NEW RESULTS ON ADDITIVE FORMS OF EVEN DEGREE*

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Abstract

This paper presents many recent results on pairs of additive forms of even degree, outlining the main techniques involved, and the proof of the following result
"A pair of additive forms of degree k, with $k = p^\tau(p-1)k_0$ and $(k_0, p) = 1$, has a common p-adic zero provided the number of variables $n \geq 2k^{2+w(p,\tau)}$, where $w(p,\tau) = 1/(\log p + \frac{1}{\tau}\log(p-1))$."

Resumo

Este artigo apresenta vários resultados recentes sobre pares de formas aditivas de grau par, descrevendo as técnicas principais, e a demonstração do seguinte resultado
"Um par de formas aditivas de grau k, onde $k = p^\tau(p-1)k_0$ e $(k_0, p) = 1$, tem um zero p-ádico comum, se o número de variáveis $n \geq 2k^{2+w(p,\tau)}$, onde $w(p,\tau) = 1/(\log p + \frac{1}{\tau}\log(p-1))$."

1. Introduction

In the beginning of this century, around the 1920's, E. Artin conjectured (see [L2]) that *Any homogeneous polynomial of degree k in n variables has a p-adic zero, provided that $n \geq k^2 + 1$.* This conjecture had been proved to be true for quadratic forms, by Hasse in 1923 (see [BS]), as part of his wonderful work called today the *Hasse-Minkowski Theorem.* And later in 1945, R. Brauer [B] proved that there exists a condition depending on the degrees of the forms that guarantees p-adic solubility, more precisely, *there is a function ϕ_p such that a system of R p-adic forms of degrees $k_1, \cdots, k_R$ in more than $\phi_p(k_1, \cdots, k_R)$ variables always has a p-adic zero.*

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The work of R. Brauer greatly stimulated the research on the subject, and independently Lewis[L1] and Demyanov[D] proved that the conjecture was also true for cubic forms. In a series of papers Lewis and Birch worked through many cases showing the validity of the conjecture provided a condition on the size of the residue class field was fulfilled. Another surprising result, due to Ax and Kochen[AK], came in 1965, showing that the conjecture is true for almost all primes, that is, *there is a function* $P(k_1, \cdots, k_R)$ *such that a system of* R *forms of degrees* $k_1, \cdots, k_R$ in $k_1^2 + \cdots + k_R^2 + 1$ *variables has a common p-adic zero for all* $p > P(k_1, \cdots, k_R)$.

But in 1966, M. G. Terjanian[T] presented a first counter-example to this conjecture. And then many papers followed, showing that for certain primes dividing the degree, there are examples of forms with the number of variables of order exponential in relation to the degree k with no p-adic zeros (see [Ark] and [LM]). It is worth to mention that the conjecture is still believed to be true for the case of degree a rational prime number.

2. Additive Forms

During the decade of 1960-70, H. Davenport and D. J. Lewis[DL1,2,3,4] published a series of papers dedicated to the study of additive (or diagonal) forms. They started with single additive forms

$$f = a_1 x_1^k + \cdots + a_n x_n^k,$$

proving that for these cases the conjecture is true. Afterwards they studied pairs of additive forms proving that the conjecture is true for pairs of odd degree. For pairs of even degree k they established that p-adic solubility occurs whenever the number of variables surpasses $7k^3$. Then came the results on general systems of additive forms.

In 1989 came an interesting result by O. Atkinson and R. Cook[AC], where it is proved that *pairs of additive forms of degree* k *in* $n \geq 4k+1$ *variables have p-adic zeros provided* $p > k^6$. This shows that the farther one gets from the divisors of k the fewer variables one needs to guarantee p-adic solubility. And later on Atkinson, Cook and Brüdern[ABC], generalized this result for systems of additive forms. The interesting case of forms with unequal degree was treated by T. Wooley (see [W] for more information).

3. Pairs of Additive Forms of Even Degree

In the paper [DL3], Davenport and Lewis stated that the main obstacle to improve their result on pairs of additive forms lies on the prime divisors p of k, such that $p(p-1)$ also divides k (these primes will be called *singular primes*), and particularly in the case $k = 2^m$. I was introduced to this theory, and to the question of verifying the conjecture for pairs of additive quartic forms, through many insightful conversations with Prof. Lewis. And I was able to prove that

"Pairs of additive quartic forms have p-adic zeros for all primes p, provided the number of variables is greater than 60."

