TEST, GENERIC AND ALMOST PRIMITIVE ELEMENTS IN FREE GROUPS

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Abstract

A **test element** in a group G is an element g with the property that if $f(g) = g$ for an endomorphism f of G to G then f must be an automorphism. A test element in a free group is called a **test word**. Nielsen gave the first example of a test word by showing that in the free group on x, y the commutator $[x, y]$ satisfies this property. T.Turner recently characterized test words as those elements of a free group contained in no proper retract. Since free factors are retracts, test words are therefore very strong forms of non-primitive elements. In this paper we give some new examples of test words and examine the relationship between test elements and several other concepts, in particular generic elements and almost-primitive elements (APE's). In particular we show that an almost primitive element which lies in a certain type of verbal subgroup must be a test word. Further using a theorem of Rosenberger on equations in free products we prove a result on APE's, generic elements and test words in certain free products of free groups. Finally we examine some ties between test elements and tame automorphisms - that is automorphisms induced by free group automorphisms.

1. Introduction

A **test element** in a group G is an element g with the property that if $f(g) = g$ for an endomorphism f of G to G then f must be an automorphism. A test element in a free group is called a **test word**. Nielsen [N] gave the first non-trivial example of a test word by showing that in the free group on x, y the commutator $[x, y]$ satisfies this property. T.Turner [T] recently characterized

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test words as those elements of a free group which do not lie in any proper retract. Using this characterization he was able to give several straight forward criteria to determine if a given element of a free group is a test word. Using these criteria, Comerford [C] proved that it is effectively decidable whether elements of free groups are test words. Since free factors are retracts, Turner's result implies that no test word can fall in a proper free factor. Therefore being a test word is a very strong form of non-primitivity.

In this paper we consider relationships between test words and two related concepts - **almost primitive elements** (APE's) and **generic elements**. We give the formal definitions in the next section where we also prove that an almost primitive element of a free group which lies in a certain type of verbal subgroup must be a test word (Theorem 1). This is quite surprising given the strong non-primitivity of test words. In section 3 we use a theorem of Rosenberger [R1] on equations in free products to prove a result on APE's, generic elements and test words in certain free products of free groups. In section 4, using Nielsen transformations, we produce a set of generic elements in the free group of rank two. Using the theorem of Rosenberger mentioned above, these examples can be extended to finding generic elements in higher rank free groups. Finally in section 5 we give some straightforward results on extensions of these concepts to arbitrary non-free groups. As pointed out by Turner the characterization of test elements in general is more subtle and difficult than in the free group case.

We note that a few versions of the results appear in the Diplomearbeit of N.Isermann [I] however the proofs we give are somewhat different.

2. Test Words, Almost Primitive Elements and Generic Elements

A **test element** in a group G is an element g with the property that if $f(g) = g$ for an endomorphism f of G to G then f must be an automorphism. A test element in a free group is called a **test word**. Nielsen [N] gave the first non-trivial example of a test word by showing that in the free group on x, y the commutator $[x, y]$ satisfies this property. Other examples of test words have been given by Zieschang [Z1,Z2],Rosenberger[R 1,R2,R3] Kalia and Rosenberger[K-R], Hill and Pride [H-P] and Durnev[D]. Gupta and Shpilrain [G-S] have studied the question as to whether the commutator $[x, y]$ is a test element in various quotients of the free group on x, y. T.Turner [T] characterized test words as those elements of a free group which do not lie in any proper retract. Using this characterization he was able to give several straightforward criteria to determine if a given element of a free group is a test word. Using these criteria, Comerford [C] proved that it is effectively decidable whether elements of free groups are test words. Since free factors are retracts, Turner's result implies that no test word can fall in a proper free factor. Therefore being a test word is a very strong form of non-primitivity. Shpilrain [Sh] defined the **rank** of an element w in a free group F as the small est rank of a free factor containing w. Clearly in a free group of rank n a test word has maximal rank n. Shpilrain conjectured that the converse was also true but Turner gave an example showing this to be false.

However Turner also proved that Shpilrain's conjecture is true if only test words for monomorphisms are considered.

