

DIRECT DECOMPOSITIONS IN ARTINIAN MODULES OVER FC -HYPERCENTRAL GROUPS

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Abstract

In this article the existence of \mathcal{X} -decomposition in artinian $\mathbb{Z}G$ -modules is established for some kinds of formations \mathcal{X} , where G is a locally soluble FC -hypercentral group.

1. Introduction.

As in Finite Group Theory a class \mathcal{X} of groups is called a formation if it satisfies the following conditions:

if $G \in \mathcal{X}$, H is a normal subgroup of G , then $G/H \in \mathcal{X}$;

if H_1, H_2 are normal subgroups of G such that $G/H_1, G/H_2 \in \mathcal{X}$, then $G/H_1 \cap H_2 \in \mathcal{X}$.

Let R be a ring, G a group, A an RG -module, B_1, B_2 RG -submodules of A , $B_1 \leq B_2$, and let \mathcal{X} be a class of groups. The factor B_1/B_2 is called \mathcal{X} -central (respectively \mathcal{X} -eccentric) if $G/C_G(B_2/B_1) \in \mathcal{X}$ (respectively $G/C_G(B_2/B_1) \notin \mathcal{X}$).

Let $a \in A$. We say that a is an $\mathcal{X}C$ -element if $G/C_G(aRG) \in \mathcal{X}$.

Put

$$\mathcal{X}C_{RG}(A) = \{a \in A \mid a \text{ is an } \mathcal{X}C\text{-element of } A\}.$$

It is easy to see that $\mathcal{X}C_{RG}(A)$ is a RG -submodule of A in the case when \mathcal{X} is a formation. The submodule $\mathcal{X}C_{RG}(A)$ is called the $\mathcal{X}C$ -center of A (more

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precisely: the $\mathcal{X}C$ - RG -center). Starting from the $\mathcal{X}C$ -center, we can construct the upper $\mathcal{X}C$ -central series of the module A . It is the following series

$$\langle 0 \rangle = A_0 \leq A_1 \leq \dots A_\alpha \leq A_{\alpha+1} \leq \dots A_\gamma,$$

where $A_1 = \mathcal{X}C_{RG}(A)$, $A_{\alpha+1}/A_\alpha = \mathcal{X}C_{RG}(A/A_\alpha)$, $\alpha < \gamma$, $\mathcal{X}C_{RG}(A/A_\gamma) = \langle 0 \rangle$.

The last term A_γ of this series is called the upper $\mathcal{X}C$ -hypercenter of the module A (more precisely, the $\mathcal{X}C$ - RG -hypercenter), and is denoted by $\mathcal{X}C_{RG}^\infty(A)$; the terms A_α of this series are called the upper $\mathcal{X}C$ -hypercenters.

If $A = A_\gamma$ then the module A is called $\mathcal{X}C$ -hypercentral; if γ is finite, then A is called $\mathcal{X}C$ -nilpotent.

If $\mathcal{X} = \mathcal{G}$ is the class of all identity groups then we come to the concept of RG -hypercentral (or hypertrivial) module.

If $\mathcal{X} = \mathcal{F}$ is the class of all finite groups then we will obtain the concept of FC -hypercentral module.

Let R be a ring, G a group, A an RG -module, and let \mathcal{X} be a formation of groups. We say that A has the \mathcal{X} -decomposition (more precisely, \mathcal{X} - RG -decomposition) if $A = \mathcal{X}C_{RG}^\infty(A) \oplus \mathcal{X}C_{RG}^*(A)$ where $\mathcal{X}C_{RG}^*(A)$ is an RG -submodule of A such that every of its non-zero RG -factor is \mathcal{X} -eccentric.

If $\mathcal{X} = \mathcal{G}$ then we obtain the Z -decomposition. This themes began from the famous Fitting lemma which is very useful in Group Theory. In the Infinite Group Theory first results on the existence of Z -decomposition were obtained by B. Hartley and M.J.Tomkinson [3].

