_{\mathcal{I}_2} \longrightarrow V_{\mathcal{I}_1}$ is an isomorphism. It is clear that the induced homomorphism

$$V_{\mathcal{I}_2}(s) \longrightarrow H^0(C_i, \mathcal{I}_2(s)|_{C_i})$$

is injective. If $\mathcal{I}_2$ satisfies the second property in the statement of the theorem as well, then we are done: put $\mathcal{L}_i := \mathcal{I}_2$. If not, then procceed as before, thereby producing extensions $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_m$ on the m-th step such that $\mathcal{I}_j \cong \mathcal{I}_{j+1}(D_j)$ for a non-zero, effective, C_i-free Cartier divisor D_j on C supported in $C(s)$, and the induced homomorphism $V_{\mathcal{I}_{j+1}} \longrightarrow V_{\mathcal{I}_j}$ is an isomorphism, for every $j < m$. In particular, we have that

$$V_{\mathcal{I}_1} = V_{\mathcal{I}_m} \subseteq H^0(C, \mathcal{I}_1(-D_1 - \ldots - D_{m-1})). \tag{1.2}$$

It follows from (1.2) that the above procedure cannot go on indefinitely. Thus there will be an $m \geq 1$ such that $\mathcal{L}_i := \mathcal{I}_m$ is as in the statement of the theorem.

The uniqueness of $\mathcal{L}_i$ is obvious from its properties. The proof of the theorem is complete.

□

Definition. We say that $\mathcal{L}_i$ is the extension of L_η *associated* to C_i (and the subvectorspace $V_\eta \subseteq H^0(C(\eta), L_\eta)$). We say that $(V_{\mathcal{L}_i}(s), \mathcal{L}_i(s))$ is the *limit linear system associated to* C_i.

Proposition 2. *If $\mathcal{I}$ is an extension of L_η to C, then $\mathcal{I} \cong \mathcal{L}_i$ if and only if*

(1) *the canonically induced homomorphism*

$$V_{\mathcal{I}}(s) \longrightarrow H^0(C_i, \mathcal{I}(s)|_{C_i}) \tag{2.1}$$

is injective;

(2) *for every irreducible component $C_j \subseteq C(s)$ with $j \neq i$, the canonically induced homomorphism*

$$V_{\mathcal{I}}(s) \longrightarrow H^0(C_j, \mathcal{I}(s)|_{C_j}) \tag{2.2}$$

is not identically zero.

Proof. Suppose that $\mathcal{I} \cong \mathcal{L}_i$. Then (2.1) is injective. Suppose by contradiction that there is an irreducible component $C_j \subseteq C(s)$ with $j \neq i$ such that (2.2) is identically zero. It follows that $V_{\mathcal{I}} = V_{\mathcal{J}}$, where $\mathcal{J} := \mathcal{I}(-C_j)$, contradicting the minimality property of $\mathcal{L}_i$.

Conversely, suppose that (2.1) is injective and (2.2) is not identically zero for every $j \neq i$. Since (2.1) is injective, then there is an effective, C_i-free

Cartier divisor D on C supported in $C(s)$ such that $\mathcal{I} \cong \mathcal{L}_i(D)$ and the induced homomorphism $V_{\mathcal{L}_i} \longrightarrow V_{\mathcal{I}}$ is an isomorphism. It follows that

$$V_{\mathcal{I}} \subseteq H^0(C, \mathcal{I}(-D)),$$

and hence every section of $V_{\mathcal{I}}(s)$ is zero on D. Since (2.2) is not identically zero for every $j \neq i$ and D is C_i-free, then $D = 0$, and hence $\mathcal{I} \cong \mathcal{L}_i$.

□

Remark 3. Ziv Ran had also studied degenerations of linear systems in [4], where he had also obtained the linear system $(V_{\mathcal{L}_i}, \mathcal{L}_i)$ of Theorem 1 for each component C_i. He called such system an "effective state with focus C_i".

Proposition 4. *Fix i, j with $i \neq j$. Let l_{im}, for $m \in \{1, \ldots, t\} \setminus \{i\}$, be the unique integers such that*

$$\mathcal{L}_i \cong \mathcal{L}_j(\sum_{m \neq i} l_{im} C_m).$$

Then $0 \leq l_{im} \leq l_{ij}$ for every m.

