SECANT VARIETIES OF ADJOINT VARIETIES

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Introduction

The purpose of this work is to show how the graded decomposition of complex simple Lie algebras $\mathfrak{g}$ can be applied to studying the secant varieties of adjoint varieties, which is part of a joint work [KOY] with M. Ohno and O. Yasukura.

Here an *adjoint variety* associated with $\mathfrak{g}$ is defined as follows: Consider a connected simple algebraic group G with Lie algebra $\mathfrak{g}$ and the adjoint representation of G on $\mathfrak{g}$. Then G naturally acts on the projective space $\mathbb{P}_*(\mathfrak{g})$. The adjoint variety X is defined to be the unique closed orbit of this action (see [Bt1], [Bt2], [W]), which is a non-degenerate, smooth projective variety in $\mathbb{P}_*(\mathfrak{g})$ and does not depend on the choice of G. The *secant variety* of a projective variety $X \subseteq \mathbb{P}$, denoted by $\operatorname{Sec} X$, is defined to be the closure of the union of lines in $\mathbb{P}$ passing through at least two points of X (see [LV], [Z]).

The key invariant here is the *secant deficiency* of a projective variety $X \subseteq \mathbb{P}$, which is defined by

$$\delta := 2\dim X + 1 - \dim \operatorname{Sec} X,$$

and a non-degenerate $X \subseteq \mathbb{P}$ is considered to have *degenerate secants* if $\delta > 0$ and $\operatorname{Sec} X \neq \mathbb{P}$. F. L. Zak proved that $\delta \leq \frac{1}{2}\dim X$ for non-degenerate smooth projective varieties $X \subseteq \mathbb{P}$ with degenerate secants, and gave a classification of X attaining the upper bound of δ, which are called *Severi varieties* (see [LV], [Z]).

Our first result is

Theorem A. *If $X \subseteq \mathbb{P}_*(\mathfrak{g})$ is an adjoint variety, then* $\dim \operatorname{Sec} X = 2\dim X$, *that is,* $\delta = 1$. *Moreover if* $\mathrm{rk}\,\mathfrak{g} \geq 2$, *then* $\mathrm{Sec}\,X \neq \mathbb{P}_*(\mathfrak{g})$.

The main part is the former (see Theorem 4.1; for the latter see Proposition 4.4), to which we give two proofs: One is based on Terracini's Lemma, and the other is on a description of the secants of the adjoint variety as the closure of an orbit, which is given below (see Proposition 4.3). Thus this result tells us that the adjoint varieties for $\mathrm{rk}\,\mathfrak{g} \geq 2$ yield an example of projective varieties with degenerate secants (see Example 4.5): note that the adjoint variety in the remaining case is of type A_1, which turns out to be a conic in $\mathbb{P}^2$ (see Proposition 3.1). This result is used in [K] to obtain a classification of homogeneous projective varieties with degenerate secants. Zak [Z] also listed such varieties. However, adjoint varieties of $\mathrm{rk}\,\mathfrak{g} \geq 2$ do not appear in his list because a certain dimensional condition is assumed there.

Now for a projective variety $X \subseteq \mathbb{P}$ and for a general point $u \in X$, we denote by C_u the contact locus of the embedded tangent space $T_u\mathrm{Sec}\,X$ to $\mathrm{Sec}\,X$, that is, the closure of the set of smooth points $v \in \mathrm{Sec}\,X$ such that $T_v\mathrm{Sec}\,X = T_u\mathrm{Sec}\,X$. Recently it has been recognized that the dimension of C_u is a significant invariant in classifying $X \subseteq \mathbb{P}$ with degenerate secants. It is easily shown that $\dim C_u \geq \delta + 1$ for a general $u \in \mathrm{Sec}\,X$ (see [FR], [Fj], [O]). Moreover for all Severi varieties and Scorza varieties (see [Z], [LV]) and for all 3-dimensional X with degenerate secants (see [Fj]), it was shown in their classifications that $\dim C_u = \delta + 1$ for a general $u \in \mathrm{Sec}\,X$.

