UNIQUENESS THEOREM AND EXACT BOUNDARY CONTROLLABILITY FOR A CLASS OF DISTRIBUTED SYSTEMS

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Abstract

Este paper apresenta resultados de unicidade e controlabilidade exata na fronteira para uma classe geral de sistemas lineares do tipo de Schrödinger. Demonstra-se, primeiramente, um resultado de observabilidade (Desigualdade Inversa via o método HUM) para o sistema homogêneo associado, para um tempo (explícito) suficientemente grande. Estabelece-se também a desigualdade direta para este sistema. Estas desigualdades são obtidas pelo método dos multiplicadores de Lagrange. Utilizando-se estas duas desigualdades e aplicando-se o método HUM, demonstra-se a controlabilidade exata no espaço natural associado ao sistema.

1. Introduction and Problem Formulation

Throughout this paper Ω is an open bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\Gamma = \partial\Omega$. We denote by $\Gamma_0 = \Gamma_0(x^0)$ the part of Γ for which the following inequality is satisfied:

$$(x - x^0, \nu(x)) > 0,$$

where $\nu(x)$ is the unit outer normal, x^0 is a point in $\mathbb{R}^n$, and $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^n$. We assign $\Gamma_1 = \Gamma \setminus \Gamma_0$.

In the cylinder $\Omega\times]0,T[$, $T > 0$, we consider the following initial boundary value problem that generalizes the Schrödinger equation:

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$$\begin{cases} i\dfrac{\partial u}{\partial t} + (-1)^\ell \displaystyle\sum_{p,q} \frac{\partial^\ell}{\partial x_{p_1}\dots\partial x_{p_\ell}}\left(A^{pq}\frac{\partial^\ell u}{\partial x_{q_1}\dots\partial x_{q_\ell}}\right) + Q(x)u = 0, \\ u(x,0) = \varphi(x), \qquad \text{in } \Omega, \\ D_x^\alpha u = 0, \qquad \text{on } \sum = \Gamma\times]0,T[, \quad |\alpha| = |(\alpha_1,\dots,\alpha_n)| \le \ell - 1, \end{cases} \tag{1}$$

where $u = (u^1,\dots,u^m)$, $u^\kappa = u^\kappa(x,t)$, $A^{pq} = (A^{qp})^*$ and $Q(x) = Q^*(x)$ are square matrices of order m with real entries, and the summation goes over all possible collections $p = (p_1,\dots,p_\ell)$ and $q = (q_1,\dots,q_\ell)$, $p_\kappa, q_r \in \{1,\dots,n\}$. We assume that $Q(x) \ge 0$, $Q(x) \in [C^1(\overline{\Omega})]^{m^2}$, and

$$\sum_{p,q} A^{pq}\zeta_q \cdot \overline{\zeta}_p \ge C_0 \sum_p |\zeta_p|^2, \qquad C_0 > 0.$$

Here $\zeta_p = (\zeta_p^1,\dots,\zeta_p^m)$ is an arbitrary, complex-valued vector whose components depend on the collections $p = (p_1,\dots,p_\ell)$, $|\zeta_p|^2 = |\zeta_p^1|^2 + \dots + |\zeta_p^m|^2$, and the dot is the inner product of vectors:

$$\zeta_p \cdot \overline{\zeta}_q = \zeta_p^1\overline{\zeta}_q^1 + \dots + \zeta_p^m\overline{\zeta}_q^m.$$

Our main purpose is to study the following exact controllability problem:

Given the initial state $\varphi(x)$, time $T > 0$, and a desired terminal state $\psi(x)$, with $\varphi(x)$ and $\psi(x)$ in appropriate function spaces, find a vector-valued function $p(x,t)$ in a suitable function space such that the solution of $(1)_1$–$(1)_2$ with the boundary condition

$$\left.\frac{\partial^\kappa u}{\partial \nu^\kappa}\right|_{\sum} = 0, \quad \kappa = 0, 1\dots\ell-2, \quad \left.\frac{\partial^{\ell-1} u}{\partial \nu^{\ell-1}}\right|_{\sum_1} = 0, \quad \left.\frac{\partial^{\ell-1} u}{\partial \nu^{\ell-1}}\right|_{\sum_0} = p(x,t), \tag{2}$$

satisfies:

$$u(x,T,p) = \psi(x). \tag{3}$$

Since the system is linear, it is equivalent to seeking controls which cause the system to rest.