An **almost primitive element** - (APE) - is an element of a free group F which is not primitive in F but which is primitive in any proper subgroup of F containing it. This can be extended to arbitrary groups in the following manner. An element $g \in G$ is **primitive** in G if g generates an infinite cyclic free factor of G, that is g has infinite order and $G = < g > \star G_1$ for some $G_1 \subset G$. g is then an APE if it is not primitive in G but primitive in any proper subgroup containing it. Rosenberger [R1] proved that in the free group $F = F(x_i, y_i, z_j); 1 \leq i \leq m, 1 \leq j \leq n$ of rank $2m + n$ the element

$$[x_1, y_1]...[x_m, y_m]z_1^{p_1} \dots z_n^{p_n}$$

where the p_i are not necessarily distinct primes, is an APE in F. Rosenberger [R1] proved, in a different setting that if A, B are arbitrary groups containing APE's a, b respectively, then the product ab is either primitive or an APE in the free product $A \star B$. This was reproved by Brunner, Burns and Oates-Williams [B-B-O] who also prove the more difficult result that if a and b are tame APE's in groups A, B respectively then their product is a tame APE in $A \star B$. An APE w in a group G ia a **tame APE** if whenever $w^\alpha \in H \subset G$ with $\alpha \geq 1$ minimal, then either w^α is primitive in H or the index $[G : H]$ is α. It follows easily that $[a_1, b_1]...[a_g, b_g], g \geq 1$ is a tame APE in the free group on $a_1, b_1...a_g, b_g$. (see [R3]). We note that Brunner, Burns and Oates-Williams give a more technical definition of a tame APE.

Let $\mathcal{U}$ be a variety defined by a set of laws $\mathcal{V}$. (We refer to the book of H.Neumann [Ne] for relevant terminology.) For a group G we let $\mathcal{V}(G)$ denote the verbal subgroup of G defined by $\mathcal{V}$. An element $g \in G$ is **$\mathcal{U}$-generic** in G if $g \in \mathcal{V}(G)$ and whenever H is a group, $f : H \to G$ a homomorphism and $w = f(u)$ for some $u \in \mathcal{V}(H)$ it follows that f is surjective. Equivalently $g \in G$ is $\mathcal{U}$-generic in G if $g \in \mathcal{V}(G) \subset G$ but $g \notin \mathcal{V}(K)$ for every proper subgroup K of G [St]. An element is **generic** if it is $\mathcal{U}$-generic for some variety $\mathcal{U}$. Let $\mathcal{U}_n$ be the variety defined by the set of laws $\mathcal{V}_n = \{[x, y], z^n\}$. Stallings [St] and Dold[Do] have given sufficient conditions for an element of a free group to be $\mathcal{U}_n$-generic. Using this it can be shown that $x_1^n x_2^n ... x_m^n$ is $\mathcal{U}_n$-generic in the free group on $x_1, ..., x_m$ for all $n \geq 2$ and if m is even $[x_1, x_2], ...[x_{m-1}, x_m]$ is $\mathcal{U}_n$-generic in the free group on $x_1, ..., x_m$ for $n = 0$ and for all $n \geq 2$. These facts are also consequences of a result of Rosenberger [R1].

Comerford [Co] points out that if G is Hopfian, which is the case if G is free, then being generic implies being a test word. Thus for free groups we have

$$\text{generic} \longrightarrow \text{test word} .$$

Comerford also shows that there is no converse. In particular he shows that in a free group of rank 3 on x, y, z the word $w = x^2[y^2, z]$ is a test word but is not generic. We can also show that in general, generic does not imply APE. Suppose $F = F(x, y)$ is the free group of rank two on x, y and let $w = x^4 y^4$. Then w is $\mathcal{U}_4$-generic but w is not an APE since $w \in < x^2, y^2 >$ and is not primitive in this subgroup while this subgroup is not all of F.

Further, in general it is not true that being an APE implies being a test word. Again let $F = F(x, y)$ and let $w = x^2yx^{-1}y^{-1}$. Brunner,Burns and Oates-Williams show that w is an APE but Turner shows that is not a test word. Since generic elements are test words this example shows further that APE does not imply generic in general. This is really to be expected since test words are strongly non-primitive. However we can prove that many APE's are indeed generic and therefore test words. Recall that a variety $\mathcal{U}$ defined by the set of laws $\mathcal{V}$ is non-trivial variety if it contains more than just the trivial group. In this case $\mathcal{V}(F) \neq F$ for any free group F.

