CUSP FORMS OF WEIGHT 2 OVER AN IMAGINARY QUADRATIC FIELD AS HOMOLOGY CLASSES

Luiz M. S. Figueiredo

1. Introduction

Let K be an imaginary quadratic field and let $\mathcal{O}_K$ denote its ring of integers. Let Γ be a discrete subgroup of $GL_2(\mathcal{O}_K)$.

There are various ways of defining cusp forms of weight 2 for Γ. One can see cusp forms as regular algebraic cuspidal automorphic representations of $GL_2(\mathbb{A}_K)$, where $\mathbb{A}_K$ is the ring of adèles of K. One can also consider cusp forms of weight 2 as certain complex harmonic functions $F : \mathcal{H}_3 \to \mathbb{C}^3$, where $\mathcal{H}_3$ denotes the hyperbolic three space. These vector valued functions correspond to cuspidal differentials in $H^*(\Gamma\backslash\mathcal{H}_3, \mathbb{C})$.

In [7] we show the connection between cusp forms of weight 2 and the cohomology $H^q(X, \mathbb{C})$, $q = 1, 2$. From there we show some sort of duality between this space and the homology space $H_1(\Gamma\backslash\mathcal{H}_3 \cup \mathbb{P}(K), \mathbb{C})$.

Let N be an ideal of $\mathcal{O}_K$. Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_K) \mid c \in N \right\} \quad \text{and}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_K) \mid c \in N,\ d \equiv \epsilon \bmod N, \text{ for some } \epsilon \in \mathcal{O}_K^* \right\}.$$

We are interested in calculating mod l cusp forms of weight 2 for $\Gamma_1(N)$ and a given character of $\Gamma_1(N)\backslash\Gamma_0(N)$. These cannot be defined as corresponding to cuspidal harmonic differentials as we do in characteristic 0. We define the mod l cusp forms of weight 2 as homology classes in the homology space $H_1(X^*, \bar{\mathbb{F}}_l)$, where X^* is the manifold with cusps $\Gamma_1(N)\backslash(\mathcal{H}_3 \cup \mathbb{P}(K))$.

Recall that in the rational case mod l cusp forms are defined as the mod l reduction of cusp forms in characteristic zero. This is not the case here. If $H_1(X^*, \mathbb{Z})^{\mathrm{TF}}$ indicates the torsion free part of $H_1(X^*, \mathbb{Z})$ then one has

$$H_1(X^*, \mathbb{Z})^{\mathrm{TF}} \otimes \bar{\mathbb{F}}_l \subset H_1(X^*, \bar{\mathbb{F}}_l) \ ,$$

but the space on the right is generally bigger.

In section 2.3 we announce some results concerning the explicit calculation of mod l cusp forms over K related to Galois representations.

2. Cusp forms of weight 2 over K

2.1. Cuspidal harmonic forms

Assume Γ has no non-trivial points of finite order. The action of Γ on $\mathcal{H}_3$ is free and discontinuous. Therefore the quotient space

$$X = \Gamma \backslash \mathcal{H}_3$$

is a Riemannian manifold, which obviously has dimension 3. Note that if Γ has points of finite order other than the identity than X is not a manifold. In this case one can proceed by considering a normal subgroup Γ' of Γ of finite index with no points of finite order other than the identity.

Consider the topology of $\mathcal{H}_3$ induced by the metric $(ds)^2 = \frac{dx^2+dy^2+dt^2}{t^2}$, where $(x+yi, t) \in \mathcal{H}_3$. This metric is invariant for the action of $GL_2(\mathbb{C})$ on $\mathcal{H}_3$.

The space X is not compact. One obtains a compactification of X by adjoining the cusps of Γ.

The topology of $\mathcal{H}_3 \cup \mathbb{P}(K)$ is given by extending the topology of $\mathcal{H}_3$ by taking as open neighborhoods of a point $\alpha \in K$ the sets $S \cup \{\alpha\}$, where S is an open sphere whose boundary is tangent to $\{(c, 0) \mid c \in \mathbb{C}\}$ at α. Take as basis of open neighborhood of ∞ the sets $\{(z, t), t > t_0\} \ \cup \ \{\infty\}$ for all $t_0 > 0$.