Several approaches are known to solving the problem of exact boundary controllability. One of them is based on Russell's controllability via stabilizability principle (cf. [13]). Another method (the Hilbert uniqueness method), introduced by J. L. Lions (cf. [7] and [8]), is based on the construction of appropriate Hilbert space structures on the space of initial data. These Hilbert structures are connected with uniqueness properties.

In this paper, the controllability problem $(1)_1$–$(1)_2$, (2)–(3) is solved using the HUM.

For some results of the general theory of exact boundary control, we can cite the works of J. Lagnese, in that he studies the exact boundary controllability of Maxwell's equations in a general region (cf. [5]), and the works of B. V. Kapitonov (cf. [3] and [4]). For specific results of the exact controllability theory applied to the Schrödinger equation, we can cite the work of E. Machtyngier, where she studies the exact controllability and the boundary stabilization for the Schrödinger equation (cf. [9]); the work of C. Fabre, where she studies the exact internal controllability for the Schrödinger equation (cf. [2]); the works of M. Milla Miranda and L. A. Medeiros (cf. [11] and [10], and finally the work of G. Lebeau (cf. [6]).

2. Well-Posedness of (1)

Let $\mathcal{H}$ be the Hilbert space consisting of $u(x)\in[H_0^\ell(\Omega)]^m$, with the inner product:

$$(u,v)_0=\int_\Omega\sum_{p,q}A^{pq}D_q^\ell u\cdot D_p^\ell\overline{v}dx+\int_\Omega Q(x)u\cdot\overline{v}dx,$$

where

$$D_p^\ell u=\frac{\partial^\ell u}{\partial x_{p_1}\dots\partial x_{p_\ell}},\quad p=(p_1,\dots,p_\ell),\quad p_\kappa\in\{1,\dots,n\}.$$

In addition, below we use the notation:

$$D_q^i u = \frac{\partial^i u}{\partial x_{q_1} \dots \partial x_{q_i}}, \quad D_q^{\ell-i} u = \frac{\partial^{\ell-i} u}{\partial x_{q_{i+1}} \dots \partial x_{q_\ell}}, \quad (i \leq \ell),$$

and

$$Au = (-1)^\ell \sum_{p,q} D_p^\ell (A^{pq} D_q^\ell u).$$

In $\mathcal{H}$, we define the unbounded operator B:

$$D(B) = \{u \in \mathcal{H}; \mathcal{A}\sqcap \in \mathcal{H}\},$$
$$Bu = i\, Au + i\, Q(x)u, \quad \text{for } u \in D(B).$$

We remark that $\mathcal{H} = [\mathcal{H}_\prime^\ell(\Omega)]^\Updownarrow$.

The following lemma is demonstrated in a standard way.

Lemma 2.1. *The operator B is skew-self-adjoint.*

From Stone's theorem, it follows that the operator B generates a one-parameter group of unitary operators $U(t)$ in $\mathcal{H}$. Moreover, $U(t)$ is strongly continuous in t and $U(t)\varphi$ is strongly differentiable with respect to t for $\varphi \in D(B)$. Furthermore,

$$\frac{d}{dt} U(t)\varphi = BU(t)\varphi, \quad \text{and} \quad \|U(t)\varphi\|_0^2 = \|\varphi\|_0^2, \quad \forall t \in \mathbb{R},$$

and $U(t)$ takes $D(B)$ over $D(B)$ and commutes with B.

It then follows that $u(x,t) = U(t)\varphi$ is the unique solution of the mixed problem (1) for $\varphi \in D(B)$, and has the following regularity:

$$u \in C^0([0,T]; D(B)) \cap C^1([0,T]; \mathcal{H}).$$

3. Uniqueness Theorem

The proof of uniqueness is based on the invariance of the system $(1)_1$, without potential $(Q(x) \equiv 0)$ relative to the one-parameter group of dilations in all variables. This property leads to the identity