Later on, motivated by these results and observations, I studied pairs with degree powers of 2, and proved (see [G2])

"Let f,g be a pair of additive forms of degree $k = 2^m$, $m \geq 2$, *in* $n \geq 16k^2 - 26k + 1$ *variables. Then f,g have common p-adic zeros for all primes p".*

My efforts since then are concentrated in proving that p-adic solubility for singular primes is guaranteed if the number of variables is of order $O(k^2)$, and for that it was necessary to find a way to understand this question as p gets larger. In this direction we have proved

Theorem 1. *A pair of additive forms of degree k, with $k = p^\tau(p-1)k_0$ and $(k_0, p) = 1$, has a common p-adic zero provided the number of variables $n \geq 2k^{2+w(p,\tau)}$, where $w(p,\tau) = 1/(\log p + \frac{1}{\tau}\log(p-1))$.*

This result is interesting for as the values of the prime p increases we approximate the result stated in the conjecture of Artin. These results show us that the whole problem really lies on degrees of the type $k = p^\tau(p-1)$ for p not "large enough".

More recently, in a joint work with C. Ripoll[GR], we were able to use the above theorem and prove that, for systems of two additive forms of degree k, p-adic solubility occurs for all primes provided $n \geq 2k^{\frac{5}{2}}$, with the possible exception of $k = 2^3, 2^4, 2^5$. This is a good improvement over the previous bound of $7k^3$.

4. p-Normalization

With a pair of additive forms

$$\begin{aligned} f &= a_1 x_1^k + \cdots + a_n x_n^k \\ g &= b_1 x_1^k + \cdots + b_n x_n^k \end{aligned} \tag{1}$$

we associate the parameter

$$\vartheta = \vartheta(f,g) = \prod_{i \neq j} (a_i b_j - a_j b_i).$$

For a given pair of forms with $\vartheta \neq 0$ and a fixed prime p, there is a related p-normalized pair (f^*, g^*) with nice properties which has a p-adic zero if only if the original pair has a p-adic zero. Davenport and Lewis proved that it is sufficient to consider pairs with $\vartheta \neq 0$ and p-normalized, so we will assume that from now on, and write

$$\begin{aligned} f &= f_0 + p f_1 + \cdots + p^{k-1} f_{k-1} \\ g &= g_0 + p g_1 + \cdots + p^{k-1} g_{k-1}, \end{aligned} \tag{2}$$

where f_i, g_i are forms in the variables x_j of f, g, satisfying $i = min(v_p(a_j), v_p(b_j))$, and v_p is the p-adic valuation.

Definition 1. *Define* $\mathbf{m}_i$ *with* $i = 0, 1, \cdots$ *to be the number of variables present in the pair* f_i, g_i *(see(2)), and as seen above, each of the* m_i *variables occurs in at least one of* f_i, g_i *with a coefficient not divisible by* p*, and such a variable will be said to be at level* i*. Further define* $\mathbf{q}_i$ *to be the minimum number of variables* *appearing with coefficients not divisible by p in any form $\lambda f_i + \mu g_i$ with λ, μ not both divisible by p.*

One of the important properties of p-normalized pairs of forms is that the numbers m_i and q_i satisfy the inequalities

$$m_0 + \cdots + m_{j-1} \geq jn/k \quad for\ j = 1, \cdots, k. \tag{3}$$

$$m_0 + \cdots + m_{j-1} + q_j \geq (2j+1)n/2k \quad for\ j = 0, \cdots, k-1. \tag{4}$$

The criterion used here to establish that (1) has a p-adic zero is stated in the next lemma also due to Davenport & Lewis (see [DL3]).

Lemma 3. *Let f,g be a pair of forms as in (1), $k = p^\tau k_0$, and let γ be defined as follows*

$$\gamma = \begin{cases} \tau + 1 & \text{if } p > 2, \\ \tau + 2 & \text{if } p = 2. \end{cases}$$

If the system

$$f \equiv 0 (mod\ p^\gamma), \qquad g \equiv 0 (mod\ p^\gamma)$$

has a solution $(x_1, \cdots, x_n)$ in rational integers for which the matrix

$$\begin{pmatrix} a_1 x_1 & \cdots & a_n x_n \\ b_1 x_1 & \cdots & b_n x_n \end{pmatrix}$$

has rank 2 modulo p (i.e. $(a_i b_j - a_j b_i) x_i x_j \not\equiv 0 \ (mod\, p)$ for some i, j), then the pair f,g in (1) has a p-adic zero.

5. Contraction of Variables

Any solution for the congruences

$$f \equiv 0 (mod\ p^\alpha), \qquad g \equiv 0 (mod\ p^\alpha)$$

for which the matrix

$$\begin{pmatrix} a_1 x_1 & \cdots & a_n x_n \\ b_1 x_1 & \cdots & b_n x_n \end{pmatrix}$$

has rank 2 modulo p, will be called a non-singular solution modulo p^α. And if a pair of subforms can be found among the variables of the pair f, g, which has a non-singular solution modulo p^α, this pair of subforms will be called a *non-singular set of level* α, denoted by $\mathbf{NS}p^\alpha$.