The space $X^* = \Gamma \backslash (\mathcal{H}_3 \cup \mathbb{P}(K))$ is a compact manifold with cusps.

Let $\beta = \{\beta_1, \beta_2, \beta_3\}$ be a basis of $\Omega^1(X)$ given by

$$\left\{ \frac{-dz}{t}, \frac{dt}{t}, \frac{d\bar{z}}{t} \right\} \ .$$

Any differential $w \in \Omega^1(X)$ can be written as

$$w = F\beta = f_1\beta_1 + f_2\beta_2 + f_3\beta_3$$

where $F\colon \mathcal{H}_3 \to \mathbb{C}^3$ is some vector valued function. We say that a harmonic differential w is *cuspidal* if $w = F\beta$ with F a cusp form of weight 2 (see [3] for a definition of cusp forms over K as vector valued functions).

It turns out that $H_1(X^*, \mathbb{C})$ is isomorphic to the space of Γ-invariant cuspidal harmonic forms, which in turn correspond to cuspidal automorphic forms of weight 2 (see [7] for the details).

If Γ has non-trivial points of finite order then by passing to a normal subgroup of finite index with no non-trivial elements of finite order and taking invariants we see that for any discrete subgroup $\Gamma \subset GL_2(\mathcal{O}_K)$ we have that $H_1(\Gamma\backslash\mathcal{H}_3 \cup \mathbb{P}(K), \mathbb{C})$ is isomorphic to the space of harmonic cusp forms on $\mathcal{H}_3$ invariant by the group Γ.

2.2. Mod l cusp forms and Galois representations

Serre's conjecture for modular forms asserts that any continuous irreducible odd representation

$$\rho\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\bar{\mathbb{F}}_l)$$

arises from a cusp eigenform f, in the sense that $\mathrm{trace}\rho(\mathrm{Frob}_p) = a_p$ and $\det\rho(\mathrm{Frob}_p) = \epsilon(p)p^{k-1}$ for all p not dividing Nl, where N, k, ϵ are respectively the level, weight and character of f. Representations arising in this way are called modular.

The conjecture also affirms that ρ should arise from a form of level $N(\rho)$, weight $k(\rho)$ and character $\epsilon(\rho)$, where the triple $(N(\rho), k(\rho), \epsilon(\rho))$ is defined by Serre solely in terms of the representation ρ.

Let K be an imaginary quadratic field. In view of Serre's conjecture, one might ask whether a continuous irreducible representation

$$\rho\colon \mathrm{Gal}(\bar{\mathbb{Q}}/K) \to GL_2(\bar{\mathbb{F}}_l)$$

is modular. Note that there is no odd/even distinction for representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$. Moreover one can also ask if a modular representation ρ of $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$ arises with the level and weight obtained in a manner similar to that of Serre's conjecture.

Let N be an ideal of $\mathcal{O}_K$. We will define mod l cusp forms over K for $\Gamma_1(N)$ as homology classes with coefficients in $\bar{\mathbb{F}}_l$, ie. homology classes in the space $H_1(N) = H_1(\Gamma_1(N)\backslash\mathcal{H}_3 \cup \mathbb{P}(K), \bar{\mathbb{F}}_l)$.

Now we have to consider cusp forms with character. Let ϵ be a character

$$\epsilon\colon \Gamma_1(N)\backslash\Gamma_0(N) \to \bar{\mathbb{F}}_l^* \ .$$

It is easy to see that $\Gamma_1(N)\backslash\Gamma_0(N) \cong (\mathcal{O}_K/N)^* /\mathcal{O}_K^*$, the isomorphism being given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d$. For any $u \in (\mathcal{O}_K/N)^* /\mathcal{O}_K^*$ denote by γ_u the matrix $\gamma_u = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$. The set of all γ_u with u running through $(\mathcal{O}_K/N)^* /\mathcal{O}_K^*$ is a set of coset representatives for $\Gamma_1(N)$ in $\Gamma_0(N)$.