Proof. Let

$$\begin{aligned} E &:= \sum_{l_{im} > 0} l_{im} C_m; \\ F &:= -\sum_{l_{im} < 0} l_{im} C_m. \end{aligned}$$

We have that $\mathcal{L}_i \cong \mathcal{L}_j(E - F)$, where E and F are disjoint, effective, C_i-free Cartier divisors on C supported in $C(s)$. Let

$$\begin{aligned} \mathcal{M} &:= \mathcal{L}_j(E); \\ \mathcal{N} &:= \mathcal{L}_j(-F). \end{aligned}$$

It is clear that $\mathcal{M} \cong \mathcal{L}_i(F)$ and $\mathcal{N} \cong \mathcal{L}_i(-E)$. Since F is C_i-free, the embedding $\mathcal{L}_i \hookrightarrow \mathcal{M}$ induces an isomorphism $V_{\mathcal{L}_i} \cong V_{\mathcal{M}}$. Since E and F are disjoint, then $V_{\mathcal{N}} = V_{\mathcal{L}_i} \cap V_{\mathcal{L}_j}$ inside $V_{\mathcal{M}}$. Thus $V_{\mathcal{N}} \cong V_{\mathcal{L}_j}$. So the induced homomorphism

$$V_{\mathcal{L}_j}(s) \longrightarrow H^0(C(s), \mathcal{L}_j(s))$$

factors through the homomorphism

$$H^0(C(s), \mathcal{N}(s)) \longrightarrow H^0(C(s), \mathcal{L}_j(s))$$

induced by the embedding $\mathcal{N} \hookrightarrow \mathcal{L}_j$. It follows from Proposition 2 that $F = 0$. So $l_{im} \geq 0$ for every m.

On the other hand, we have that

$$\mathcal{L}_j \cong \mathcal{L}_i(-\sum l_{im} C_m) \cong \mathcal{L}_i(l_{ij} C_i + \sum_{m \neq j} (l_{ij} - l_{im}) C_m).$$

Applying the result of the previous paragraph to the above situation, we obtain that $l_{ij} - l_{im} \geq 0$ for every $m \neq j$. The proof of the proposition is complete.

□

Definition. We say that l_{ij} is the *connecting number* of $\mathcal{L}_i$ and $\mathcal{L}_j$.

Note that l_{ij} depends only on the specializations $\mathcal{L}_i(s)$ and $\mathcal{L}_j(s)$.

Corollary 5. *Let $\mathcal{I}$ be an extension of L_η to C. If the canonically induced homomorphism*

$$V_{\mathcal{I}}(s) \longrightarrow H^0(C_m, \mathcal{I}(s)|_{C_m})$$

is injective for $m = i, j$, then $\mathcal{L}_i \cong \mathcal{L}_j$.

Proof. It follows from Theorem 1 that there is an effective, C_m-free Cartier divisor D_m on C supported in $C(s)$ such that $\mathcal{I} \cong \mathcal{L}_m(D_m)$, for each $m = i, j$. Thus

$$\mathcal{L}_i \cong \mathcal{L}_j(D_j - D_i).$$

Since D_i is C_i-free and D_j is C_j-free, it follows easily from Proposition 4 that $D_i = D_j$. The proof is complete.

□

Proposition 6. *Let $C_i, C_j \subseteq C(s)$ be two irreducible components intersecting at $p \in C_i \cap C_j$. For each $m = i, j$, let $\epsilon_0^m(p), \ldots, \epsilon_r^m(p)$ be the increasing sequence* *of orders of vanishing at p of the linear system*

$$(V_{\mathcal{L}_m}(s), \mathcal{L}_m(s)|_{C_m}).$$

Then

$$\epsilon^i_h(p) + \epsilon^j_{r-h}(p) \geq l_{ij}$$

for every $h = 0, \ldots, r$.

Proof. The proof is analogous to the one given by Eisenbud and Harris in [1, Prop. 2.1, p. 348].

□

3. Ramification points

Set-up: Assume from now on that the characteristic of the residue field $k(s)$ is 0.