Theorem 1. *Let F be a free group and $\mathcal{B}$ an non-trivial variety defined by the set of laws $\mathcal{V}$. Let $w \in \mathcal{V}(F)$. If w is an APE then w is $\mathcal{B}$- generic. In particular w is a test word.*

Proof. Let $w \in \mathcal{V}(F)$ be an APE and let $\phi : H \to F$ be a homomorphism with $\phi(u) = w$ for some $u \in \mathcal{V}(H)$. As in the statement of the theorem, $\mathcal{V}$ is the set of laws defining the non-trivial variety $\mathcal{B}$. Let K be a proper subgroup of F. If $w \notin K$ then clearly $w \notin \mathcal{V}(K)$. If $w \in K$ then since w is an APE, w is primitive in K since K is a proper subgroup of F. Further since $\mathcal{B}$ is an non-trivial variety and K is free we have that $K \neq \mathcal{V}(K)$. It follows then from the primitivity of w in K that $w \notin \mathcal{V}(K)$. Therefore $w \in \mathcal{V}(F)$ and for any proper subgroup K of F we have $w \notin \mathcal{V}(K)$ and hence w is $\mathcal{B}$-generic. Since free groups are Hopfian, w must then be a test word.

In particular let $F(n)$ be the subgroup of the free group F generated by all commutators and nth powers, that is $F(n) = \mathcal{V}_n(F)$. Then:

Corollary 1. *Let $w \in F(n)$. If w is an APE then w is $\mathcal{U}_n$-generic and w is a test word.*

3. APE's in Certain Free Products of Free Groups

In this section we give a result on APE's, generic elements and test words on certain free products of free groups. The result depends on the following theorem of Rosenberger [R1]. The proof of this result as well as the proofs of Theorems 3,4 and 5 use the Nielsen cancellation method in both free groups and free products. A good general reference for this method is the article [F-R-S].

Theorem 2. [R1] *Let* $G = H_1 \star \ldots H_n$, $n \geq 2$, *the free product of groups* $H_1, \ldots, H_n$. *Let* $a_j \in H_j, a_j \neq 1$ *and let* p *be the number of* a_j *which are proper powers in* $H_j, (1 \leq j \leq n)$. *Let* $\{x_1, ..., x_m\} \subset G, m \geq 1$, *and let* H *be the subgroup of* G *generated by* $x_1, ..., x_m$. *If* $a = a_1...a_n \in H$ *then one of the following cases holds:*

(1) *There is a Nielsen transformation from* $\{x_1, ..., x_m\}$ *to a system* $\{y_1, ..., y_m\}$ *with* $y_1 = a_1...a_n$.

(2) *It is* $m \geq 2n - p$, *and there is a Nielsen transformation from* $\{x_1, ..., x_m\}$ *to a system* $\{y_1, ..., y_m\}$ *with* $y_i \in H_j, 1 \leq j \leq n, 1 \leq i \leq 2n - p$; *and moreover* a_j *can be written as a word in those* $y_k, 1 \leq k \leq m$, *which are contained in* H_j,$1 \leq j \leq n$.

Using Theorem 2 as well as its proof techniques we can prove the following result on APE's in free products of free groups.

Theorem 3. *Let* F *be a finitely generated free group with basis* B. *Let* $B_1, ..., B_n$, $n \geq 2$ *be pairwise disjoint, non-empty subsets of* B *and let* F_j *be the subgroup of* F *generated by* B_j, $1 \leq j \leq n$. *Let* $a_j \in F_j$ *with* $a_j \neq 1$, $1 \leq j \leq n$ *and let* $a = a_1...a_n$. *Then:*

(1) *If each* a_j *is an APE in* F_j *then* a *is an APE in* F .

(2) *Let* $\mathcal{U}$ *an non-trivial variety defined by the set of laws* $\mathcal{V}$.

(a) *Let* $a_j \in \mathcal{V}(F_j)$ *for each* j. *If each* a_j *is* $\mathcal{U}$*-generic in* F_j *then* $a \in \mathcal{V}(\mathcal{F})$ *and* a *is* $\mathcal{U}$*-generic in* F.

(b) *Let* $a \in \mathcal{V}(F)$. *If* a *is* $\mathcal{U}$*-generic in* F *then each* $a_j \in \mathcal{V}(\mathcal{F}_|)$ *and each*

a_j is $\mathcal{U}$-generic in F_j.