Artinian modules are very good extensions of finite modules. In his paper [9] D.I. Zaitsev proved that every artinian $\mathbb{Z}G$ -module has the Z -decomposition for every hypercentral group G .

The next natural formation is the class \mathcal{F} of all finite groups. The first result

about the \mathcal{F} -decomposition was obtained by D.I.Zaitsev [10], who proved that every artinian $\mathbb{Z}G$ -module over locally soluble hyperfinite group G has the \mathcal{F} -decomposition. The condition of local solubility is not necessary (Z.Duan [2]). But for non-torsion FC -hypercentral groups we have only some initial results (D.I.Zaitsev [11], Z.Duan [1]).

In this paper we consider the question about the existence of \mathcal{X} -decomposition not only for the formation \mathcal{F} but for some extension of \mathcal{F} .

We say that the formation \mathcal{X} is overfinite if \mathcal{X} satisfies the following conditions:

- (1) $\mathcal{F} \leq \mathcal{X}$;
- (2) if $G \in \mathcal{X}$, H is a normal subgroup of finite index, then $H \in \mathcal{X}$;
- (3) if G is a group, H is a normal subgroup of G such that $|G : H|$ is finite and $H \in \mathcal{X}$, then $G \in \mathcal{X}$.

Our main result is the following theorem.

Theorem. *Let G be a locally soluble FC -hypercentral group, and let A be an artinian $\mathbb{Z}G$ -module. If \mathcal{X} is an overfinite formation of groups then A has the \mathcal{X} -decomposition.*

Corollary. *Let G be a locally soluble FC -hypercentral group, A be an artinian $\mathbb{Z}G$ -module. Then A has the \mathcal{X} -decomposition for the following formations \mathcal{X} :*

- (1) $\mathcal{X} = \mathcal{F}$, the formation of all finite groups;
- (2) $\mathcal{X} = \mathcal{LF}$, the formation of all polycyclic-by-finite groups;
- (3) $\mathcal{X} = \mathcal{C}$, the formation of all Chernikov groups;
- (4) $\mathcal{X} = \mathcal{S}_2\mathcal{F}$, the formation of all soluble-by-finite minimax groups;
- (5) $\mathcal{X} = \hat{\mathcal{S}}\mathcal{F}$, the formation of all soluble-by-finite groups of finite special (Mal'cev-Prüfer) rank;
- (6) $\mathcal{X} = \mathcal{S}_0\mathcal{F}$, the formation of all soluble-by-finite groups of finite section rank.

2. Some preliminary results.

The first two lemmas and their corollaries are almost obvious and we omit their proofs.

Lemma 1. *Let R be a ring, G a group, A an RG -module, B an RG -submodule of A , and let \mathcal{X} be a formation of groups. Then $\mathcal{X}_{RG}^\infty(B) \leq \mathcal{X}_{RG}^\infty(A)$.*

Lemma 2. *Let R be a ring, G a group, A an RG -module, B_1 and B_2 RG -submodules of A , $B_1 \leq B_2$, and let \mathcal{X} be a formation of groups. If every non-zero RG -factor of B_1 and of B_2/B_1 is \mathcal{X} -eccentric, then every non-zero RG -factor of B_2 is \mathcal{X} -eccentric.*

Corollary 1. *Let $\{B_\alpha | \alpha \leq \gamma\}$ be an ascending chain of RG -submodules of A satisfying the following condition:*

if E, C are RG -submodules such that $B_\alpha \leq C < E \leq B_{\alpha+1}$, $C \neq E$, $\alpha < \gamma$, then E/C is \mathcal{X} -eccentric.

Then every non-zero RG -factor of the submodule B_γ is \mathcal{X} -eccentric.