Denote by $\mathcal{R}_\epsilon$ the ideal of $\mathbb{Z}\Gamma_1(N)$ generated by the relations $\gamma_u - \epsilon(u)I$, for all $u \in (\mathcal{O}_K/N)^* /\mathcal{O}_K^*$. These relations will be called character relations.

A cusp form of weight 2 for $\Gamma_1(N)$ with character ϵ will be defined as the coinvariant space of $H_1(\Gamma_1(N)\backslash\mathcal{H}_3^*, \bar{\mathbb{F}}_l)$ by the character relations, denoted by $H_1(\Gamma_1(N))_\epsilon$.

Then for each prime ideal $\wp$ of $\mathcal{O}_K$ not dividing N we define a Hecke operator $T_\wp$ acting on the homology space (see [7]). Homology classes which are simultaneous eigenvectors for all Hecke operators $T_\wp$ are called eigenforms.

The space $H_1(\Gamma_1(N))_\epsilon$ can be calculated using the modular symbols method (see [4] and [7]). John Cremona has written a program in C++ that calculates cusp forms of weight 2 and trivial character for $K = \mathbb{Q}(\sqrt{-d})$, $d = 1, 2, 3, 7$ and 11. The author then extended these programs to calculate forms of arbitrary character and to work over finite fields.

2.3. Serre's conjecture over K

We now formulate an analog over K of Serre's conjecture. Let $G_K = \mathrm{Gal}(\bar{\mathbb{Q}}/K)$

and let l be a prime integer. Let $\rho\colon G_K \to GL_2(\bar{\mathbb{F}}_l)$ be a continuous irreducible representation.

Define the level $N(\rho)$ as the prime to l part of the Artin conductor of ρ (see [14]).

For any prime λ of $\mathcal{O}_K$ lying above l we define an integer $\delta_\lambda = 0, 1, 2$ according to the restriction of ρ to the inertia group at λ. This is the power of λ that we need to put in the level so as to reduce the weight to 2 (see [7]).

Let $\det \rho = \epsilon(\rho)\chi^h$, where χ^h is some power of the mod l cyclotomic character and $\epsilon(\rho)$ is a character

$$\epsilon(\rho)\colon \left(\mathcal{O}_K/\tilde{N}(\rho)\mathcal{O}_K\right)^* / \mathcal{O}_K^* \to \bar{\mathbb{F}}_l^*$$

where

$$\tilde{N}(\rho) = N(\rho) \prod_{\lambda | l} \lambda^{\delta_\lambda} .$$

Then we ask if there is a homology class

$$v \in H_1^*(\Gamma_1(\tilde{N}(\rho)), \bar{\mathbb{F}}_l)_{\epsilon(\rho)}$$

such that v is a common eigenvector for the Hecke operators and for all prime ideals $\wp$ not dividing $\tilde{N}l$

$$\mathrm{trace}\rho(\mathrm{Frob}_\wp) = a_\wp$$

where $a_\wp$ is the eigenvalue of v for the Hecke operator $T_\wp$.

We have produced three examples of Galois representations of G_K and tested if they were modular. The answer seems to be positive in all three cases. We have computed cusp forms of suitable level whose Hecke eigenvalues, calculated for more then one hundred primes, match the traces of Frobenius of the representations, thus making it extremely likely that these are positive examples. The details of the examples are in my thesis [7] and in the paper [6].

These examples were all representations obtained by restricting to G_K even continuous irreducible representations $\rho\colon G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}_l)$ with $l > 2$. These are guaranteed not to correspond to mod l holomorphic cusp forms over $\mathbb{Q}$.