(3) (a) *Let $a_j \in F_j^q F_j'$, $q = 0$ or $q = 2$ for each $j, 1 \leq j \leq n$. If each a_j is a test word in F_j then a is a test word in F.*

(b) *Let $a \in F^q F'$, $q = 0$ or $q = 2$. If a is a test word in F then each a_j is a test word in F_j.*

Proof. The proof of Theorem 3 uses Theorem 2 and the Nielsen cancellation method. We sketch part of the argument. Complete proofs of both of Theorems 2 and 3 are in [F-R-S-S]. We give the argument for part (1).

Let $a_j \in F_j$ with $a_j \neq 1$, $1 \leq j \leq n$, and let $a = a_1 ... a_n$. Then a cannot be primitive in F because in that case at least one a_j has to be primitive in F_j contradicting that each a_j is an APE. Let K be a proper subgroup of F with $a \in K$. From Theorem 2, a is primitive in K or without loss of generality, we may assume that K has a finite basis X which is the disjoint union of n subsets X_j of F_j such that $a_j \in K_j \subset F_j$ for each $j, 1 \leq j \leq n$, where K_j is the subgroup generated by X_j. We consider this latter situation. If $K_j = F_j$ for each j then $K =< K_1, ..., K_n > =< F_1, ..., F_n >= F$ contradicting the fact that $K \neq F$. Hence K_j is a proper subgroup of F_j for at least one j. Suppose $K_1 \subset F_1, K_1 \neq F_1$. Then a_1 is primitive in K_1 since a_1 is an APE in F_1 and hence $a = a_1 ... a_n$ is primitive in K. This completes part (1).

THe other parts are handled in a similar manner using Theorem 2 and Nielsen cancellation methods.

From Theorem 3 and Example 4 in Turner's paper we easily get the following corollary.

Corollary 2. *Let $F =< x_1, y_1 ..., x_g, y_g; >, g \geq 1$. Let $a_j = a_j(x_j, y_j) \neq 1$ for $j = 1, ..., g$ and let both x_j and y_j occur in the freely reduced expression of a_j. Let $|a_j|_{x_j}$ be the total exponent of x_j in a_j and let $|a_j|_{y_j}$ be the total exponent of y_j in a_j.*

(1) *Let each a_j be not a proper power in F_j. Then a is a testword in F if and only if $gcd(|a_j|_{x_j}, |a_j|_{y_j}) \neq 1$ for each j.*

(2) *Let a be an element of the commutator subgroup of F and suppose a is a product $a = a_1 a_2 ... a_j$ where each a_j is a non-trivial element of the commutator subgroup in F_j. Then a is a testword.*

4. A Class of Generic Elements

Again using Nielsen cancellation methods a class of examples of generic elements in free groups can be obtained.

Theorem 4. *Let F be a free group on a, b and let $X =< x_1, ..., x_k >, k \geq 1$, be a finitely generated subgroup of F. Suppose that X contains some element $[a^n, b^m]$ for positive integers n, m. Then $\{x_1, ..., x_k\}$ can be carried by a Nielsen transformation into a free basis $\{y_1, ..., y_p\}$, $1 \leq p \leq k$ for X for which one of the following cases occurs.*

(1) $y_1 = [a^n, b^m]$ *is a primitive element of* X*;*

(2) $y_1 = a^\alpha, 1 \leq \alpha \leq n, \alpha | n$ *and*
$y_2 = b^\beta, 1 \leq \beta \leq m, \beta | m$;

(3) $y_1 = a^\alpha, 1 \leq \alpha \leq n, \alpha | n$ *and*
$y_2 = b^m a^\beta b^{-m}, 1 \leq \beta \leq n, \beta | n$;

(4) $y_1 = b^\alpha, 1 \leq \alpha < m, \alpha | m$ *and*
$y_2 = a^n b^\beta a^{-n}, 1 \leq \beta \leq m, \beta | m$;

(5) $y_1 = a^\alpha, 1 \leq \alpha \leq n, \alpha | n$ *and*
$y_2 = b^m a^\beta, 1 \leq \beta \leq \alpha$;

(6) $y_1 = b^\alpha, 1 \leq \alpha \leq m, \alpha | m$ *and*
$y_2 = a^n b^\beta, 1 \leq \beta < \alpha$;

(7) $y_1 = a^n b^m$*,* $y_2 = a^\alpha$*,* $1 \leq \alpha \leq 2n, \alpha | 2n$ *and*
$y_3 = b^\beta, 1 \leq \beta \leq 2m, \beta | 2m$

Proof. The proof follows the general outline of the proof of Theorem 2 as in [R1] and [F-R-S-S] and uses the Nielsen cancellation method in free products.