Corollary 2. *Let $\{B_\lambda | \lambda \in \Lambda\}$ be a family of RG -submodules of A such that every non-zero RG -factor of B_λ is \mathcal{X} -eccentric for any $\lambda \in \Lambda$, and $B = \sum_{\lambda \in \Lambda} B_\lambda$. Then every non-zero RG -factor of B is \mathcal{X} -eccentric.*

Corollary 3. *Let $\{B_\lambda | \lambda \in \Lambda\}$ be a family of RG -submodules of A such B_λ has the \mathcal{X} - RG -decomposition for any $\lambda \in \Lambda$, and $B = \sum_{\lambda \in \Lambda} B_\lambda$. Then B has the \mathcal{X} - RG -decomposition.*

Corollary 4. *The module A has the largest RG -submodule having the \mathcal{X} - RG -decomposition.*

Lemma 3. *Let R be a ring, G a group, A a RG -module, H a normal subgroup of G such that G/H is finite, and let B be a RH -submodule of A such that $A = BRG$. If \mathcal{X} is an overfinite formation of groups and B has the \mathcal{X} - RH -decomposition then A has the \mathcal{X} - RG -decomposition.*

Proof. Let $\{g_1, \dots, g_n\}$ be a transversal for H in G . Then $A = Bg_1 + \dots + Bg_n$. If C, D are RH -submodules, $C \geq D, L = C_H(C/D), g \in G$, then $C_H(Cg/Dg) = g^{-1}Lg$. Therefore $H/C_H(Cg/Dg) = H/g^{-1}Lg \cong H/L$. It follows that if the RH -factor C/D is \mathcal{X} -central (respectively, \mathcal{X} -eccentric) then the RH -factor Cy/Dy is \mathcal{X} -central (respectively, \mathcal{X} -eccentric) too. This means that the RH -submodule Bg has the \mathcal{X} - RH -decomposition.

Let $a \in \mathcal{X}C_{RH}(A), A_0 = aRH, A_1 = aRG$. Then $A = A_0g_1 + \dots + A_0g_n$. Put $U = C_H(A_0)$, then $H/U \in \mathcal{X}$. It follows that $H/C_H(A_0g) \in \mathcal{X}$ for each $g \in G$, so that $ag \in \mathcal{X}C_{RH}(A)$. In particular, $\mathcal{X}C_{RH}(A)$ is an RG -submodule of A . Further, $C_H(A_1) = g_1^{-1}Ug_1 \cap \dots \cap g_n^{-1}Ug_n$, hence $H/C_H(A_1) \in \mathcal{X}$. From the definition of overfinite formation we obtain that $G/C_G(A_1) \in \mathcal{X}$, i.e. $\mathcal{X}C_{RH}(A) \leq \mathcal{X}C_{RG}(A)$. The converse inclusion is also valid, so that $\mathcal{X}C_{RH}(A) = \mathcal{X}C_{RG}(A)$. It follows from transfinite induction that $\mathcal{X}C_{RH}^\infty(A) = \mathcal{X}C_{RG}^\infty(A)$.

Let $B = B_1 \oplus B_2$ where $B_1 = \mathcal{X}C_{RH}^\infty(B), B_2 = \mathcal{X}C_{RH}^*(B)$. Then $B_2g_1 + \dots + B_2g_n$ is an RG -submodule of A . Corollary 2 of Lemma 2 implies that $B_2g_1 + \dots + B_2g_n \leq \mathcal{X}C_{RH}^*(A)$. Lemma 1 yields that $B_1g_1 + \dots + B_1g_n \leq \mathcal{X}C_{RH}^\infty(A)$, hence $\mathcal{X}C_{RH}^*(A) = B_2g_1 + \dots + B_2g_n$, in particular, $\mathcal{X}C_{RH}^*(A)$ is an RG -submodule of A . Let U and V be RG -submodules of $\mathcal{X}C_{RH}^*(A)$ such that $U \geq V$ and $U \neq V$. Then U/V is a non-zero RH -factor of $\mathcal{X}C_{RH}^*(A)$, so that $H/C_H(U/V) \notin \mathcal{X}$. From the definition of overfinite formation we obtain that $G/C_G(U/V) \notin \mathcal{X}$. This proves the equation $\mathcal{X}C_{RH}^*(A) = \mathcal{X}C_{RG}^*(A)$.

The following lemma is well-known.