The second result is

Theorem B. *If* $X \subseteq \mathbb{P}_*(\mathfrak{g})$ *is an adjoint variety, then* $\dim C_u = 2$ *for general* $u \in \mathrm{Sec}\,X$.

In fact, we explicitly describe C_u in terms of 3-dimensional Lie subalgebras of $\mathfrak{g}$ (see Theorem 5.1).

1. Graded Decomposition of Simple Lie Algebras

Let $\mathfrak{g}$ be a complex simple Lie algebra with the Killing form B, $\mathfrak{h}$ a Cartan subalgebra, and R the set of roots with respect to $\mathfrak{h}$. For any $\alpha \in R$, there exist

$T_\alpha \in \mathfrak{h}$ and $Y_\alpha \in \mathfrak{g}$ such that $B(T_\alpha, H) = \alpha(H)$ and $[H, Y_\alpha] = \alpha(H)Y_\alpha$ for all $H \in \mathfrak{h}$. Then $B(T_\alpha, T_\alpha) \neq 0$ and set

$$H_\alpha := \frac{2}{B(T_\alpha, T_\alpha)} T_\alpha.$$

Fix a system of simple roots, $\{\alpha_1, \ldots, \alpha_r\}$, and define an ordering on R. Let λ be the highest root with respect to the ordering, and let R^+ be the set of positive roots. For each $\alpha \in R$, choose $X_\alpha \in \mathbb{C} \cdot Y_\alpha$ such that

$$[X_\alpha, X_{-\alpha}] = H_\alpha.$$

Then $\{H_{\alpha_i}, X_\alpha | 1 \leq i \leq r, \alpha \in R\}$ forms a Chevalley basis of $\mathfrak{g}$. From a standard fact (see, for example, [Hm1; 25.2]) one obtains

Proposition 1.1. *Let $\mathfrak{g}$ be a complex simple Lie algebra, and consider eigenspaces of* ad H_λ*:*

$$\mathfrak{g}_j := \{Y \in \mathfrak{g} | (\mathrm{ad}\ H_\lambda)Y = jY\}.$$

Then we have a decomposition,

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with the following properties:

(1)
$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+ \setminus (R_\lambda \cup \{\lambda\})} (\mathbb{C} \cdot X_\alpha \oplus \mathbb{C} \cdot X_{-\alpha}), \\ \mathfrak{g}_1 &= \bigoplus_{\alpha \in R_\lambda} \mathbb{C} \cdot X_\alpha, \quad \mathfrak{g}_{-1} = \bigoplus_{\alpha \in R_\lambda} \mathbb{C} \cdot X_{-\alpha}, \\ & \mathfrak{g}_2 = \mathbb{C} \cdot X_\lambda, \quad \mathfrak{g}_{-2} = \mathbb{C} \cdot X_{-\lambda}, \end{aligned}$$

where we set $R_\lambda := \{\alpha \in R^+ | \lambda - \alpha \in R\}$.

(2) $[\mathfrak{g}_k, \mathfrak{g}_l] \subseteq \mathfrak{g}_{k+l}$ *with $\mathfrak{g}_j = 0$ for $|j| > 2$.*

(3) $\mathfrak{g}_{\pm 1} \neq 0$ *if and only if* rk $\mathfrak{g} \geq 2$.

The decomposition above is called the *graded decomposition of complex contact type* for $\mathfrak{g}$ in case of rk $\mathfrak{g} \geq 2$ (see [A1], [A2], [W; §2, Theorem 4.2]).