$$
\begin{aligned}
&2\mathrm{Re}\left\{\left(\ell\overline{u}_t+(x-x^0,\nabla)\overline{u}+\frac{n}{2}\overline{u}\right)\cdot\left[iu_t+(-1)^\ell D_p^\ell(A^{pq}D_q^\ell u)+Q(x)u\right]\right\} \qquad (4)\\
&\quad=\frac{\partial}{\partial t}\left[2\ell t\left(\sum_{p,q}A^{pq}D_q^\ell u\cdot D_p^\ell\overline{u}+Q(x)u\cdot\overline{u}\right)+\mathrm{Im}\,\overline{u}\cdot(x-x^0,\nabla)u\right]\\
&\quad-\frac{\partial}{\partial x_\kappa}\left[-(x_\kappa-x_\kappa^0)\left(\sum_{p,q}A^{pq}D_q^\ell u\cdot D_p^\ell\overline{u}+Q(x)u\cdot\overline{u}+\mathrm{Im}\,\overline{u}_t\cdot u\right)\right]\\
&\quad-(-1)^\ell\sum_{p,q}\sum_{\sigma=1}^{\ell}(-1)^\sigma\frac{\partial}{\partial x_{p_\sigma}}\left[2\mathrm{Re}\left(D_p^{\sigma-1}(x_\kappa-x_\kappa^0)\overline{u}_{x_\kappa}+2\ell tD_p^{\sigma-1}\overline{u}_t\right.\right.\\
&\quad\left.\left.+\frac{n}{2}D_p^{\sigma-1}\overline{u}\right)\cdot D_p^{\ell-\sigma}(A^{pq}D_q^\ell u)\right]-\left[(x_\kappa-x_\kappa^0)Q_{x_\kappa}u\cdot\overline{u}+2\ell Q(x)u\cdot\overline{u}\right].
\end{aligned}
$$

Let $u(x,t)$ be a solution of (1). It is easy to verify that

$$\int_\Omega|u(x,t)|^2dx=\int_\Omega|\varphi(x)|^2dx,\qquad t\geq 0, \tag{5}$$

and

$$
\begin{aligned}
&\int_\Omega\left(\sum_{p,q}A^{pq}D_q^\ell u\cdot D_p^\ell\overline{u}+Qu\cdot\overline{u}\right)dx \qquad (6)\\
&\quad=\int_\Omega\left(\sum_{p,q}A^{p,q}D_p^\ell\varphi\cdot D_p^\ell\overline{\varphi}+Q\varphi\cdot\overline{\varphi}\right)dx,\quad t\geq 0.
\end{aligned}
$$

The integration of (4) over $\Omega\times(0,T)$ gives us

$$
\begin{aligned}
&T\int_\Omega\left[2\ell\left(\sum_{p,q}A^{pq}D_q^\ell u\cdot D_p^\ell\overline{u}+Q(x)u\cdot\overline{u}\right)\right]\Bigg|_{t=T}dx \qquad (7)\\
&\quad+\int_\Omega\mathrm{Im}\,\overline{u}\cdot(x-x^0,\nabla)u\Big|_{t=T}dx-\int_\Omega\mathrm{Im}\,\overline{\varphi}\cdot(x-x^0,\nabla)\varphi\,dx\\
&\quad-\int_0^T\int_\Omega\left[(x_\kappa-x_\kappa^0)Q_{x_\kappa}u\cdot\overline{u}+2\ell Q(x)u\cdot\overline{u}\right]dx\,dt\\
&\quad-\int_0^T\int_{\Gamma_1}(x-x^0,\nu)\sum_{p,q}A^{pq}D_q^\ell\overline{u}\cdot D_p^\ell\overline{u}\,d\Gamma\,dt\\
&=\int_0^T\int_{\Gamma_0}(x-x^0,\nu)\sum_{p,q}A^{pq}D_q^\ell u\cdot D_p^\ell\overline{u}\,d\Gamma\,dt.
\end{aligned}
$$

Denote by C_1 the least positive constant for which

$$\int_\Omega (|u|^2 + |\nabla u|^2)dx \le C_1 \int_\Omega \sum_p |D_p^\ell u|^2 dx, \quad \text{for } u \in [H_0^\ell(\Omega)]^m.$$

Then we obtain the estimate

$$\left| \int_\Omega \operatorname{Im} \overline{u} \cdot (x - x^0, \nabla) u \, dx \right|_{t-0}^{t=T} \le R \frac{C_1}{C_0} \left(\sum_{p,q} A^{pq} D_q^\ell u \cdot D_p^\ell \overline{u} + Q(x) u \cdot \overline{u} \right) dx,$$

where $R = \max_{x \in \overline{\Omega}} |x - x^0|$.

