A SHORT INTRODUCTION TO HAMILTONIAN PDE'S

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1. Introduction

This is an expository paper on Hamiltonian methods for PDE's. The aim is to stress few basic ideas by means of simple examples, rather than to discuss actual specific problems. The paper starts from an elementary review of the concept of Hamiltonian vector field in Mechanics, and it ends by showing how the Hamiltonian techniques can be used to solve a special Cauchy problem for the Korteweg–de Vries equation. In a rather loose sense, it can also be considered as a very cursory introduction to Sato's theory of KP equations [**2**]. This theme, however, is only touched on in this paper, mainly to provide a perspective of the potentialities of the Hamiltonian techniques.

The basic concept around which the paper is constructed is that of Poisson pencil introduced in Section 2. It is central in the theory of Hamiltonian integrable systems, according to a beautiful result of Gel'fand and Zakharevich [**4**]. In Section 3 this concept is used to define the KdV hierarchy and to show its Hamiltonian properties. Section 4 is a brief detour towards the KP equations, here approached as the system of conservation laws associated with the KdV equation. The paper ends by a simple application to the study of a particular class of solution (the "n–gaps solutions") of the KdV equation. According to the spirit of the paper, the style is informal, and the proofs are omitted to keep the paper into a reasonable size. However, a certain effort has been done to isolate the main ideas from a congeries of disturbing details. The paper is an attempt to discuss these ideas in a concrete and direct way, by means of selected examples.

2. The evolution of the concept of Hamiltonian vector field

Let us start with the classical definition. If q^i and p_i are the coordinates and the momenta of a mechanical system with n degrees of freedom, and H is the Hamiltonian function, the equations of motion are written in the well–known form

$$\dot{q} = H_p \qquad \dot{p} = -H_q. \tag{1}$$

Passing to the differentials, these equations are readily put in the coordinate–free form

$$\dot{p}dq - \dot{q}dp = -dH, \tag{2}$$

sometimes called the "central equation" of Mechanics [**5**, p.233 and p.288]. At this point we can recognize on the left–hand side the 1–form obtained by contracting the symplectic 2–form

$$\omega = dp \wedge dq \tag{3}$$

with the Hamiltonian vector field

$$X = \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p}. \tag{4}$$

In this way we arrive to the third intrinsic definition of Hamiltonian vector field, generally accepted in the framework of the so–called symplectic geometry. A vector field on a symplectic manifold is Hamiltonian if the image of X by ω is an exact 1–form:

$$\omega(X, \cdot) = -dH. \tag{5}$$

The interesting feature of this definition is that it clearly points out that the concept of Hamiltonian vector field is relative to the choice of a symplectic 2–form. Usually the 2–form is assumed to be given. But we can also take the other point of view, and ask how many symplectic forms on the manifold make a given vector field Hamiltonian in the previous sense. Along this paper we shall try to convince the reader that it is this second point of view which is worthwhile of interest when looking to PDE's (since, in this case, we do not have, usually, a symplectic form given at the beginning).

To pass to PDE's, however, it is still required to move one step further, by solving equation (5) with respect to the vector field X. This can be done in several ways. A particular terse way has been suggested by Poisson and developed by Lie. Following them, we introduce the Poisson bracket of the functions H and K,

$$\{H, K\} := X_H(K) = \omega(X_H, X_K), \tag{6}$$

defined as the derivative of the function K along the vector field X_H. By this simple step, we are exploiting the duality between the concepts of state and observable, and we are introducing a description of the dynamics on forms rather than on points of a manifold. By choosing any system of local coordinates x^j, and setting $K = x^j$, equations (6) are readily written in the form

$$\dot{x}^j = \sum_{k=1}^{2n} P^{jk}(x^1, \ldots, x^{2n}) \frac{\partial H}{\partial x^k}, \tag{7}$$

where the functions $P^{jk}(x^1, \ldots, x^{2n})$ are the "fundamental Poisson brackets"

$$P^{jk}(x^1, \ldots, x^{2n}) := \{x^j, x^k\}. \tag{8}$$

Since the Poisson bracket verifies the well–known Jacobi identity, it is easily recognized that the functions $P^{jk}(x^1, \ldots, x^{2n})$ obey the cyclic condition

$$\sum_{l=1}^{2n} \left(P^{jl} \frac{\partial P^{hk}}{\partial x^l} + P^{kl} \frac{\partial P^{jh}}{\partial x^l} + P^{hl} \frac{\partial P^{kj}}{\partial x^l} \right) = 0. \tag{9}$$

It is presently common to refer the skewsymmetric solutions of this condition as to the *Poisson bivectors* of the manifold M.