2. Adjoint Varieties

Let G be a complex, connected, simple algebraic group with the Lie algebra $\mathfrak{g}$, V a finite-dimensional complex vector space, and $G \to GL(V)$ an irreducible representation of G (see, for example, [Hm2; 31.3 and 33.6]). Then G naturally acts on the complex projective space $\mathbb{P}_*(V)$ of one-dimensional subspaces of V through the canonical projection,

$$\pi : V \setminus \{0\} \to \mathbb{P}_*(V); v \mapsto \mathbb{C} \cdot v.$$

From any non-zero $v \in V$ one obtains a smooth, quasi-projective variety,

$$X := \pi(G \cdot v) = G \cdot x \subseteq \mathbb{P}_*(V),$$

where we set $x := \pi(v)$ (see, for example, [Hm2; 8.3]). Since the action is irreducible, $X \subseteq \mathbb{P}_*(V)$ is non-degenerate, that is, not contained in any hyperplanes of $\mathbb{P}_*(V)$. Moreover if v is a highest weight vector of the representation, then the orbit X is closed, hence projective (see, for example, [Hm2; 31.3, Theorem and 21.3, Corollary B]). Conversely a closed orbit in $\mathbb{P}_*(V)$ is unique and obtained from a highest weight vector (see, for example, [FH; Claim 23.52]).

In general for a variety X in a projective space $\mathbb{P}$ or in an affine space $\mathbb{A}$, we denote by T_xX the embedded tangent space to X at x in $\mathbb{P}$ or in $\mathbb{A}$: On the other hand, we denote by t_xX the Zariski tangent space to X at x (see, for example, [Hr, Lecture 14], where our embedded tangent spaces in $\mathbb{P}$ and in $\mathbb{A}$ are respectively called the projective and affine tangent spaces).

Lemma 2.1. *For $X = \pi(G \cdot v) = G \cdot x \subseteq \mathbb{P}_*(V)$ with $\pi(v) = x$, we have*

$$T_xX = \begin{cases} \mathbb{P}_*(\mathfrak{g} \cdot v), & (v \in \mathfrak{g} \cdot v) \\ \mathbb{P}_*(\mathfrak{g} \cdot v \oplus \mathbb{C} \cdot v), & (v \notin \mathfrak{g} \cdot v), \end{cases}$$

where note that $\mathfrak{g}$ naturally acts on V by the differential of the representation $G \to GL(V)$.

Proof. It suffices to show that in either case $T_xX = \pi((\mathfrak{g} \cdot v + v) \setminus \{0\})$, where $\mathfrak{g} \cdot v + v$ is an affine subspace of $\mathfrak{g}$ which is a translation of the vector subspace

$\mathfrak{g}\cdot v$ by v. We see that $\mathfrak{g}\cdot v = t_v(G\cdot v)$ as vector subspaces via the natural isomorphism of vector spaces, $V \simeq t_vV$. Therefore we have $T_v(G\cdot v) = \mathfrak{g}\cdot v + v$ as affine subspaces in V, and the result follows since $T_xX = \pi(T_v(G\cdot v)\setminus\{0\})$. □

Assume that the representation is adjoint. Then the orbit of the highest root vector $X_\lambda \in \mathfrak{g}$ yields a non-degenerate, smooth projective variety,

$$X = \pi(G\cdot X_\lambda) = G\cdot x_\lambda \subseteq \mathbb{P}_*(\mathfrak{g}),$$

where we set $x_\lambda := \pi(X_\lambda)$. The varieties $X \subseteq \mathbb{P}_*(\mathfrak{g})$ obtained in this way are called *adjoint varieties*. Note that adjoint varieties $X \subseteq \mathbb{P}_*(\mathfrak{g})$ are determined by the Lie algebra $\mathfrak{g}$ and independent of the choice of algebraic groups G with Lie algebra $\mathfrak{g}$: Indeed for any such G, the image of the adjoint representation $G \to GL(\mathfrak{g})$ is equal to Int $(\mathfrak{g})$.

Proposition 2.2. *If $X \subseteq \mathbb{P}_*(\mathfrak{g})$ is an adjoint variety, then we have*

$$T_{x_\lambda}X = \mathbb{P}_*(\mathbb{C}\cdot H_\lambda \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2) = \langle h_\lambda, \mathbb{P}_*(\mathfrak{g}_1), x_\lambda\rangle,$$

where $x_\lambda = \pi(X_\lambda), h_\lambda = \pi(H_\lambda)$ and $\langle * \rangle$ denotes the linear span of $*$ in $\mathbb{P}_*(\mathfrak{g})$. In particular, we have $\dim X = \dim \mathfrak{g}_1 + 1 = \#R_\lambda + 1$.