Regard F as the free product $F =< a > \star < b >$ together with the length L and order with respect to this factorization. We may assume $\{x_1, ..., x_k\}$ is

Nielsen reduced with $x_i \neq 1$ for all i.

Further we may assume from the start that there is no Nielsen transformation from $\{x_1, ..., x_k\}$to a system $\{y_1, ..., y_k\}$ with $[a^n, b^m] \in < y_1, ..., y_{k-1} >$, that is k is minimal with respect to this property.

For this system we then have an equation

$$\prod_{k=1}^{q} x_{\nu_k}^{\epsilon_k} = [a^n, b^m] \tag{1}$$

where $\epsilon_k = \pm 1, \epsilon_k = \epsilon_{k+1}$ if $\nu_k = \nu_{k+1}$.

Among the equations as in (1) there is one for which q is minimal and let us assume that this is the case in equation (1). Further we may also assume that each $x_i \neq 1$ and that each x_i occurs in (1). If some x_i occurs only once in (1) as either x_i or x_i^{-1} then case (1) of the theorem holds. We then assume that case(1) does not hold.

Using this minimal solution and assuming that case (1) does not hold together with a detailed analysis of the Nielsen length we can reduce to the remaining 6 cases of the theorem. A complete proof is in [F-R-S-S].

Using the theorem we first obtain the following corollaries. The first is due to Comerford and Edmonds [C-M] and the second due to Comerford [C].

Corollary 3. *Let F be the free group on x, y and let $[x_1, x_2] = [x^n, y^m]$. Then $\{x_1, x_2\}$ is Nielsen equivalent to a pair $\{y_1, y_2\}$ with either $y_1 = x^n$ and $y_2 = y^m x^\alpha, 0 \leq \alpha < n$ or $y_1 = y^m$ and $y_2 = x^n y^\beta, 1 \leq \beta < m$.*

Corollary 4. *The element $[x^n, y^m]$ is a test word in the free group of rank two on x, y for any $n, m \geq 1$.*

Recall that $\mathcal{U}_n$ is the variety generated by the laws $\mathcal{V}_n = \{[x, y], z^n\}, n = 0$ or $n \geq 2$. We let $\mathcal{L}_n$ be the variety generated by the laws $\mathcal{W}_n = \{[x^n, y^n]\}$. We then obtain the following class of generic elements.

Corollary 5. *Let F be a free group of rank 2 on x, y. Then $[x^n, y^n]$ is $\mathcal{L}_n$-generic in F but for $n \geq 2$ it is not $\mathcal{U}_n$-generic in F.*

Corollary 6. *Let F be a free group of rank 2 on x, y. Then the test element $[x^n, y^m]$, $n, m \geq 1$ is an APE if and only if $n = m = 1$.*

Recall that in general it is not true that being an APE implies being a test word. As mentioned earlier if $F = F(x, y)$ and $w = x^2yx^{-1}y^{-1}$ then Brunner, Burns and Oates-Williams show that w is an APE but Turner shows that it is not a test word. Since generic elements are test words this example shows further that APE does not imply generic in general. However using the same techniques as in Theorems 2 and 4 we can generalize the fact that the element w above is an APE to obtain further examples of APE's and testwords.

Theorem 5. *Let $F =< a, b; >$ and let $X =< x_1, ..., x_k > \subset F$, $k \geq 1$. Suppose $a^nba^{-1}b^{-1} \in X, n \geq 2$. Then there is a Nielsen transformation from $\{x_1, ..., x_k\}$ to a basis $\{y_1, ..., y_p\}$,$1 \leq p \leq k$ of X such that one of the following cases holds:*

(1) $y_1 = a^nba^{-1}b^{-1}$ *or*

(2) $y_1 = a, y_2 = b$.

From this theorem and Theorem 1 we get the following corollary.