By the process outlined before we have obtained a new interpretation of the concept of Hamiltonian vector field. A given vector field on a manifold M is

Hamiltonian if it can be "factorized" in a Poisson bivector acting on an exact 1–form:

$$X^j(x^1,\ldots,x^m)=\sum_{l=1}^m P^{jl}(x^1,\ldots,x^m)\frac{\partial H}{\partial x^l}. \tag{10}$$

From this new perspective, it is no longer necessary to assume the Poisson bracket to be invertible. This gives us a lot of additional freedom which will be essential when dealing with PDE's. As a special consequence of this generalization, there presently exist functions K commuting with all the other functions or, what is the same, generating the null vector field:

$$\sum_{l=1}^m P^{jl}(x^1,\ldots,x^m)\frac{\partial K}{\partial x^l}=0. \tag{11}$$

These functions are called *Casimir functions* of the Poisson bivector. They are the first element of the geometry of a "Poisson manifold". The second element of interest are the level surfaces of these functions. They enjoy the peculiar property of being symplectic submanifolds of M. Thus a Poisson manifold can also be viewed as a collection of symplectic leaves, whose symplectic forms join together in a coherent way into a single Poisson bracket defined on the whole manifold. The main purpose of this paper is to display, by selected examples, the interplay between PDE's, Poisson bivectors, Casimir functions, and their symplectic leaves.

3. Poisson pencils

We presently exploit some unexpected consequences of the "relativization" of the concept of Hamiltonian vector field. We use a concrete example. In the space $M=\mathbb{R}^6$, referred to coordinates (a_l,b_l), $l=1,2,3$, we consider the vector field

$$\begin{aligned}\dot a_l &= a_l(b_{l+1}-b_l)\\ \dot b_l &= a_l-a_{l-1}\end{aligned} \tag{12}$$

with the periodicity condition

$$a_{3+l}=a_l, \qquad b_{3+l}=b_l. \tag{13}$$

These equations define the so–called three–particle Toda lattice. (The general n–particle system is recovered by replacing every where 3 by n). Our aim is to show that this vector field admits several Hamiltonian formulations: actually an infinite number of them.

As a first step, we remark that equations (12) can be written in the form

$$\begin{aligned} \dot{a}_l &= a_l \left(\frac{\partial K}{\partial b_{l+1}} - \frac{\partial K}{\partial b_l} \right) \\ \dot{b}_l &= a_l \frac{\partial K}{\partial a_l} - a_{l-1} \frac{\partial K}{\partial a_{l-1}} \end{aligned} \tag{14}$$

where

$$K = \frac{1}{2}(b_1^2 + b_2^2 + b_3^2) + a_1 + a_2 + a_3. \tag{15}$$

This remark shows that the Toda equations are Hamiltonian with respect to the linear Poisson bracket

$$\begin{aligned} \{F, K\}_1 &= \sum_{l+1}^{3} \left(\dot{a}_l \frac{\partial F}{\partial a_l} + \dot{b}_l \frac{\partial F}{\partial b_l} \right) \\ &= \sum_{l=1}^{3} \left[a_l \left(\frac{\partial F}{\partial a_l} \frac{\partial K}{\partial b_{l+1}} - \frac{\partial F}{\partial b_{l+1}} \frac{\partial K}{\partial a_l} \right) + a_l \left(\frac{\partial F}{\partial b_l} \frac{\partial K}{\partial a_l} - \frac{\partial F}{\partial a_l} \frac{\partial K}{\partial b_l} \right) \right]. \end{aligned} \tag{16}$$