Proof. By virtue of Proposition 1.1 we describe $(\mathrm{ad}\ \mathfrak{g})X_\lambda$ as follows:

$$\begin{aligned}(\mathrm{ad}\ \mathfrak{g})X_\lambda &= [\mathfrak{g}_{-2}, X_\lambda] \oplus [\mathfrak{g}_{-1}, X_\lambda] \oplus [\mathfrak{g}_0, X_\lambda] \oplus [\mathfrak{g}_1, X_\lambda] \oplus [\mathfrak{g}_2, X_\lambda] \\ &= \mathbb{C}\cdot[X_{-\lambda}, X_\lambda] \oplus [\mathfrak{g}_{-1}, X_\lambda] \oplus [\mathfrak{g}_0, X_\lambda] \oplus 0 \oplus 0.\end{aligned}$$

Since ad $X_\lambda : \mathbb{C}\cdot X_{-\alpha} \to \mathbb{C}\cdot X_{\lambda-\alpha}$ is an isomorphism for $\alpha \in R_\lambda$, ad X_λ : $\mathfrak{g}_{-1} \to \mathfrak{g}_1$ is an isomorphism, and we have $[\mathfrak{g}_{-1}, X_\lambda] = \mathfrak{g}_1$. Moreover we have $[\mathfrak{g}_0, X_\lambda] = \mathfrak{g}_2$. Thus we find

$$(\mathrm{ad}\ \mathfrak{g})X_\lambda = \mathbb{C}\cdot H_\lambda \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Obviously we have $X_\lambda \in (\mathrm{ad}\ \mathfrak{g})X_\lambda$. Therefore the result follows from Lemma 2.1.

□

3. In case of A_1

Let $\mathfrak{g}$ be the Lie algebra,

$$\mathfrak{sl}_2\mathbb{C} = \mathbb{C}\cdot X_+ \oplus \mathbb{C}\cdot H \oplus \mathbb{C}\cdot X_-,$$

where $[X_+, X_-] = H, [H, X_+] = 2X_+$ and $[H, X_-] = -2X_-$. We concretely set

$$X_+ := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X_- := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We indentify $\mathbb{P}_*(\mathfrak{sl}_2\mathbb{C})$ with $\mathbb{P}^2$ by $\pi(\xi X_+ + \eta H + \zeta X_-) = (\xi : \eta : \zeta)$, and consider a conic in $\mathbb{P}^2$:

$$Q := \{\pi(Y) \in \mathbb{P}_*(\mathfrak{sl}_2\mathbb{C}) | \det Y = 0\} = \{(\xi : \eta : \zeta) \in \mathbb{P}^2 | \xi\zeta + \eta^2 = 0\}.$$

Set $x_+ := \pi(X_+)$, $h := \pi(H)$ and $x_- := \pi(X_-)$, where $\pi : \mathfrak{sl}_2\mathbb{C} \setminus \{0\} \to \mathbb{P}_*(\mathfrak{sl}_2\mathbb{C}) = \mathbb{P}^2$ is the canonical projection.

Proposition 3.1. (a) *The adjoint action on* $\mathfrak{sl}_2\mathbb{C}$ *has two orbits as follows:*

$$G \cdot x_+ = Q, \quad \textit{and} \quad G \cdot h = \mathbb{P}^2 \setminus Q.$$

(b) *In particular,* Q *is an adjoint variety, and we have* $\operatorname{Sec} Q = \mathbb{P}^2$.