Let ω be the canonical sheaf on C relative to S. If $\mathcal{L}$ is an extension of L_η to C, then we can associate (cf. [3]) to the inclusion $V_{\mathcal{L}} \hookrightarrow H^0(C, \mathcal{L})$ a section

$$s_{\mathcal{L}} \in H^0(C, \mathcal{L}^{\otimes r+1} \otimes \omega^{\otimes \binom{r+1}{2}}),$$

called the *ramification section of the linear system* $(V_{\mathcal{L}}, \mathcal{L})$. Let $Z_{\mathcal{L}}$ denote the zero scheme of $s_{\mathcal{L}}$. The subscheme $Z_{\mathcal{L}} \subseteq C$ is called the *ramification subscheme of the linear system* $(V_{\mathcal{L}}, \mathcal{L})$ on C. The intersection $Z_\eta := Z_{\mathcal{L}} \cap C(\eta)$ is the ramification subscheme of (V_η, L_η). Thus $Z_{\mathcal{L}}$ does not contain $C(\eta)$. For every $i = 1, \ldots, t$, let $n_i^{\mathcal{L}}$ denote the multiplicity of C_i in $Z_{\mathcal{L}}$. It is clear that $s_{\mathcal{L}}$ factors through a section

$$s^*_{\mathcal{L}} \in H^0(C, \mathcal{L}^{\otimes r+1} \otimes \omega^{\otimes \binom{r+1}{2}} \otimes \mathcal{O}_C(-n_1^{\mathcal{L}} C_1 - \ldots - n_t^{\mathcal{L}} C_t)).$$

Moreover, the zero scheme Z of $s^*_{\mathcal{L}}$ is the unique relative Cartier divisor on C over S such that $Z_\eta = Z \cap C(\eta)$. Of course,

$$Z_{\mathcal{L}} = Z + \sum_{i=1}^{t} n_i^{\mathcal{L}} C_i.$$

Note in particular that, if $V_{\mathcal{L}}(s) \subseteq H^0(C_i, \mathcal{L}(s)|_{C_i})$ for a certain irreducible component $C_i \subseteq C(s)$, then $Z \cap C_i^* = Z_{\mathcal{L}} \cap C_i^*$.

Set-up: Let Z denote the relative Cartier divisor on C over S whose generic fibre $Z(\eta)$ is the ramification subscheme of (V_η, L_η). We call $Z(s)$ the *limit ramification divisor.* (If (V_η, L_η) is the canonical system, then we call $Z(s)$ the *limit Weierstrass divisor.*) For every $q \in Z(s)$, we let w_q denote the weight of q in $Z(s)$.

Theorem 7. *For each $i = 1, \dots, t$, let $Z_i \subseteq C_i$ be the ramification subscheme of*

$$(V_{\mathcal{L}_i}(s), \mathcal{L}_i(s)|_{C_i}).$$

Let $q \in C(s)$. For every irreducible component $C_i \subseteq C(s)$ containing q, we let w_q^i denote the weight of q in Z_i. Then:

(1) *if $q \in C_i^*$, then $w_q = w_q^i$.*

(2) *if $q \in C_i \cap C_j$ for $i \neq j$, then*

$$w_q = w_q^i + w_q^j + (r - l_{ij})(r+1).$$

Proof. As we remarked before, we have that $Z \cap C_i^* = Z_{\mathcal{L}_i} \cap C_i^*$. On the other hand, it is clear that $Z_{\mathcal{L}_i}(s) \cap C_i^* = Z_i \cap C_i^*$. Thus, if $q \in C_i^*$ then $w_q = w_q^i$.

Suppose now that $q \in C_i \cap C_j$ for $i \neq j$. For $m = i, j$, let ω_m denote the dualizing sheaf on C_m. We have a canonical embedding

$$\omega_m \longrightarrow \omega(s)|_{C_m}$$

of invertible sheaves whose cokernel has length 1 at q. Thus

$$w_q^m = a_m - \binom{r+1}{2}, \tag{7.1}$$

where a_m is the order of vanishing at q of the restriction of $s_{\mathcal{L}_m}$ to C_m, for $m = i, j$. For each $m = i, j$, let b_m be the order of vanishing at q of the restriction of $s^*_{\mathcal{L}_m}$ to C_m. Since Z is equal to the zero scheme of $s^*_{\mathcal{L}_m}$ for $m = i, j$, then $s^*_{\mathcal{L}_i} = s^*_{\mathcal{L}_j}$ and

$$w_q = b_i + b_j. \tag{7.2}$$

On the other hand, it is clear that

$$\begin{aligned} b_i &= a_i - n_j^{\mathcal{L}_i}, \\ b_j &= a_j - n_i^{\mathcal{L}_j}. \end{aligned} \tag{7.3}$$

On one hand, since $s^*_{\mathcal{L}_i} = s^*_{\mathcal{L}_j}$, then

$$\mathcal{L}_i^{\otimes r+1} \cong \mathcal{L}_j^{\otimes r+1} \otimes \mathcal{O}_C(\sum_{h=1}^{t}(n_h^{\mathcal{L}_i} - n_h^{\mathcal{L}_j})C_h). \tag{7.4}$$