The three cases considered are the representations given by the Galois group of the polynomials x^4-7x^2-3x+1, $x^4-x^3-24x^2+x+11$ and $x^4-x^3-7x^2+2x+9$. The discriminant of the number field given by these polynomials are $3^2\times 61^2$, $3^4\times 79^2$ and 163^2, respectively. The three of them have Galois group $PSL_2(\mathbb{F}_3)$, so all the examples considered are mod 3 examples.

The conductors of the cusp forms corresponding to the representations given by the polynomials above depend on the field. For example, for the field $K=\mathbb{Q}(\sqrt{-1})$, the conductors are $N=183$, $N=237$ and $N=489$, respectively.

The representation arising from the first polynomial was tested for the fields $K=\mathbb{Q}(\sqrt{-d})$, $d=1,2,3$ and 7. The representations arising from the other 2 polynomials were tested for the fields $K=\mathbb{Q}(\sqrt{-1})$ and $K=\mathbb{Q}(\sqrt{-3})$.

Note that *odd* continuous irreducible representations are (conjecturally) modular and so by base change there is a mod l cusp form over K with the same set of eigenvalues. Therefore testing whether the restricted representation ρ_K is modular would be nothing else then testing Serre's conjecture for $\mathbb{Q}$ itself, for which there is already a large amount of evidence. Thus we have only considered even representations ρ.

The results obtained so far are very encouraging: all three examples appear to be modular, arising from an eigenform at the level and character obtained in analogy to Serre's conjecture.

It would be very nice to have more representations one could test, but the program can only calculate forms of levels with small norm, which restricts the number of examples we can process.

Note that we are not actually proving these representations are modular. All we have is a list of traces of Frobenius for hundreds of primes and a corresponding list of eigenvalues which match, thus giving very strong evidence that the representations are modular.

References

[1] Baily, W. L., *The decomposition theorem for V-manifold*, Amer. J. Math.

78 (1956), 862–888.

[2] Borel, A., *Cohomology of arithmetic groups*, Proc. Int. Congress Math. Vancouver, Vancouver, (1974) 435–442.

[3] Cremona, J. E., *Modular symbols*, Ph.D. thesis, University of Oxford, 1981.

[4] ———, *Hyperbolic tessellations, modular symbols, and elliptic curves over complex quadratic fields*, Comp. Math. 51 (1984), 275–323.

[5] ———, *Modular symbols for* $\Gamma_1(N)$ *and elliptic curves with everywhere good reduction*, Math. Proc. Camb. Phil. Soc. 111 (1992), 199–218.

[6] Figueiredo, L., *Serre's conjecture over imaginary quadratic fields*, paper in preparation.

[7] ———, *Serre's conjecture over imaginary quadratic fields*, Ph.D. thesis, University of Cambridge, 1995.

[8] Grunewald, F. J. and Schwermer, J., *Arithmetic quotients of hyperbolic 3-space, cusp forms and link complements*, Duke Math. Journal 48 (1981), no. 2, 351–358.

[9] Harder, G., *On the cohomology of* $SL_2(\mathcal{O})$, Liegroups and their representations, Hilger, 1975, pp. 139–150.

[10] ———, *Eisenstein cohomology of arithmetic groups. The case* GL_2, Invent. math. 89 (1987), 37–1987.

[11] Matsushima, Y. and Murakami, S., *On the vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds*, Annals of Mathematics 78 (1963), no. 2, 365–416.

[12] Ribet, K., *On modular representations of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *arising from modular forms*, Invent. Math. 100 (1990), 431–476.

[13] ———, *Report on mod l representations of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, Proc. Symp. Pure Math., vol. 55, 1994, pp. 639–676.

[14] Serre, J.-P., *Sur les représentations modulaires de degré* 2 *de* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, Duke Math. J. 54 (1987), 179–230.

[15] Taylor, R., *l-adic representations associated to modular forms over imaginary quadratic fields. II*, Invent. Math. 116 (1994), 619–643.

Instituto de Matemática Pura e Aplicada
Estrada Dona Castorina, 110
Jardim Botânico
22460-320, Rio de Janeiro, RJ