Lemma 4. *Let G be a FC-hypercentral group, L be a finitely generated subgroup of G . Then L is nilpotent-by-finite. In particular, G is a locally (polycyclic-by-finite) group.*

Lemma 5. *Let G be a polycyclic-by-finite group, $1 \neq g \in \zeta(G)$, and let A be a finitely generated $\mathbb{Z}G$ -module. If A is a monolithic module with the monolith M and $M(g-1) = \langle 0 \rangle$, then $A(g-1)^m = \langle 0 \rangle$ for some $m \in \mathbb{N}$.*

Proof. Since M is a simple $\mathbb{Z}G$ -submodule then $pM = \langle 0 \rangle$ for some prime p ([6], theorem 9.55). Let T be the torsion part of A . Since A is a monolithic module then T is a p -subgroup. Since $\mathbb{Z}G$ is a noetherian ring and A is a finitely generated module then A is a noetherian $\mathbb{Z}G$ -module. In particular, T is finitely generated. It follows that there exists a number $n \in \mathbb{N}$ such that $p^n T = \langle 0 \rangle$. Then $p^n A$ is torsion-free, therefore, $M \cap p^n A = \langle 0 \rangle$. This means that $p^n A = \langle 0 \rangle$, i.e. $A = T$. In other words A is a p -group and $p^n A = \langle 0 \rangle$.

We can consider the submodule $A_1 = \Omega_1(A)$ as $\mathbb{F}_p G$ -module. Let $R = \mathbb{F}_p \langle x \rangle$ be the group algebra of an infinite cyclic group $\langle x \rangle$ over \mathbb{F}_p . Put $ax = ag$ for each $a \in A_1$. Then A is an RG -module and R is a principal ideal domain. Since $M(g-1) = M(x-1) = \langle 0 \rangle$ then the $(x-1)$ -component of A_1 is non-zero. Using the same arguments we obtain that A_1 coincides with its $(x-1)$ -component. Since A is a noetherian RG -module then this implies that $A_1(x-1)^{m_1} = \langle 0 \rangle$ for some $m_1 \in \mathbb{N}$.

The mapping $\varphi : \Omega_2(A) \rightarrow \Omega_1(A)$ defining by the rule $a\varphi = pa, a \in \Omega_2(A)$, is a $\mathbb{Z}G$ -homomorphism, so $Im\varphi$ and $Ker\varphi = \Omega_1(A)$ are $\mathbb{Z}G$ -submodules. Since $Im\varphi \leq A_1$ then $Im\varphi(x-1)^{m_1} = \langle 0 \rangle$. Similarly $Ker\varphi(x-1)^{m_1} = \langle 0 \rangle$. Therefore $\Omega_2(A)(x-1)^{2m_1} = \langle 0 \rangle$.

Using a simple induction and the equality $A = \Omega_n(A)$, we obtain that $A(x-1)^m = \langle 0 \rangle$ where $m = nm_1$. Thus $A(g-1)^m = \langle 0 \rangle$.

Corollary. *Let G be a polycyclic-by-finite group, $1 \neq g \in \zeta(G)$, A be a finitely generated $\mathbb{Z}G$ -module. If $C_A(g) \neq \langle 0 \rangle$ then $A \neq A(g-1)$.*

Proof. Let $0 \neq a \in C_A(g)$. Since $g \in \zeta(G)$ then $C_A(g)$ is a $\mathbb{Z}G$ -submodule. Let Ba be a maximal $\mathbb{Z}G$ -submodule of A with the property $a \notin Ba$. Then A/Ba is a monolithic $\mathbb{Z}G$ -module with the monolith $a\mathbb{Z}G + Ba/Ba = M/Ba$. Then $(M/Ba)(g-1) = \langle 0 \rangle$, so we can apply lemma 5 to the module A/Ba .

Lemma 6. *Let G be a locally (polycyclic-by-finite) group, $1 \neq g \in \zeta(G)$, A a*

finitely generated $\mathbb{Z}G$ -module. If $C_A(g) \neq \langle 0 \rangle$ then $A \neq A(g-1)$.