Proof. (a) This is an easy exercise of linear algebra since the action is conjugate and one may assume $G = SL_2\mathbb{C}$. It follows from the definition of Q that $G \cdot x_+ \subseteq Q$ and $G \cdot h \subseteq \mathbb{P}^2 \setminus Q$ since $\det X_+ = 0$ and $\det H = -1 \neq 0$. Conversely, let Y be a non-zero element in $\mathfrak{sl}_2\mathbb{C}$. Since $\operatorname{tr} Y = 0$, the set of eigenvalues of Y is $\{a, -a\}$ for some $a \in \mathbb{C}$. If $a = 0$, then the Jordan canonical form of Y is equal to X_+, which means that $\pi(Y) \in G \cdot x_+$. Thus $Q \subseteq G \cdot x_+$. If $a \neq 0$, then Y is diagonalizable, hence $\pi(Y) \in G \cdot h$. Thus $\mathbb{P}^2 \setminus Q \subseteq G \cdot h$.

(b) The conic Q must be an adjoint variety since it is a unique closed orbit of adjoint action, so that $\operatorname{Sec} Q = \mathbb{P}^2$ clearly follows.

□

There is another proof for the orbit of h without diagonalization, that is, tangent lines to Q are available for a proof of $\mathbb{P}^2 \backslash Q \subseteq G \cdot h$, which geometrically illustrates relationship among x_+, x_- and h.

For an arbitrary point $z \in \mathbb{P}^2 \backslash Q$, consider tangent lines to Q passing through z, and let $x, y \in Q$ be the points of contact, with $x \neq y$. It is easy to obtain a parametric representaion for the orbit of (X_+, X_-) in $\mathfrak{sl}_2\mathbb{C}^{\oplus 2}$. Eliminating the parameters from this, we find that

$$G \cdot (x_+, x_-) = Q \times Q \setminus \Delta$$

in $\mathbb{P}^2 \times \mathbb{P}^2$, where Δ is the diagonal set of $Q \times Q$: this part amounts to the digonalization and a weaker statement still holds for a general case (see Lemma 4.2). This implies that there exists $g \in G$ such that $g \cdot (x_+, x_-) = (x, y)$. Since the action on $\mathbb{P}^2$ is linear, it follows that $g \cdot T_{x_+}Q = T_x Q$ and $g \cdot T_{x_-}Q = T_y Q$. On the other hand, we have $T_{x_+}Q \cap T_{x_-}Q = \{h\}$, which follows from the defining equation of Q. Therefore $g \cdot h = z$ since $T_x Q \cap T_y Q = \{z\}$.

Remark 3.2. It turns out that the geometric illustration of relationship among x_+, x_- and h extends to the case of general adjoint varieties (see the forthcoming paper [KY]).

4. Secant Varieties

In this section we prove

Theorem 4.1. *If $X \subseteq \mathbb{P}_*(\mathfrak{g})$ is an adjoint variety, then* $\dim \mathrm{Sec}\, X = 2 \dim X$.

We give two proofs: One is based on Terracini's Lemma (see, for example, [FR; §2]), and the other is based on a realization of Sec X, which is given below. In either case, the following observation is essential:

Lemma 4.2. *The orbit $G \cdot (x_\lambda, x_{-\lambda})$ in $X \times X$ is a dense open subset of $X \times X$.*

Proof. According to [Hm2; 8.3, Proposition], each orbit is locally closed. On the other hand, it follows from [Z; p. 51] that $\overline{G \cdot (x_\lambda, x_{-\lambda})} = X \times X$. Thus the claim follows.

□

Proof 1 of Theorem 4.1. According to Terracini's Lemma, we have $T_z\mathrm{Sec}\ X = \langle T_xX, T_yX\rangle$ for a general point $z \in \mathrm{Sec}\ X$ such as $z \in \langle x, y\rangle$ for general points $x, y \in X$. So we have $\dim \mathrm{Sec}\ X = \dim\langle T_xX, T_yX\rangle$ for general $x, y \in X$. It follows from Lemma 4.2 that $\dim\langle T_xX, T_yX\rangle = \dim\langle T_{x_\lambda}X, T_{x_{-\lambda}}X\rangle$ for general $x, y \in X$. It follows from Proposition 2.2 that

$$\langle T_{x_\lambda}X, T_{x_{-\lambda}}X\rangle = \mathbb{P}_*(\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathbb{C}\cdot H_\lambda \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2),$$

whose dimension is equal to $2(\dim \mathfrak{g}_1 + 1) = 2\dim X$. This completes the proof.