Thus, (7) implies the inequality

$$\begin{aligned} &\left(2\ell T - R\frac{C_1}{C_0}\right) \int_\Omega \left(\sum_{p,q} A^{pq} D_q^\ell u \cdot D_p^\ell \overline{u} + Q u \cdot \overline{u} \right) dx \\ &\quad \le \int_0^T \int_\Omega \left[(x_\kappa - x_\kappa^0) Q_{x_\kappa} u \cdot \overline{u} + 2\ell Q u \cdot \overline{u} \right] dx \\ &\qquad + \int_0^T \int_{\Gamma_0} (x - x^0, \nu) \sum_{p,q} A^{pq} D_q^\ell u \cdot D_q^\ell \overline{u} \, d\Gamma \, dt. \end{aligned} \tag{8}$$

Let us assume that $Q(x)$ satisfies the following assumption:

(i) $(x_\kappa - x_\kappa^0) Q_{x_\kappa} \zeta \cdot \overline{\zeta} + \mu Q \zeta \cdot \overline{\zeta} \le 0$, $\zeta = (\zeta_1, \dots, \zeta^m) \in \mathbb{C}^m$, for some $0 < \mu < 1$,

In view of this assumption, the inequality (8) gives us the estimate:

$$\begin{aligned} &\left(T\mu - R\frac{C_1}{C_0}\right) \int_\Omega \left(\sum_{p,q} A^{pq} D_q^\ell u \cdot D_p^\ell \overline{u} + Q(x) u \cdot \overline{u} \right) dx \\ &\quad \le R \int_0^T \int_{\Gamma_0} \sum_{p,q} A^{pq} D_q^\ell u \cdot D_p^\ell \overline{u} \, d\Gamma \, dt. \end{aligned} \tag{9}$$

We arrive at the following assertion:

Theorem 3.1. *Suppose that the matrices A^{pq} and Q satisfy the assumption made earlier. Let a vector function $u(x,t)$ satisfying the following conditions:*

$$\begin{cases} i\dfrac{\partial u}{\partial t} + Au + Q(x)u = 0, & in\ \Omega\times]0,T[, \\ u, \dfrac{\partial u}{\partial \nu}, \cdots, \dfrac{\partial^{\ell-1} u}{\partial \nu^{\ell-1}} = 0, & on\ \Gamma\times]0,T[, \\ \dfrac{\partial^{\ell} u}{\partial \nu^{\ell}} = 0, & on\ \Gamma_0\times]0,T[. \end{cases}$$

If $T > T_0 = R\frac{C_1}{\mu C_0}$, *then* $u(x,t) \equiv 0$, *for* $(x,t) \in \Omega\times]0,T[$.

Proof. It suffices to observe that

$$\sum_{p,q} A^{pq} D_q^{\ell} u \cdot D_q^{\ell} \overline{u} = \sum_{p,q} A^{pq} \nu_{q_1} \cdots \nu_{q_\ell} \frac{\partial^\ell u}{\partial \nu^\ell} \cdot \nu_{p_1} \ldots \nu_{p_\ell} \frac{\partial^\ell \overline{u}}{\partial \nu^\ell},$$

since $u(x,t)$ vanishes on Γ_0 together with the normal derivatives up to order ℓ.

□

Another consequence of (9) or Theorem 3.1 is the following assertion: for $T > T_0$, the expression

$$\left(\int_0^T \int_{\Gamma_0} \sum_{p,q} A^{pq} D_q^{\ell} u \cdot D_p^{\ell} \overline{u}\, d\Gamma\, dt \right)^{\frac{1}{2}} \tag{10}$$

determines a norm on the set $\varphi(x)$ of the initial data (in (10), $u(x,t)$ is the solution of (1) with initial data $\varphi(x)$).

Denote by $\mathcal{F}$ the Hilbert space resulting from completion of the set of the functions $\varphi(x) \in D(B)$ in norm (10). By virtue of the inequality (9), we have $\mathcal{F} \subset [\mathcal{H}_{\prime}^{\ell}(\Omega)]^{\Updownarrow}$.

The following inequality for the solutions of (1) can be obtained by analogy with the inference of (9) by replacing the operator $(x - x^0, \nabla)$ by $(\nabla\varphi, \nabla)$ in the identity (4). We choose the function $\varphi = \varphi(x)$ as follows: $\varphi \in C^{\ell+1}(\Omega)$ and $\frac{\partial \varphi}{\partial \nu} \geq \sigma > 0$, for $x \in \Gamma$. From the corresponding integral identity, we obtain the inequality

$$\int_0^T \int_{\Gamma_0} \sum_{p,q} A^{pq} D_q^{\ell} u \cdot D_p^{\ell} \overline{u} d\Gamma\, dt \leq \int_\Omega \left(\sum_{p,q} A^{pq} D_q^{\ell} u \cdot D_p^{\ell} \overline{u} + Q(x) u \cdot \overline{u} \right) dx\, dt.$$

Thus, $\mathcal{F} = [\mathcal{H}_{\prime}^{\ell}(\Omega)]^{\Updownarrow}$ and the following scalar product is defined over the space $[H_0^\ell(\Omega)]^m$:

$$(\varphi, \psi)_{\mathcal{F}} = \int_0^T \int_{\Gamma_0} \sum_{p,q} A^{pq} D_p^\ell u \cdot D_q^\ell \overline{v}\, d\Gamma\, dt,$$

where $T > T_0$, u is the solution of (1), and v is the solution of (1) with the initial data $\psi(x)$.