Proof. Let $A = a_1\mathbb{Z}G + \dots + a_n\mathbb{Z}G$. Assume that $A = A(g-1)$. Then there are elements $b_1, \dots, b_n \in A$ such that $a_i = b_i(g-1), 1 \leq i \leq n$. Let $0 \neq c \in C_A(g)$. Choose in G a finitely generated subgroup H with the properties $g \in H, c, b_1, \dots, b_n \in a_1\mathbb{Z}H + \dots + a_n\mathbb{Z}H = B$. Let $b \in B$ then $b = a_1x_1 + \dots + a_nx_n = b_1(g-1)x_1 + \dots + b_n(g-1)x_n = (b_1x_1 + \dots + b_nx_n)(g-1)$.

It follows that $B = B(g-1)$. On the other hand, $c \in C_A(g) \cap C_B(g)$, in particular, $C_B(g) \neq \langle 0 \rangle$. Corollary to lemma 5 implies that $B \neq B(g-1)$. This is a contradiction.

3. Proof of the main theorem.

If $G \in \mathcal{X}$ then $A = \mathcal{X}C_{\mathbb{Z}G}^\infty(A)$. Therefore we can assume that $G \notin \mathcal{X}$.

Suppose that A does not have the \mathcal{X} - $\mathbb{Z}G$ -decomposition. Put $\mathcal{M} = \{B \mid B \text{ is a } \mathbb{Z}G\text{-submodule such that } B \text{ does not have the } \mathcal{X}\text{-}\mathbb{Z}G\text{-decomposition}\}$. Then $A \in \mathcal{M}$, in particular, $\mathcal{M} \neq \emptyset$. Since A is an artinian $\mathbb{Z}G$ -module, then \mathcal{M} has a minimal element C . Corollary 4 of lemma 2 implies that C includes the largest $\mathbb{Z}G$ -submodule M having the \mathcal{X} - $\mathbb{Z}G$ -decomposition. From the choice of C we obtain that M includes every proper $\mathbb{Z}G$ -submodule of C , in particular, M is a maximal $\mathbb{Z}G$ -submodule of C .

Let $M = M_1 \oplus M_2$ where $M_1 = \mathcal{X}C_{\mathbb{Z}G}^\infty(M), M_2 = \mathcal{X}C^*_{\mathbb{Z}G}(M)$. Assume first that $G/C_G(C/M) \notin \mathcal{X}$, and consider the factor-module C/M_2 . In other words, we can assume that $M = \mathcal{X}C_{\mathbb{Z}G}^\infty(M)$. We can assume also that $C_G(C) = \langle 1 \rangle$.

Put $S = Soc_{\mathbb{Z}G}(C)$. Since S has the \mathcal{X} - $\mathbb{Z}G$ -decomposition then $S \leq M$. It follows that $G/C_G(S) \in \mathcal{X}$, in particular, $C_G(S) \neq \langle 1 \rangle$. Since G is a FC -hypercentral group then $C_G(S) \cap FC(G) \neq \langle 1 \rangle$ [4, lemma 3]. Let $1 \neq x \in C_G(S) \cap FC(G)$, then $\langle x \rangle^G$ is central-by-finite and the index $|G : C_G(\langle x \rangle^G)|$ is finite. Therefore either $\langle x \rangle^G$ includes a finite minimal G -invariant subgroup X

or a G -invariant torsion-free finitely generated subgroup X . If X is finite then X is also abelian, because G is locally soluble. Put $H = C_G(X)$. In these both cases $|G : H|$ is finite and $X \leq \zeta(H)$.