□

For another proof of the claim, we describe Sec X as the closure of an orbit under the adjoint action. For a root α of $\mathfrak{g}$, let $\mathfrak{s}_\alpha$ be a Lie subalgebra of type A_1 in $\mathfrak{g}$ as follows:

$$\mathfrak{s}_\alpha = \mathbb{C}\cdot X_\alpha \oplus \mathbb{C}\cdot H_\alpha \oplus \mathbb{C}\cdot X_{-\alpha}.$$

Proposition 4.3. *If $X \subseteq \mathbb{P}_*(\mathfrak{g})$ is an adjoint variety, then the orbit $G\cdot h_\lambda$ is a dense open subset of* Sec X*, where λ is the highest root of $\mathfrak{g}$ and $h_\lambda = \pi(H_\lambda)$. In particular, we have* $\mathrm{Sec}\ X = \overline{G\cdot\mathbb{P}_*(\mathfrak{s}_\lambda)}$.

Proof. For the former part, it is enough to show that $\mathrm{Sec}\ X = \overline{G\cdot h_\lambda}$ (see the proof of Lemma 4.2). In case of A_1 this follows from Proposition 3.1.

For a general case, let G_λ be the algebraic subgroup of G corresponding to the Lie subalgebra $\mathfrak{s}_\lambda$, and let Q_λ be the orbit $G_\lambda\cdot x_\lambda$ of x_λ by the action restricted to G_λ. It follows from Proposition 3.1 that Q_λ is a conic in $\mathbb{P}_*(\mathfrak{s}_\lambda)$ passing through x_λ and $x_{-\lambda}$. We first show that $G\cdot h_\lambda \subseteq \mathrm{Sec}\ X$. It is sufficient to show that $h_\lambda \in \mathrm{Sec}\ X$ since X is homogeneous, and this follows from $h_\lambda \in \mathrm{Sec}\ Q_\lambda \subseteq \mathrm{Sec}\ X$.

Next we show the converse. It suffices to show that a general point z of Sec X lies in the orbit of h_λ under the adjoint action. By virtue of Lemma 4.2, we may assume that $z \in \langle x, y\rangle$ and $(x, y) = g\cdot(x_\lambda, x_{-\lambda})$ for some $g \in G$. It follows that $g^{-1} \cdot z \in \langle x_\lambda, x_{-\lambda} \rangle \subseteq \mathbb{P}_*(\mathfrak{s}_\lambda)$. Then we may assume $g^{-1} \cdot z \notin Q_\lambda$ since z is general. Thus it follows from Proposition 3.1 that $g^{-1} \cdot z = g' \cdot h_\lambda$ for some $g' \in G_\lambda$. Then we have that $z = gg' \cdot h_\lambda \in G \cdot h_\lambda$.

For the latter part, since it follows from Proposition 3.1 that $G \cdot h_\lambda = G \cdot G_\lambda \cdot h_\lambda = G \cdot (\mathbb{P}_*(\mathfrak{s}_\lambda) \setminus Q_\lambda)$, we have $\overline{G \cdot h_\lambda} = \overline{G \cdot \mathbb{P}_*(\mathfrak{s}_\lambda)}$.

□

Proof 2 of Theorem 4.1. It follows from Proposition 4.3 that $\dim \mathrm{Sec}\, X = \dim G \cdot h_\lambda$. On the other hand, by virtue of Proposition 1.1 we have

$$\begin{aligned}(\mathrm{ad}\ \mathfrak{g})H_\lambda &= [\mathfrak{g}_{-2}, H_\lambda] \oplus [\mathfrak{g}_{-1}, H_\lambda] \oplus [\mathfrak{g}_0, H_\lambda] \oplus [\mathfrak{g}_1, H_\lambda] \oplus [\mathfrak{g}_2, H_\lambda] \\ &= \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus 0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.\end{aligned}$$

Since $H_\lambda \notin (\mathrm{ad}\ \mathfrak{g})H_\lambda$, it follows from Lemma 2.1 that

$$T_{h_\lambda}(G \cdot h_\lambda) = \mathbb{P}_*(\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathbb{C} \cdot H_\lambda \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2),$$

whose dimension is equal to $2 \dim X$ (see Proposition 2.2). This completes the proof.