On the other hand,

$$\mathcal{L}_i^{\otimes r+1} \cong \mathcal{L}_j^{\otimes r+1}((r+1)l_{ij}C_j + (r+1)E_{ij}), \tag{7.5}$$

where E_{ij} is a C_i-free and C_j-free effective Cartier divisor on C with support in $C(s)$. Combining (7.4) and (7.5) we get that

$$n_j^{\mathcal{L}_i} + n_i^{\mathcal{L}_j} = (r+1)l_{ij}. \tag{7.6}$$

Combining (7.1), (7.2), (7.3) and (7.6), we have the equality in the statement (2) of the theorem. The proof is complete.

□

Corollary 8. *For each $i = 1, \ldots, t$, let d_i denote the degree of $\mathcal{L}_i(s)$ on C_i. Then*

$$\sum_{i=1}^{t} d_i = d + \sum_{i<j} \delta_{ij} l_{ij}.$$

Proof. The above formula follows from the Plücker formulas giving the degrees of $Z, Z_1, \ldots, Z_t$ and the formula in item 2 of Theorem 7.

□

Corollary 9. *Let $C_i, C_j \subseteq C(s)$ be irreducible components that intersect at a certain $q \in C_i \cap C_j$. For $m = i, j$, let $\epsilon_0^m(q), \ldots, \epsilon_r^m(q)$ be the increasing sequence of orders of vanishing at q of the linear system*

$$(V_{\mathcal{L}_m}(s), \mathcal{L}_m(s)|_{C_m}).$$

Then $q \notin Z$ if and only if

$$\epsilon^i_h(q) + \epsilon^j_{r-h}(q) = l_{ij}$$

for all $h = 0, \dots, r$.

Proof. It follows from item 2 of Theorem 7 that $q \notin Z$ if and only if

$$w^i_q + w^j_q = (l_{ij} - r)(r + 1).$$

On the other hand, since

$$w^m_q = \sum_{h=0}^{r} (\epsilon^m_h(q) - h)$$

for $m = i, j$, then

$$w^i_q + w^j_q = \sum_{h=0}^{r} (\epsilon^i_h(q) + \epsilon^j_{r-h}(q)) - r(r + 1).$$

Hence, it follows from Proposition 6 that

$$w^i_q + w^j_q = (l_{ij} - r)(r + 1)$$

if and only if

$$\epsilon^i_h(p) + \epsilon^j_{r-h}(p) = l_{ij}$$

for all $h = 0, \dots, r$. Combining the above two if-and-only-if statements we finish the proof.

□

4. Cases

Case 10. (Curves of compact type) We do not have much control over the multidegrees of the limit linear systems. We know their ranges: it follows from Proposition 2 that

$$0 \leq \deg_{C_j} \mathcal{L}_i(s) \leq d$$

for all i, j. If $C(s)$ is of compact type, then Eisenbud and Harris [1] developed a theory of limit linear series: for every irreducible component $C_i \subseteq C(s)$, they associated the unique limit linear system (V_i, L_i) with $\deg_{C_j} L_i = d\delta_{ij}$ for all j, where δ_{ij} is the Kronecker symbol. (Of course, such choice may not be possible if $C(s)$ is not of compact type.) Eisenbud and Harris called the collection $\{(V_i, L_i|_{C_i})|1 \leq i \leq t\}$ a crude limit series. It is also possible, in a way analogous to Theorem 7, to determine the limit ramification divisor Z out of the crude limit series. (Though Eisenbud and Harris [1] prefer to blow up the degeneration family at the nodes of the special fibre in order to swerve the degenerating ramification points away from these nodes.) Very seldom does the crude limit series coincide with the collection $\{(V_{\mathcal{L}_i}(s), \mathcal{L}_i(s)|_{C_i})|1 \leq i \leq t\}$. In fact, if the above two collections coincide, then (V_i, L_i) does not have base points at any of the nodes of intersection of two components.

Case 11. (Planar curves) Let $E \subset \mathbf{P}^2$ be the planar curve of degree d consisting of an irreducible nodal curve Q of degree $d-1$ and one of its general secants M. Let $p_1, p_2, \ldots, p_{d-1}$ be the points of Q where M intersects.