Since C/M is a simple $\mathbb{Z}G$ -module then $C/M = \bigoplus_{1 \leq i \leq n} (B/M)g_i$ where B/M is a simple $\mathbb{Z}H$ -module, $g_1, \dots, g_n \in G$ [8, lemma]. If we assume that $H/C_H(B/M) \in \mathcal{X}$ then from the equation $H/C_H((B/M)g) = H/g^{-1}C_H(B/M)g \cong H/C_H(B/M)$ we obtain that $H/C_H((B/M)g_i) \in \mathcal{X}$ for any $i, 1 \leq i \leq n$. It follows that $H/C_H(C/M) \in \mathcal{X}$. This contradiction shows that $H/C_H(B/M) \notin \mathcal{X}$. Since $B \not\leq M^G$ then $B \mathbb{Z}G = C$. If we assume that B has the \mathcal{X} - $\mathbb{Z}G$ -decomposition then $C = B \mathbb{Z}G$ has the \mathcal{X} - $\mathbb{Z}G$ -decomposition by lemma 3, a contradiction. Hence B does not have the \mathcal{X} - $\mathbb{Z}G$ -decomposition.

Put $\mathcal{C} = \{Q | Q \text{ is a } \mathbb{Z}G\text{-submodule of } B \text{ such that } Q \text{ does not have the } \mathcal{X}\text{-}\mathbb{Z}G\text{-decomposition}\}$. Since $B \in \mathcal{C}$ then $\mathcal{C} \neq \emptyset$. By theorem A of paper [7] \mathcal{C} is an artinian $\mathbb{Z}G$ -module. Thus \mathcal{C} has a minimal element E . Lemma 3 yields that $E \not\leq M$, so that $B = E + M$. Corollary 4 of lemma 2 shows that E includes the largest $\mathbb{Z}H$ -submodule E_1 having the \mathcal{X} - $\mathbb{Z}G$ -decomposition. Lemma 3 shows that $E_1 \mathbb{Z}G \leq M$, i.e. $E_1 \leq M$. Since B/M is a simple $\mathbb{Z}H$ -module then $E_1 = E \cap M$. Moreover, $E/E_1 = E/E \cap M \cong_{DH} E + M/M = B/M$, so that $H/C_H(E/E_1) \notin \mathcal{X}$.

Let $S_1 = Soc_{\mathbb{Z}G}(C)$. Since C is an artinian $\mathbb{Z}H$ -module then $S_1 = L_1 \oplus \dots \oplus L_s$ for some simple $\mathbb{Z}H$ -submodules $L_i, 1 \leq i \leq s$. Since $X \leq \zeta(H)$ then $L_i(\omega \mathbb{Z}X)$ is a $\mathbb{Z}H$ -submodule of $L_i, 1 \leq i \leq s$, here $\omega \mathbb{Z}X$ is the augmentation ideal of $\mathbb{Z}X$. This means that either $L_i(\omega \mathbb{Z}X) = L_i$ or $L_i(\omega \mathbb{Z}X) = \langle 0 \rangle$ because L_i is a simple $\mathbb{Z}G$ -module of $C, 1 \leq i \leq s$. Consequently, $S_1 = C_{S_1}(X) \oplus S_1(\omega \mathbb{Z}X)$. Since S_1 is a $\mathbb{Z}G$ -submodule of C and X is a normal subgroup then $S_1(\omega \mathbb{Z}X)$ is a $\mathbb{Z}G$ -submodule of C . If we assume that $S_1(\omega \mathbb{Z}X) \neq \langle 0 \rangle$ then $S_1(\omega \mathbb{Z}X) \cap Soc_{\mathbb{Z}G}(C) \neq \langle 0 \rangle$. On the other hand, $X \leq C_G(Soc_{\mathbb{Z}G}(C))$, so that $S_1(\omega \mathbb{Z}X) \cap Soc_{\mathbb{Z}G}(C) \leq S_1(\omega \mathbb{Z}X) \cap C_{S_1}(X) = \langle 0 \rangle$. This contradiction shows that $S_1 \leq C_G(X)$. Hence $E \cap C_G(X) \neq \langle 0 \rangle$. If $e \in E \setminus E_1$ then $e \mathbb{Z}H \not\leq E_1$, so