In the same vein, one can remark that the Toda equations can also be written in the form

$$\begin{aligned} \dot{a}_l &= a_l \left(a_{l+1} \frac{\partial H}{\partial a_{l+1}} - a_{l-1} \frac{\partial H}{\partial a_{l-1}} + b_{l+1} \frac{\partial H}{\partial b_{l+1}} - b_l \frac{\partial H}{\partial b_l} \right) \\ \dot{b}_l &= b_l \left(a_l \frac{\partial H}{\partial a_l} - a_{l-1} \frac{\partial H}{\partial a_{l-1}} \right) + a_l \frac{\partial H}{\partial b_{l+1}} - a_{l-1} \frac{\partial H}{\partial b_{l-1}}, \end{aligned} \tag{17}$$

where

$$H = b_1 + b_2 + b_3. \tag{18}$$

This remark points out a second Hamiltonian factorization of the Toda equations, relative to the quadratic Poisson bracket

$$
\begin{aligned}
\{F,G\}_2 \;&=\; \sum_{l=1}^{3} a_l a_{l+1}\left(\frac{\partial F}{\partial a_l}\frac{\partial G}{\partial a_{l+1}} - \frac{\partial F}{\partial a_{l+1}}\frac{\partial G}{\partial a_l}\right)\\
&+\; \sum_{l=1}^{3} a_l b_{l+1}\left(\frac{\partial F}{\partial a_l}\frac{\partial G}{\partial b_{l+1}} - \frac{\partial F}{\partial b_{l+1}}\frac{\partial G}{\partial a_l}\right)\\
&+\; \sum_{l=1}^{3} a_l b_l\left(\frac{\partial F}{\partial a_l}\frac{\partial G}{\partial b_l} - \frac{\partial F}{\partial b_l}\frac{\partial G}{\partial a_l}\right)\\
&+\; \sum_{l=1}^{3} a_l\left(\frac{\partial F}{\partial b_l}\frac{\partial G}{\partial b_{l+1}} - \frac{\partial F}{\partial b_{l+1}}\frac{\partial G}{\partial b_l}\right).
\end{aligned}
\tag{19}
$$

So, the Toda equations are an example of "bihamiltonian equations". The possibility of giving the same equation different Hamiltonian factorizations is a new feature previously unnoticed in Classical Mechanics. One of the reason is that to find these factorizations from the mere knowledge of the equations is a rather nontrivial problem, as shown by the example. However, the existence of these multiple Hamiltonian factorizations is a powerful clue to solve the equations. We shall presently try to prove this statement, following a procedure suggested by Gel'fand and Zakharevich.

To put the procedure into work, we still have to remark that the second Hamiltonian H is a Casimir function of the first Poisson bracket (16). This entails that the Toda equations can be written in the "mixed form"

$$
\begin{aligned}
\dot a_l &= a_l\left(a_{l+1}\frac{\partial H}{\partial a_{l+1}} - a_{l-1}\frac{\partial H}{\partial a_{l-1}} + b_{l+1}\frac{\partial H}{\partial b_{l+1}} - b_l\frac{\partial H}{\partial b_l}\right) - \lambda a_l\left(\frac{\partial H}{\partial b_{l+1}} - \frac{\partial H}{\partial b_l}\right)\\
\dot b_l &= b_l\left(a_l\frac{\partial H}{\partial a_l} - a_{l-1}\frac{\partial H}{\partial a_{l-1}}\right) + a_l\frac{\partial H}{\partial b_{l+1}} - a_{l-1}\frac{\partial H}{\partial b_{l-1}} - \lambda\left(a_l\frac{\partial H}{\partial a_l} - a_{l-1}\frac{\partial H}{\partial a_{l-1}}\right)
\end{aligned}
\tag{20}
$$

containing an arbitrary real parameter λ. This suggests to consider the one–parameter family of brackets

$$
\{F,G\}_\lambda = \{F,G\}_2 - \lambda\{F,G\}_1. \tag{21}
$$

The surprising feature is that this bracket verifies the Jacobi identity for any value of the parameter λ. When this happens, we say that the brackets $\{F, G\}_1$ and $\{F, G\}_2$ are compatible.