Definition 4. *Let*

$$\begin{array}{l} a_1x_1^k + \cdots + a_\mu x_\mu^k \\ b_1x_1^k + \cdots + b_\mu x_\mu^k, \end{array}$$

be a pair of subforms of f, g, with its variables found among the variables of f_j, g_j (see (2)), and assume they have a common zero $\xi = (\xi_1, \cdots, \xi_\mu)$ modulo p^i for $i > j$. Multiply ξ by a new variable T, and we will have

$$\begin{array}{rcll} (a_1\xi_1^k + \cdots + a_\mu\xi_\mu^k)T^k & \equiv & p^i\alpha T^k & (mod\ p^{i+1}) \\ (b_1\xi_1^k + \cdots + b_\mu\xi_\mu^k)T^k & \equiv & p^i\beta T^k & (mod\ p^{i+1}). \end{array}$$

The replacement of $(x_1, \cdots, x_\mu)$ by $(\xi_1 T, \cdots, \xi_\mu T)$ is called **contraction of μ variables at the level j to a variable T at the level i or higher.**

Remark 5. *Since an $\mathbf{NS}p^i$ is a pair of subforms with a non-singular zero modulo p^i, it makes sense to speak of* **contracting an $\mathbf{NS}p^i$ set to a new variable T at the i level**, *but in this case we are going to assume that the zero used in the contraction procedure described in definition above is the non-singular zero modulo p^i, ensured by definition of an NSp^i set.*

6. Outline of the Proof of Theorem 1

The main idea is to construct, as many as possible, NSp sets. And after *contracting* each one of them, we will have some new special variables at level 1 (at least). Again this process maybe continued by constructing as many as possible NSp^2 sets, using the NSp sets and all the other variables present at this level (see (2)). Our goal is to produce an NSp^γ set, since this will guarantee a p-adic zero for f, g, by lemma 3.

On the other hand we have to keep track of the number of variables in each level, and this is done through the inequalities (see (3) and (4))

$$\begin{array}{ll} m_0 \geq \frac{n}{k} & q_0 \geq \frac{n}{2k} \\ m_0 + m_1 \geq \frac{2n}{k} & m_0 + q_1 \geq \frac{3n}{2k} \\ \vdots & \vdots \\ m_0 + \cdots + m_{k-1} = n & m_0 + \cdots + q_{k-1} \geq \frac{(2k-1)n}{2k}. \end{array}$$

The hardest case to consider is the possibility of having q_0 small, since the number of NSp sets is closely related to the value of q_0, and m_0 very large, leaving not enough variables at upper levels to help in the construction of NSp^i sets.

The main strategy then was to construct new variables at higher levels, using the exceeding variables from m_0. And this was made based upon the observation that it is possible to have the exceeding variables from m_0 in the form

$$\begin{array}{c} A_1 x_1^k + \cdots + A_r x_r^k \\ B_1 x_1^k + \cdots + B_r x_r^k, \end{array}$$

such that $(A_i, p) = 1$ and $(B_i, p) = p$ for $1 \leq i, j \leq r$. At this point the following ideas were introduced:

Let $\mathcal{A} = \{z_i\}_{i=1}^m$ be a finite sequence of integers, not necessarily distincts, all co-primes with a fixed prime p. We will say that a subsequence $\{z'_j\}$ of $\mathcal{A}$ is a "primary subsequence", denoted by PSS, if it has the following property

$$\sum_j z'_j \equiv 0 \pmod{p} \quad \text{but} \quad \sum_j z'_j \not\equiv 0 \pmod{p^2}.$$

The importance of these ideas lies on the fact that each PPS will later give rise to a new variable at level 1. In this direction it was proved:

Theorem 6. *Let $\mathcal{A}$ be as above. If $m = p(p-1)$ and $p \geq 5$ then there can be found among the elements of $\mathcal{A}$, two disjoint "primary subsequences" (PSS).*

Theorem 7. *Let $\mathcal{A}$ be as above and $p = 3$. If $m = 8$ then there can be found two disjoint PSS among the elements of $\mathcal{A}$.*

For the proof of these two theorems, we have looked at the elements of the sequence $\mathcal{A}$ modulo p^2, writing its m elements (all co-primes with p) as

$$z_i = \tau_i + p\lambda_i + p^2\mu_i$$

where

$$1 \leq \tau_i \leq p-1, \quad 0 \leq \lambda_i \leq p-1 \qquad \text{and} \qquad \mu_i \in \mathbb{Z} \; .$$

And to deal with the combinatorics involved here, we translated all this information on a matricial form (a_{ij}) with $i = 1, \ldots, p-1$ and $j = 1, \ldots, p$ where the a_{ij} entry represents the number of elements of $\mathcal{A}$ congruent to $i + (j-1)p$ modulo p^2. The conclusion of these proofs came through considerations over the lines and columns of this matrix.

As a final remark, the power $k^{w(\tau,p)}$, appearing on the enunciate of theorem 1, came on the scene to give the "extra room" sufficient to reach the desired NSp^γ, and to show that this is less and less necessary as the value of p increases.

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