4. Exact Controllability

We consider the homogeneous mixed problem:

$$\begin{cases} iv_t + Av + Q(x)v = 0, & \text{in } \Omega\times]0,T[, \\ \dfrac{\partial^\kappa v}{\partial \nu^\kappa} = 0, \quad \kappa = 0,\dots,\ell-1, & \text{in } \sum = \Gamma\times]0,t[, \\ v(x,0) = \psi(x), & \text{in } \Omega. \end{cases} \tag{11}$$

From this problem, we resolve the following non-homogeneous retrograde problem:

$$\begin{cases} iw_t + Aw + Q(x)w = 0, \quad \text{in } \Omega\times]0,T[, \\ \dfrac{\partial^\kappa w}{\partial \nu^\kappa}\Big|_{\sum=\Gamma\times]0,T[} = 0, \quad \kappa = 0,\dots,\ell-2; \quad \dfrac{\partial^{\ell-1} w}{\partial \nu^{\ell-1}}\Big|_{\sum_1=\Gamma_1\times]0,T[} = 0, \\ \dfrac{\partial^{\ell-1} w}{\partial \nu^{\ell-1}}\Big|_{\sum_0=\Gamma_0\times]0,T[} = \dfrac{\partial^\ell v}{\partial \nu^\ell}\Big|_{\sum_0}, \qquad w(x,T) = 0, \quad \text{in } \Omega, \end{cases} \tag{12}$$

and, with the solution $w = w(x,t)$ of (12), we define the application:

$$\Lambda\psi = iw(0).$$

The following identity can be verified by direct computations:

$$\int_\Omega \psi(x)\cdot \overline{iw(x,0)}dx = \int_0^T \int_{\Gamma_0} \sum_{p,q} A^{pq} D_q^\ell v \cdot \nu_{p_1}\dots\nu_{p_\ell} \frac{\partial^{\ell-1}\overline{w}}{\partial \nu^{\ell-1}} d\Gamma\, dt.$$

Taking $\frac{\partial^{\ell-1}w}{\partial\nu^{\ell-1}} = \frac{\partial^{\ell}v}{\partial\nu^{\ell}}$, on $\sum_0 = \Gamma_0\times]0,T[$, we get:

$$\langle\Lambda\psi,\psi\rangle_{\mathcal{F}',\mathcal{F}} = \|\psi\|^2_{\mathcal{F}} = \|\psi\|^2_{[H_0^\ell(\Omega)]^m}, \quad \text{for } T > T_0.$$

Consequently, the operator Λ is an isomorphism of the space $[H_0^\ell(\Omega)]^m$ onto the dual space $[H^{-\ell}(\Omega)]^m$.

Consider the following control problem: Let $\varphi(x)\in[H^{-\ell}(\Omega)]^m$ be an arbitrary initial data of the problem

$$\begin{cases} iu_t + Au + Q(x)u = 0, \quad \text{in } \Omega\times]0,T[, \\ \left.\dfrac{\partial^\kappa u}{\partial\nu^\kappa}\right|_{\sum} = 0, \quad \kappa = 0,\ldots,\ell-2; \quad \left.\dfrac{\partial^{\ell-1}u}{\partial\nu^{\ell-1}}\right|_{\sum_1} = 0, \\ \left.\dfrac{\partial^{\ell-1}u}{\partial\nu^{\ell-1}}\right|_{\sum_0} = p(x,t) \\ u(x,0) = \varphi(x), \quad \text{in } \Omega. \end{cases} \tag{13}$$

Find $p(x,t)\in L^2(\Gamma_0\times]0,T[)$ such that

$$u(x,T,p) = 0, \qquad \text{in } \Omega \tag{14}$$

for a sufficiently large $T>0$.

The control $p(x,t)$ is constructed as follows: Determine the unique solution $\psi\in[H_0^\ell(\Omega)]^m$ for the equation:

$$\Lambda\psi = i\varphi.$$

Let $v(x,t)$ be the solution of (11) with $\psi = \Lambda^{-1}(i\varphi)$. Then we put $p(x,t) = \frac{\partial^\ell v}{\partial\nu^\ell}$. It is obvious that, by choosing $p(x,t)$, $u(x,t)$, the solution of (13) satisfies (14).

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