Definition 1. *A Poisson pencil is a one–parameter family of compatible Poisson brackets.*

We are interested in the study of the Casimir functions of the Poisson pencil. In our example they are:

$$\begin{aligned} C &= a_1 a_2 a_3 \\ D(\lambda) &= (b_1-\lambda)(b_2-\lambda)(b_3-\lambda) - a_1(b_3-\lambda) - a_2(b_1-\lambda) - a_3(b_2-\lambda). \end{aligned} \tag{22}$$

The first one does not depend on λ. This means that C is a common Casimir for all the brackets of the pencil. The second depends polynomially on λ. Its expansion in powers of λ,

$$D(\lambda) = -\lambda^3 + D_0\lambda^2 - D_1\lambda + D_2, \tag{23}$$

gives the three functions

$$\begin{aligned} D_0 &= b_1 + b_2 + b_3 \\ D_1 &= b_1b_2 + b_2b_3 + b_3b_1 - a_1 - a_2 - a_3 \\ D_2 &= b_1b_2b_3 - a_1b_3 - a_2b_1 - a_3b_2. \end{aligned} \tag{24}$$

It is easily checked (and it could be easily proved) that the four functions (C, D_0, D_1, D_2) are in involution with respect to all the brackets of the pencil (21). Furthermore, the Hamiltonians H and K of the Toda system are given by

$$\begin{aligned} H &= D_0 \\ K &= \tfrac{1}{2}D_0^2 - D_1. \end{aligned} \tag{25}$$

This leads to the following simple geometric construction.

In the six–dimensional manifold M, consider a four–dimensional level surface S of the pair of functions C and D_0. It is a symplectic leaf of the brackets $\{F, G\}_1$. Restrict the remaining two functions D_1 and D_2 to S. Since D_1

and D_2 are in involution with respect to $\{F,G\}_1$, their restrictions will be in involution with respect to the symplectic 2–form defined on S. Therefore, the functions D_1 and D_2 define a Lagrangean foliation on S. By condition (25), the Toda vector field is tangent to this foliation, and hence it fulfills the conditions of the Arnold–Liouville theorem on the complete integrability of a Hamiltonian vector field on a symplectic manifold.

The main conclusion of this section is, therefore, that Poisson pencils are a useful tool to construct integrable Hamiltonian systems.

4. The KdV equation

In this section we show, by a concrete example, how the technique of Poisson pencils can be extended to partial differential equations. We consider the KdV equation

$$u_t = \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x \tag{26}$$

with periodic boundary conditions

$$u(0,t) = u(2\pi,t). \tag{27}$$

We note that this equation admits the following two factorizations:

$$\begin{aligned} u_t &= [-2\partial_x]\left[\tfrac{1}{8}(u_{xx}-3u^2)\right] \\ u_t &= \left[-\tfrac{1}{2}\partial_{xxx}+2u\partial_x+u_x\right]\left[-\tfrac{1}{2}u\right], \end{aligned} \tag{28}$$

where the "1–forms" $\alpha = -\frac{1}{2}u$ and $\beta = \frac{1}{8}(u_{xx}-3u^2)$ are the (Lagrangean) derivatives of the functionals

$$\begin{aligned} H(u) &= -\tfrac{1}{4}\int_0^{2\pi} u^2\,dx \\ K(u) &= -\tfrac{1}{16}\int_0^{2\pi} ({u_x}^2+2u^3)\,dx. \end{aligned} \tag{29}$$