□

Proposition 4.4. *If $X \subseteq \mathbb{P}_*(\mathfrak{g})$ is an adjoint variety, then*

$$\mathrm{codim}\ (\mathrm{Sec}\ X, \mathbb{P}_*(\mathfrak{g})) = \dim \mathfrak{g}_0 - 1.$$

In particular, if $\mathrm{rk}\ \mathfrak{g} \geq 2$*, then* $\mathrm{Sec}\ X \neq \mathbb{P}_*(\mathfrak{g})$.

Proof. From Propositions 1.1, 2.1 and Theorem 4.1 we see that $\dim \mathbb{P}_*(\mathfrak{g}) - \dim \mathrm{Sec}\ X = (\dim \mathfrak{g} - 1) - 2(\dim \mathfrak{g}_1 + 1) = \dim \mathfrak{g}_0 - 1 \geq \dim \mathfrak{h} - 1 = \mathrm{rk}\ \mathfrak{g} - 1$.

□

Example 4.5. We give a table of adjoint varieties. It follows from Theorem 4.1 and Proposition 4.4 that those varieties in case of rk ≥ 2 yield an example of projective varieties $X \subseteq \mathbb{P}$ with degenerate secants, that is, $\dim \mathrm{Sec}\ X < 2 \dim X + 1$ and $\mathrm{Sec}\ X \neq \mathbb{P}$.

TABLE OF ADJOINT VARIETIES

type	highest root	$X \subseteq \mathbb{P}$	$\dim \mathbb{P} + 1$
$A_{l\geq 1}$	$\omega_1 + \omega_l$	$\mathbb{P}(T_{\mathbb{P}^l}) = \mathbb{P}^l \times \mathbb{P}^l \cap (1)$	$(l+1)^2 - 1$
$B_{l\geq 2}$	ω_2	$\mathbb{F}_1(Q^{2l-1})^{4l-5}$	$2l^2 + l$
$C_{l\geq 3}$	$2\omega_1$	$v_2(\mathbb{P}^{2l-1})$	$2l^2 + l$
$D_{l\geq 4}$	ω_2	$\mathbb{F}_1(Q^{2l-2})^{4l-7}$	$2l^2 - l$
E_6	ω_2	X^{20+1}	78
E_7	ω_1	X^{32+1}	133
E_8	ω_8	X^{56+1}	248
F_4	ω_1	X^{14+1}	52
G_2	ω_2	X^{4+1}	14

In the table, ω_i denotes the i-th fundamental weight as in [Br], v_2 the Veronese embedding, Q^n a quadric hypersurface of dimension n, $\mathbb{F}_m(Q^n)$ the Fano variety of m-planes in Q^n, and $\cap(1)$ cutting by a general hyperplane (see [FH], [Hr] for the definitions).

5. Contact Loci

In this section we prove

Theorem 5.1. *If $X \subseteq \mathbb{P}_*(\mathfrak{g})$ is an adjoint variety, then $\dim C_u = 2$ for a general point $u \in \mathrm{Sec}\, X$.*

By virtue of the Proposition 4.3 it suffices to consider the case $u = h_\lambda$, so that the claim follows from

Proposition 5.2. *If $X \subseteq \mathbb{P}_*(\mathfrak{g})$ is an adjoint variety, then $C_{h_\lambda} = \mathbb{P}_*(\mathfrak{s}_\lambda)$.*

Proof. Take a general $z \in \mathbb{P}_*(\mathfrak{s}_\lambda)$. Then $\langle h_\lambda, z\rangle \cap Q_\lambda \neq \emptyset$, and we may assume that $\langle h_\lambda, z\rangle \cap Q_\lambda = \{x, y\}$ with $x \neq y$, where Q_λ is a conic in the projective plane $\mathbb{P}_*(\mathfrak{s}_\lambda)$ obtained as the orbit $G_\lambda \cdot x_\lambda$, as before. According to Terracini's Lemma, we have $T_z\mathrm{Sec}\, X = \langle T_xX, T_yX\rangle = T_{h_\lambda}\mathrm{Sec}\, X$, which implies $z \in C_{h_\lambda}$. Thus we have $\mathbb{P}_*(\mathfrak{s}_\lambda) \subseteq C_{h_\lambda}$.