$e\mathbb{Z}H = E$. In particular, E is a finitely generated $\mathbb{Z}H$ -module. It follows from lemmas 4 and 6 that $E(g-1) \leq E_1$ for each $g \in X$. As in lemma 3 we can prove that $E_1 = \mathcal{X}C_{\mathbb{Z}H}^{\infty}(E)$. Consider the mapping $\theta : a + E_1 \rightarrow a(g-1) + E_1(g-1)$, $a \in E$. If $E(g-1) \neq E_1(g-1)$ then from $E(g-1) \leq E_1 = \mathcal{X}C_{\mathbb{Z}H}^{\infty}(E)$ we obtain that $H/C_H(E(g-1)/E_1(g-1)) \in \mathcal{X}$. However $E(g-1)/E_1(g-1) \cong_{\mathbb{Z}H} E/E_1$, in particular, $C_H(E/E_1) = C_H(E(g-1)/E_1(g-1))$. But $H/C_H(E/E_1) \notin \mathcal{X}$. This contradiction shows that $E(g-1) = E_1(g-1)$. This means that $E = C_E(g) + E_1$. It follows from the choice of E that $E = C_E(g)$ because $C_E(g)$ is a $\mathbb{Z}H$ -submodule of E . Since this is true for each $g \in X$ then $E \leq C_C(X)$, in particular, $C_C(X) \not\leq M$. Since X is a normal subgroup of G then $C_C(X)$ is a $\mathbb{Z}G$ -submodule. Hence $C = C_C(X)$, i.e. $X \leq C_G(C) = \langle 1 \rangle$. This is a contradiction.

Now consider the case when $G/C_G(C/M) \in \mathcal{X}$. We will consider the factor-module C/M_1 . In other words, we can assume that $M = \mathcal{X}C_{\mathbb{Z}G}^*(M)$. Again we can assume that $C_G(C) = \langle 1 \rangle$.

Since $G/C_G(C/M) \in \mathcal{X}$ then $C_G(C/M) \neq \langle 1 \rangle$. Therefore $C_G(C/M) \cap FC(G) \neq \langle 1 \rangle$.

Let $1 \neq y \in C_G(C/M) \cap FC(G)$ then $Y = \langle y \rangle^G$ is central-by-finite and $|G : R|$ is finite where $R = C_G(Y)$. Again we can assume that Y is abelian, i.e. $Y \leq \zeta(R)$.

Let $\mathcal{C}_1 = \{Q \mid Q \text{ is a } \mathbb{Z}R\text{-submodule of } C \text{ such that } Q \not\leq M\}$. Since $C \in \mathcal{C}_1$ then $\mathcal{C}_1 \neq \emptyset$. By theorem A of paper [7] C is an artinian $\mathbb{Z}R$ -module. Hence the set \mathcal{C}_1 has a minimal element U . Since $Y \leq \zeta(R)$ then $U(g-1)$ and $C_U(g)$ are $\mathbb{Z}R$ -submodules for any $g \in Y$. Moreover, $U(g-1) \cong U/C_U(g)$. As in lemma 3 we can prove that $M = \mathcal{X}C_{\mathbb{Z}R}^*(M)$. Since $g \in C_G(C/M)$ then $U(g-1) \leq M$. If we assume that $C_U(g) \leq M$ then $U/C_U(g)$ has one non-zero \mathcal{X} -central $\mathbb{Z}R$ -factor. On the other hand, from $U(g-1) \cong_{\mathbb{Z}R} U/C_U(g)$ and $U(g-1) \leq \mathcal{X}C_{\mathbb{Z}R}^*(M)$ we obtain that every non-zero $\mathbb{Z}R$ -factor of $U/C_U(g)$

is \mathcal{X} -eccentric. This means that $C_U(g) \not\leq M$. It follows from the choice of U that $C_U(g) = U$. It is valid for every $g \in Y$. Therefore $U \leq C_C(Y)$. In particular, $C_C(Y) \not\leq M$. Since Y is normal in G then $C_C(Y)$ is a $\mathbb{Z}G$ -submodule. Then $C = C_C(Y)$ because M includes every proper $\mathbb{Z}G$ -submodule of C . Consequently, $Y \leq C_G(C) = \langle 1 \rangle$, so that in this case we also obtain a contradiction. This final contradiction completes the proof.

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