To conclude that equations (28) are two Hamiltonian factorizations of the KdV equation, we have to check that the linear operators

$$\begin{aligned} Q_u &= -2\partial_x \\ P_u &= -\tfrac{1}{2}\partial_{xxx}+2u\partial_x+u_x \end{aligned} \tag{30}$$

are skewsymmetric and verify the Jacobi identity. In the present infinite–dimensional context the latter condition must be interpreted as follows: if

$$P'_u(\beta, P_u\gamma) = \frac{d}{dt}_{|t=0} P_{u+tP_u\gamma}(\beta) \tag{31}$$

is the (Gateaux) derivative of the bivector P_u along the vector field $u_t = P_u\gamma$, and if the evaluation form is defined by

$$\langle \alpha, u_t \rangle = \int_0^{2\pi} \alpha(x) u_t(x)\, dx, \tag{32}$$

then the cyclic sum

$$\langle \alpha, P'_u(\beta, P_u\gamma)\rangle + \langle \gamma, P'_u(\alpha, P_u\beta)\rangle + \langle \beta, P'_u(\gamma, P_u\alpha)\rangle \tag{33}$$

must vanish for any choice of the 1–forms α, β, γ. This condition is easily checked by integrating by parts and by using the periodic boundary conditions (27). Furthermore, it is not difficult to check that the Poisson bivectors P_u and Q_u are compatible. So the KdV equation shares the same Hamiltonian features previously displayed for the periodic Toda lattice.

We can, consequently, proceed as in the previous section by investigating the Casimir functions of the Poisson pencil

$$P(z) = P_u - z^2 Q_u. \tag{34}$$

It is explicitly defined by

$$u_t = -\frac{1}{2}\alpha_{xxx} + 2(u + z^2)\alpha_x + u_x\alpha. \tag{35}$$

The problem is to find a functional

$$H(z) = 2\int_0^{2\pi} h(u, u_x, \ldots; z)\, dx, \tag{36}$$

depending on the parameter z of the pencil, whose differential annihilates the right–hand side of equation (35). It can be shown that this condition is equivalent to ask that the corresponding Hamiltonian density $h(u, u_x, \ldots; z)$ verifies the Riccati equation

$$h_x + h^2 = u + z^2. \tag{37}$$

This equation can be formally solved by Laurent expansion,

$$h(z) = z + \sum_{l\geq 1} \frac{h_l}{z^l}. \tag{38}$$

One finds

$$\begin{aligned} h_1 &= \tfrac{1}{2}u \\ h_2 &= -\tfrac{1}{4}u_x \\ h_3 &= \tfrac{1}{8}(u_{xx} - u^2) \\ h_4 &= -\tfrac{1}{16}(u_{xxx} - 4uu_x) \\ h_5 &= \tfrac{1}{32}(u_{xxxx} - 6uu_{xx} - 5u_x^2 + 2u^3) \end{aligned} \tag{39}$$

and so on. By integrating these functions on the circle we get the functionals

$$\begin{aligned} H_1 &= \int_0^{2\pi} u\,dx \\ H_2 &= 0 \\ H_3 &= -\tfrac{1}{4}\int_0^{2\pi} u^2\,dx \\ H_4 &= 0 \\ H_5 &= -\tfrac{1}{16}\int_0^{2\pi} ({u_x}^2 + 2u^3)\,dx. \end{aligned} \tag{40}$$

They are the Hamiltonian functions of the KdV theory. The KdV hierarchy is defined by

$$\frac{\partial u}{\partial t_j} = -\frac{1}{2}\left(\frac{\delta H_j}{\delta u}\right)_{xxx} + 2u\left(\frac{\delta H_j}{\delta u}\right)_x + u_x \frac{\delta H_j}{\delta u}. \tag{41}$$

The first equations are

$$\begin{aligned} \frac{\partial u}{\partial t_1} &= u_x \\ \frac{\partial u}{\partial t_3} &= \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x \\ \frac{\partial u}{\partial t_5} &= \frac{1}{16}u_{xxxxx} - \frac{5}{8}uu_{xxx} - \frac{5}{4}u_x u_{xx} + \frac{15}{8}u^2 u_x. \end{aligned} \tag{42}$$

They are examples of commuting bihamiltonian vector fields, on an infinite–dimensional phase–space, described by PDE's and defined by the Casimir function of a suitable Poisson pencil.

In the rest of the paper, our aim is to use the Hamiltonian techniques to discover some important properties of these equations.