Conversely, let z be a general point in C_{h_λ}. Then it follows from Proposition 4.3 that there exists an element $g \in G$ such that $z = g \cdot h_\lambda \in \mathbb{P}_*(g \cdot \mathfrak{s}_\lambda)$. It follows from Terracini's Lemma that $T_{h_\lambda}\mathrm{Sec}\ X = \langle T_{x_\lambda}X, T_{x_{-\lambda}}X \rangle$. On the other hand, $\langle T_{x_\lambda}X, T_{x_{-\lambda}}X \rangle = \mathbb{P}_*([\mathfrak{g}, \langle X_\lambda, X_{-\lambda} \rangle])$ since $T_{x_{\pm\lambda}}X = \mathbb{P}_*([\mathfrak{g}, X_{\pm\lambda}])$. Therefore, denoting by L_λ the subspace $\mathfrak{g}_{-2} \oplus \mathfrak{g}_2 = \langle X_\lambda, X_{-\lambda} \rangle$ of $\mathfrak{g}$, we have $T_{h_\lambda}\mathrm{Sec}\ X = \mathbb{P}_*([\mathfrak{g}, L_\lambda])$, so that $T_z\mathrm{Sec}\ X = \mathbb{P}_*([\mathfrak{g}, g \cdot L_\lambda])$. Since $z \in C_{h_\lambda}$, these linear spaces coincide in $\mathbb{P}_*(\mathfrak{g})$ and we have $[\mathfrak{g}, L_\lambda] = [\mathfrak{g}, g \cdot L_\lambda]$.

Let $Z = Z_0 + \sum_{\alpha \in R} c_\alpha X_\alpha$ be a vector in $g \cdot L_\lambda$, where $Z_0 \in \mathfrak{h}$ and $c_\alpha \in \mathbb{C}$. The $\mathfrak{h}$-component of the vector $[X_{-\alpha}, Z]$ is $-c_\alpha H_\alpha$. Since $[X_{-\alpha}, Z] \in [\mathfrak{g}, g \cdot L_\lambda] = [\mathfrak{g}, L_\lambda]$ and $[\mathfrak{g}, L_\lambda] = \mathfrak{g}_{-1} \oplus \mathfrak{s}_\lambda \oplus \mathfrak{g}_1$, we have $c_\alpha = 0$ unless $\alpha = \pm\lambda$, where one should note that $\mathbb{C} \cdot H_\alpha \neq \mathbb{C} \cdot H_\beta$ if $\alpha \neq \pm\beta$. Thus we have $g \cdot L_\lambda \subseteq L_\lambda \oplus \mathfrak{h}$. This implies that $[g \cdot L_\lambda, g \cdot L_\lambda] \subseteq \mathfrak{s}_\lambda$. On the other hand, since it follows that $\mathfrak{s}_\lambda = [[L_\lambda, L_\lambda], L_\lambda] \oplus [L_\lambda, L_\lambda]$, we have

$$g \cdot \mathfrak{s}_\lambda = [[g \cdot L_\lambda, g \cdot L_\lambda], g \cdot L_\lambda] \oplus [g \cdot L_\lambda, g \cdot L_\lambda].$$

Combining these formulas, we see that $g \cdot \mathfrak{s}_\lambda \subseteq \mathfrak{s}_\lambda$, so that $z \in \mathbb{P}_*(\mathfrak{s}_\lambda)$. Hence $C_{h_\lambda} \subseteq \mathbb{P}_*(\mathfrak{s}_\lambda)$.

□

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