Corollary 7. *Let $F =< a, b; >$. Then*

(1) $a^nba^{-1}b^{-1}$, $n \geq 2$ *is an APE;*

(2) $a^nba^{-1}b^{-1}$, $n \geq 3$ *is $\mathcal{U}_{n-1}$-generic;*

(3) $a^nba^{-1}b^{-1}$, $n \geq 3$ *is a testword in F;*

(4) $a^2ba^{-1}b^{-1}$ *is not a testword in F.*

The corollary follows easily from the Theorem. If $w = a^nba^{-1}b^{-1}$ is in any proper subgroup of F then condition (2) of the theorem cannot hold and hence condition (1) must hold, that is w is primitive. Therefore w is an APE. If $n \geq 3$ then $w \in \mathcal{V}_{n-1}(F)$ where $\mathcal{V}_n$ is the set of laws $\mathcal{V}_n = \{[x, y], z^n\}$. As before if $\mathcal{U}_n$

is the variety defined by this set of laws, then this is an non-trivial variety and it follows that that w is an APE, that w is $\mathcal{U}_{n-1}$-generic and hence a testword. Finally part(4) comes from Turner.

5. A Result on Varieties and Primitive Elements

The following result relates when the laws determined by a single element generate a trivial variety and being in a retract.

Theorem 6. *Let F be the free group on $x_1, ..., x_n$ with $n \geq 2$ and let w be a freely reduced non-empty word in the generators of F which does not define a proper power of F. Then if the law $w = 1$ determines the trivial variety (consisting only of trivial groups) then w is a primitive in a retract of F.*

The proof of Theorem 6 depends on the fact that if $\mathcal{A}$ is the abelian variety and $\mathcal{E}$ is the trivial variety then $\mathcal{B} \cap \mathcal{A} = \mathcal{E}$ implies that $\mathcal{B} = \mathcal{E}$. The next result whose proof is straightforward completely characterizes the varieties such as $\mathcal{A}$ with this property. Recall that a variety $\mathcal{V}$ has **exponent** n if it satisfies the law $X^n = 1$. If $\mathcal{V}$ has no finite exponent it has **infinite exponent**.

Theorem 7. *Let $\mathcal{V}$ be a variety. Then $\mathcal{V}$ has the property that $\mathcal{B} \cap \mathcal{V} = \mathcal{E}$ implies that $\mathcal{B} = \mathcal{E}$ for an arbitrary variety $\mathcal{B}$ if and only if $\mathcal{V}$ has infinite exponent.*

6. Extensions to Arbitrary Groups

As pointed out by Turner the characterization of test elements in arbitrary non-free groups is much more subtle and complicated than in free groups. The following straightforward propositions give some results.

Proposition 1. *$w \in F$ is a test word if and only if whenever $f : F \to F$ is an endomorphism with $f(w) = w_1$ with w_1 Whitehead related to w then f is an automorphism.*

Proposition 2. *Let w be a test word in the free group F and let $N \triangle F$.*

Suppose that whenever $w \equiv w_1(N)$ it follows that w is Whitehead related to w_1. Let $p : F \to F/N$ be the natural projection and let $g = p(w)$. Then g is a test element in $G = F/N$.

Let $G =< g_1, ..., g_n >= F/N$ where F is free on $x_1, ..., x_n$ and $p : x_i \to g_i$ is the natural projection. If $\alpha : G \to G$ is an automorphism given by $g_i \to W_i(g_1, ..., g_n), i = 1, ..., n$ then the lift $\alpha^\star$ given by $x_i \to W_i(x_1, ..., x_n)$ induces a homomorphism of F. If there exists some lift of α, $x_i \to U_i(x_1, .., x_n)$ with $p(W_i) = p(U_i)$ which is an automorphism then α is **tame**. We say α is **supertame** if each lift of α to an endomorphism α^* of F is an automorphism.

Proposition 3. *Suppose $G = F/N$ as above and suppose each automorphism of G is super tame. Then each test element of G (if any) is the image of a test word.*

Test elements are thus closely tied to the existence of tame automorphisms. It is known that every automorphism of the orientable surface group $S_g =< x_1, y_1, ..., x_g, y_g; \prod_{i=1}^{g} [x_i, y_i] = 1 >$ is tame relative to this generating system. The next proposition gives a criteria for certain one-relator groups to have this property. Examples by Zieschang [Z3] and McCool and Pietrowski [M-P] show that it is not true that if

$$< x_1, ..., x_n; w_1 > \cong < x_1, ..., x_n; w_2 >$$

then w_1 and w_2 must be Whitehead related. However for those testwords w where this is true we get tame automorphisms in the resulting one-relator group.

Proposition 4. *Suppose $w \in F =< x_1, ..., x_n; >$ is a test word and suppose that $F/N(w) \equiv F/N(w_1)$ implies that w_1 is Whitehead related to w. Then each automorphism of $< x_1, ..., x_n; w >$ is tame.*

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