5. The KP equations

In this section we discuss the conservation laws associated with the KdV equation. The starting remark is that the coefficients of the Casimir functions commute in pairs with respect to all the brackets of the Poisson pencil. So, each of them is a constant of motion for all the flows of the hierarchy:

$$\frac{dH_k}{dt_j} = 0. \tag{43}$$

In terms of the associated Hamiltonian density $h(z)$, these equations assume the form of local conservation laws

$$\frac{\partial h(z)}{\partial t_j} = \partial_x H^{(j)}(z). \tag{44}$$

The first currents $H^{(j)}$ can be easily computed by direct inspection by using the Riccati equation (37). They are

$$\begin{aligned} H^{(1)} &= h(z) \\ H^{(2)} &= h_x + h^2 - 2h_1 \\ H^{(3)} &= h_{xx} + 3hh_x + h^3 - 3h_1 h - 3(h_2 + h_{1x}). \end{aligned} \tag{45}$$

The reader is referred to **[1]** for the general formula of the currents $H^{(j)}$. The point to be stressed here is that the expression of the currents $H^{(j)}$ is a direct outcome of the form of the Poisson pencil (35). By inserting the expressions

(45) into the local conservation laws (44) we get the equations

$$\begin{aligned}
\frac{\partial h}{\partial t_1} &= (h)_x \\
\frac{\partial h}{\partial t_2} &= (h_x + h^2 - 2h_1)_x \qquad (46) \\
\frac{\partial h}{\partial t_3} &= (h_{xx} + 3hh_x + h^3 - 3h_1 h - 3h_2 - 3h_{1x})_x.
\end{aligned}$$

Similar equations are obtained for the higher–order times. If h is a solution of the Riccati equation (37), written in the form

$$H^{(2)} = h_x + h^2 - 2h_1 = z^2, \qquad (47)$$

these equations are the local conservation laws associated with the KdV equation for the function

$$u = 2h_1. \qquad (48)$$

If h is a solution of the second order equation

$$H^{(3)} = h_{xx} + 3hh_x + h^3 - 3h_1 h - 3(h_2 + h_{1x}) = z^3, \qquad (49)$$

equations (46) become the local conservation laws for the Boussinesq hierarchy **[6]**, in the pair of functions

$$u = 3h_1 \qquad v = 3(h_2 + h_{1x}). \qquad (50)$$

If we do not impose any constraint on h, we can regard equations (46) as an infinite system of PDE's in an infinite number of field functions, namely the coefficients of the Laurent series

$$h(z) = z + \sum_{l \geq 1} \frac{h_l}{z^l}. \qquad (51)$$

These equations are a possible form of the celebrated KP equations **[6]**. To recognize this fact some additional work is required, since the KP equations are usually defined by an algebraic procedure (the Lax representation in the algebra of pseudo–differential operators) which tends to hide the Hamiltonian origin of these equations. However, it would be inappropriate to try to explain here, in more details, the intricacies of all these connections. Our claim is that the Hamiltonian methods, suitably pushed far, are powerful enough to cover the main parts of the modern theory of those integrable PDE's called soliton equations [**1**].

6. The restricted KdV flows

We now want to show that the Hamiltonian techniques can be used to solve a special class of Cauchy problems for the KdV equation

$$u_t = \frac{1}{4} u_{xxx} - \frac{3}{2} u u_x \tag{52}$$

on the circle. We assume that the initial condition

$$u(x, 0) = f(x) \tag{53}$$

verifies the ordinary differential equation

$$\frac{1}{16} f_{xxxxx} - \frac{5}{8} f f_{xxx} - \frac{5}{4} f_x f_{xx} + \frac{15}{8} f^2 f_x = 0. \tag{54}$$

This means that f is an equilibrium point of the third equation of the KdV hierarchy (relative to the time t_5).

Although strange from an analytical point of view, this restriction is natural from the Hamiltonian point of view. Since the flow relative to the times t_3 and t_5 commute, one can show that the solution $u(x, t)$ verifies itself the constraint (54). So, our Cauchy problem amounts to solve the KdV equation on a finite–dimensional invariant submanifold. On this submanifold the KdV equation becomes a system of ordinary differential equations.