There are several ways we can view E as a limit of smooth plane curves. In general, let $G(t) \in \mathbf{C}[x, y, z, t]$ be a polynomial in the variables x, y, z, t that is homogeneous of degree d in x, y, z. Let $C \subset \mathbf{P}^2 \times \mathbf{A}^1$ be the zero scheme of G. Assume that C is regular and flat over $\mathbf{A}^1$ in a neighbourhood of the fibre $C(0)$. Assume that the fibre $C(\lambda)$ is smooth for a general specialization $\lambda \in \mathbf{A}^1$. Finally, assume that $G(0) = F$.

We are concerned with computing the limit of the sets of inflection points on the curves $C(\lambda)$ as λ tends to 0. (Of course, the inflection points of $C(\lambda)$ are the ramification points of the linear system of hyperplanes of $\mathbf{P}^2$ restricted to $C(\lambda)$.) It is clear that the restriction $| \mathcal{O}_{\mathbf{P}^2}(1)\big|_E |$ of the complete linear system of hyperplanes on $\mathbf{P}^2$ is a limit linear system. Moreover, it follows from the characterization given by Proposition 2 that $| \mathcal{O}_{\mathbf{P}^2}(1)\big|_E |$ is the limit linear system associated to the component Q of E. This linear system has degree $d-1$ on Q and 1 on the secant M. Since Q has degree $d-1$, then the divisor of inflection points Z_Q associated to Q has degree $3(d^2-4d+3)$. To get the limit linear system associated with M, we twist $\mathcal{O}_{\mathbf{P}^2}(1)\big|_E$ by $\mathcal{O}_C(-M)|_E$. (Further twisting is not possible, since we would obtain an invertible sheaf with negative degree on Q.) Therefore, the limit linear system (V,L) on E associated to M has degrees $\deg_Q L = 0$ and $\deg_M L = d$. Restricting (V,L) to M we get a linear system of degree d and rank 2 that yields a ramification divisor Z_M of degree $3(d-2)$. Since the connecting number of Q and M is 1, then it follows from Theorem 7 that we have an extra weight contribution of 3 at each of the $p_1, p_2, \ldots, p_{d-1}$. The upshot is that the limit of the ramification divisors on the smooth planar curves degenerating to E is

$$Z = 3p_1 + 3p_2 + \ldots + 3p_{d-1} + Z_Q + Z_M.$$

We note, however, that Z_M depends on the particular degeneration to E. For an example, let $F := x(y^2+x^2-z^2)$. For each pair $c=(c_1,c_2)\in \mathbf{A}^2$, let $G_c(t) := F + t(c_1y^3+c_2y^2z)$. Thus our degeneration depends on the parameter c. Computing the limit ramification divisor Z_c on E, we get that

$$Z_c = 3p_1 + 3p_2 + (0:y_1:z_1) + (0:y_2:z_2) + (0:y_3:z_3),$$

where the $(y_i:z_i)$, for $i=1,2,3$, are the zeros of the polynomial

$$H := c_1y^3 + 3c_2y^2z + 3c_1yz^2 + c_2z^3.$$

Case 12. (Curves with no singular ramification points) In case $C(s)$ is of compact type, it is possible that the limit ramification divisor Z does not contain any singular points. (In fact, even if this is not the case, Eisenbud and Harris [1] pointed out that it is possible to make all degenerating ramification points converge to smooth points of $C(s)$ replacing $C(s)$ by a semistably equivalent curve.) In general, though, singular points of the special fibre tend to attract ramification points. In fact, it is easy to show from Corollary 9 that, if the limit ramification divisor Z contains only smooth points of $C(s)$, then:

(1) C_i^* is smooth for every i;

(2) $l_{ij} \geq r$ for all i, j;

(3) $l_{ij}\delta_{ij} \leq d$ for all i, j.

In particular, if $r = d/2$ (the case of the canonical system), then $\delta_{ij} \leq 2$ for all i, j.

As a matter of fact, there should be more restrictions on $C(s)$, especially if we consider degenerations of the canonical system. For instance, if $C(s)$ is the union of only two components C_1, C_2, meeting at nodes p_1, p_2, then the limit Weierstrass divisor Z has no singular points only if $\omega_i \cong \mathcal{O}_{C_i}((g_i - 1)(p_1 + p_2))$ for $i = 1, 2$. (We denote by ω_i the canonical sheaf on C_i, and by g_i the genus of C_i, for $i = 1, 2$.) It would be interesting to have a characterization of the nodal curves whose canonical limit Weierstrass divisors do not include singular points.

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