To construct this system we use the first five coefficients $(h_1, h_2, h_3, h_4, h_5)$ of the Hamiltonian density of the Casimir function of the Poisson pencil (35). They are related to the function u and its first four space derivatives $\{\partial_x^i u\}_{i=1,\dots,4}$ by equation (39). We derive these equations with respect to the times t_1 and t_3, and we use the KdV equation to convert the time derivatives of the function

u into space derivatives. Then we use the constraint (54) and its differential consequences to eliminate the derivatives $(\partial_x^5 u, \partial_x^6 u, \partial_x^7)$ which appear at the previous step. Finally, we eliminate the variables $(u, u_x, u_{xx}, u_{xxx}, u_{xxxx})$ in favour of the variables $(h_1, h_2, h_3, h_4, h_5)$ by using backwards the relations (39). The outcome are the equations

$$\begin{aligned}
\frac{\partial h_1}{\partial t_1} &= -2h_2 \\
\frac{\partial h_2}{\partial t_1} &= -(2h_3 + h_1^2) \\
\frac{\partial h_3}{\partial t_1} &= -(2h_4 + 2h_1 h_2) \\
\frac{\partial h_4}{\partial t_1} &= -(2h_5 + 2h_1 h_3 + h_2^2) \\
\frac{\partial h_5}{\partial t_1} &= -(4h_1 h_4 + 4h_2 h_3 - 2h_1^2 h_2)
\end{aligned} \tag{55}$$

and

$$\begin{aligned}
\frac{\partial h_1}{\partial t_3} &= -2h_4 + 2h_1 h_2 \\
\frac{\partial h_2}{\partial t_3} &= -2h_5 + h_2^2 + h_1^3 \\
\frac{\partial h_3}{\partial t_3} &= -2h_1 h_4 - 2h_2 h_3 + 4h_1^2 h_2 \\
\frac{\partial h_4}{\partial t_3} &= -2h_3^2 + h_1^2 h_3 + h_1^4 + 2h_1 h_2^2 - 2h_2 h_4 \\
\frac{\partial h_5}{\partial t_3} &= -4h_3 h_4 + 2h_1^3 h_2 + 2h_1^2 h_4.
\end{aligned} \tag{56}$$

They represent the restriction of the KdV hierarchy on the five–dimensional invariant submanifold defined by equation (54).

Not surprisingly, these equations are bihamiltonian. They can be written either in the form

$$
\begin{aligned}
\dot h_1 &= 2\frac{\partial H}{\partial h_2} + 2h_1\frac{\partial H}{\partial h_4} + 2h_2\frac{\partial H}{\partial h_5} \\
\dot h_2 &= -2\frac{\partial H}{\partial h_1} - 4h_1\frac{\partial H}{\partial h_3} - 2h_2\frac{\partial H}{\partial h_4} - (2h_3 + h_1^2)\frac{\partial H}{\partial h_5} \\
\dot h_3 &= 4h_1\frac{\partial H}{\partial h_2} + (2h_3 + 2h_1^2)\frac{\partial H}{\partial h_4} + (2h_4 + 2h_1h_2)\frac{\partial H}{\partial h_5} \\
\dot h_4 &= -2h_1\frac{\partial H}{\partial h_1} + 2h_2\frac{\partial H}{\partial h_2} - 2(h_3 + h_1^2)\frac{\partial H}{\partial h_3} + (h_2^2 + 2h_1^3)\frac{\partial H}{\partial h_5} \\
\dot h_5 &= -2h_2\frac{\partial H}{\partial h_1} + (2h_3 + h_1^2)\frac{\partial H}{\partial h_2} - 2(h_1h_2 + h_4)\frac{\partial H}{\partial h_3} \\
&\quad -(2h_5 - 6h_1h_3 + h_2^2 + 2h_1^3)\frac{\partial H}{\partial h_4}
\end{aligned}
\tag{57}
$$

or in the form

$$
\begin{aligned}
\dot h_1 &= 2\frac{\partial K}{\partial h_4} \\
\dot h_2 &= -2\frac{\partial K}{\partial h_3} - 4h_1\frac{\partial K}{\partial h_5} \\
\dot h_3 &= 2\frac{\partial K}{\partial h_2} + 4h_1\frac{\partial K}{\partial h_4} \\
\dot h_4 &= -2\frac{\partial K}{\partial h_1} - 4h_1\frac{\partial K}{\partial h_3} - (4h_3 + 2h_1^2)\frac{\partial K}{\partial h_5} \\
\dot h_5 &= 4h_1\frac{\partial K}{\partial h_2} + (4h_3 + 2h_1^2)\frac{\partial K}{\partial h_4}
\end{aligned}
\tag{58}
$$

by a suitable choice of the Hamiltonian functions H and K. For instance, the Hamiltonians of the second equation (56) are

$$
\begin{aligned}
H &= h_2h_4 - \tfrac{1}{2}h_1h_2^2 - h_1h_5 - \tfrac{1}{2}h_3^2 - \tfrac{1}{2}h_1^4 + \tfrac{3}{2}h_1^2h_3 \\
K &= \tfrac{1}{2}h_2^2h_3 - h_3h_5 + \tfrac{1}{2}h_1^5 + h_1h_3^2 - h_1h_2h_4 - \tfrac{3}{2}h_1^2h_3 + h_1^2h_5 + \tfrac{1}{2}h_4^2.
\end{aligned}
\tag{59}
$$

This bihamiltonian structure can be used to solve the equations (55) and (56) by separation of variables. The main steps are the following. First, one considers a symplectic leaf S of the bivector (58). It is a four–dimensional level surface of the Casimir function

$$L = h_5 - 2h_1h_3 + h_1^3. \tag{60}$$

It inherits a Poisson pencil from the ambient space. This Poisson pencil defines a system of *canonical coordinates* on S, namely four coordinates $(\lambda_1, \lambda_2, \mu_1, \mu_2)$ which verify the commutation relations

$$\{\lambda_i, \lambda_j\}_1 = 0 \qquad \{\lambda_i, \mu_j\}_1 = \delta_{ij} \qquad \{\mu_i, \mu_j\}_1 = 0 \tag{61}$$

and

$$\{\lambda_i, \lambda_j\}_2 = 0 \qquad \{\lambda_i, \mu_j\}_2 = \lambda_i\delta_{ij} \qquad \{\mu_i, \mu_j\}_2 = 0 \tag{62}$$

with respect to the Poisson brackets induced by (58) and (57) respectively. It can be checked that the coordinates λ_i are the zeroes of the polynomial

$$\lambda^2 = h_1\lambda + (h_3 - h_1^2) \tag{63}$$

while the coordinates μ_i are the values of the polynomial

$$\mu = h_2\lambda + h_4 - 2h_1h_2 \tag{64}$$

at $\lambda = \lambda_i$. Finally, one writes the Hamiltonians H and K in these coordinates and finds that they are in separable form according to the Stäckel theorem. This is not a surprising result. Indeed there is a close connection between the Hamiltonians H and K and the Poisson pencil defined by the bivectors (57) and (58). The Casimir function of the Poisson pencil is indeed

$$H(\lambda) = H + \lambda K + \lambda^2 L. \tag{65}$$

Thus the moral is that a Poisson pencil separates its Casimir functions.

Coming back to the initial Cauchy problem, here is the recipe for its solution:

1. plug the initial condition f into the relations (39) to compute the variables h_j, $j = 1, \ldots, 5$, as functions of x;

2. change x in t_1: the previous functions $h_j(t_1)$ become a solution of the first equation (55);

3. by using the separation of variables, solve the Cauchy problem for the second equation (56) with initial condition coinciding with the previous solution $h_j(t_1)$ of equation (55);

4. change backwards t_1 in x in the solution $h_j(t_1, t_3)$ found at the previous step: the function

$$u(x, t_3) = 2h_1(x, t_3) \tag{66}$$

is the solution of the initial Cauchy problem for the KdV equation.

This method is usually described in the language of algebraic geometry **[3]**, by using the Lax formalism. In that approach the coordinates λ_i and μ_i must be guessed. Our point is that the Hamiltonian approach gives a more systematic procedure for the construction of these coordinates, and allows to plug the new techniques into well established schemes of Classical Mechanics.

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