A FREE–BOUNDARY VALUE PROBLEM FOR THE NAVIER–STOKES EQUATIONS OF COMPRESSIBLE SHEAR FLOWS

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1. Introduction

We consider the Navier – Stokes equations of a viscous compressible fluid [1] under the assumption that, given a Cartesian coordinate system ξ, η, and ζ, solutions depend on time and the vertical coordinate ξ only. Such solutions satisfy the reduced system

$$\rho D_t u = -p_\xi + \nu u_{\xi\xi}, \quad \rho D_t \mathbf{v} = \mu \mathbf{v}_{\xi\xi}, \quad D_t\rho + \rho u_\xi = 0, \quad p = b\rho\theta, \tag{1}$$

$$\rho D_t e = \kappa\theta_{\xi\xi} - p u_\xi + \nu|u_\xi|^2 + \mu|\mathbf{v}_\xi|^2, \quad e = d\theta, \quad D_t = \frac{\partial}{\partial t} + u\frac{\partial}{\partial \xi}.$$

Here u is the projection of the velocity vector on the ξ-axis, $\mathbf{v}$ is the two-dimensional vector of the horizontal velocity with the components v_1 and v_2 along the $\eta-$ and $\zeta-$ axes, ρ is the density, p is the pressure, θ is the temperature, and e is the internal energy. The set of the positive constants (ν, μ, κ, b, d) defines a 5-dimensional vector $\mathbf{f}$ which corresponds to a fluid.

In the class of shear flows that are governed by system (1), we study a joint motion in the layer $|\xi| < 1$ of two fluids defined by vectors $\mathbf{f}^+$ and $\mathbf{f}^-$ respectively.

To formulate a corresponding free-boundary problem, we incorporate an interface function $\Gamma(t)$ such that equations (1) should be satisfied in the domain $\{t > 0, \Gamma(t) < \xi < 1\}$, with $\mathbf{f} = \mathbf{f}^+$, and in the domain $\{t > 0, -1 < \xi < \Gamma(t)\}$,

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with $\mathbf{f} = \mathbf{f}^-$. To control the interface motion, we put at $\xi = \Gamma(t)$ the no-jump conditions for velocity vector, energy, heat flux, and tensions:

$$[u] = [e] = [-p + \nu u_\xi] = [\kappa\theta_\xi] = 0, \quad [\mathbf{v}] = [\mu \mathbf{v}_\xi] = 0, \quad \Gamma^{'}(t) = u(\Gamma(t), t); \tag{2}$$

here the brackets are used to denote a jump, for example $[d\theta] = d^+\theta(\Gamma(t)+, t) - d^-\theta(\Gamma(t)-, t)$ and etc. The last condition in (2) implies that the interface does not propagate through the medium.

We formulate boundary conditions at $|\xi| = 1$ as follows

$$u = \theta_\xi = 0, \quad \mathbf{v} - \frac{\xi + 1}{2}\mathbf{a} = 0, \tag{3}$$

where $\mathbf{a} = (a_1, a_2)$ is a two-dimensional vector depending on time with the components a_1 and a_2 along the $\eta-$ and $\zeta-$ axes. The boundary conditions correspond to a flow between two parallel horizontal solid plates, with the upper one, $\xi = 1$, moving irrotationally at a constant distance, equal 2, from the lower plate $\xi = -1$, being fixed. It is assumed that the layer $|\xi| < 1$ of the liquids is heat insulated and the liquids stick to the bounding plates

.

Given functions $u_0(\xi), \mathbf{v}_0(\xi), \rho_0(\xi), \theta_0(\xi)$, and a constant $\Gamma_0, |\Gamma_0| < 1$, we set the initial conditions

$$(u, \mathbf{v}, \rho, \theta, \Gamma)|_{t=0} = (u_0, \mathbf{v}_0, \rho_0, \theta_0, \Gamma_0). \tag{4}$$

Let us normalize the initial data by assuming for simplicity that

$$\int_{-1}^{\Gamma_0} \rho_0 d\xi = \int_{\Gamma_0}^{1} \rho_0 d\xi = 1. \tag{5}$$

The flow under consideration can also be treated in the Lagrangian coordinates. By defining $x = L(\xi, t)$,

$$L(\xi, t) = \int_{\Gamma(t)}^{\xi} \rho(y, t) dy, \tag{6}$$

system (1) in the coordinates (x, t) takes the form

$$u_t = \sigma_x, \quad \mathbf{v}_t = \tau_x, \quad e_t = q_x + \sigma u_x + \mu\rho|\mathbf{v}_x|^2, \quad \rho_t + \rho^2 u_x = 0, \tag{7}$$

$$\sigma = \nu\rho u_x - p, \quad \tau = \mu\rho\mathbf{v}_x, \quad q = \kappa\rho\theta_x, \quad e = d\theta, \quad p = b\rho\theta.$$

The interface becomes fixed by the equation $x = 0$ with the following no-jump conditions on it

$$[u] = [e] = [\sigma] = [q] = 0, \quad [\mathbf{v}] = [\tau] = 0. \tag{8}$$

It follows from (1) and (5) that $L(\pm 1, t) = \pm 1$ for any t, so equations (7) are defined for $x \in (0,1) \equiv \Omega_+$ and $t > 0$, with $\mathbf{f} = \mathbf{f}^+$, and for $x \in (-1,0) \equiv \Omega_-$ and $t > 0$, with $\mathbf{f} = \mathbf{f}^-$.

The boundary and initial conditions remain the same in the new coordinates, with the substitution x for ξ in (3) and (4).

We look for solutions of problem (7),(8),(3), and (4) (A-problem) in the domain $Q = Q_+ \cup Q_-$ where $Q_\pm = \Omega_\pm \times I$ and I is a time interval $0 < t < T$. To give precise statements of our results we require that the initial and boundary data satisfy the smoothness conditions

$$\|u_0, v_{i0}, \theta_0\|_{C^{2,\alpha}(\Omega_\pm)} < \infty, \quad \|\rho_0\|_{C^{1,\alpha}(\Omega_\pm)} < \infty, \quad \|a_i(t)\|_{C^2(I)} < \infty, \tag{9}$$

for some $\alpha \in (0,1)$. Here we entered into the following agreement. Given functions $u_1, u_2, \ldots$ in the same function space equipped with some norm $\|\cdot\|$, the notation $\|u_1, u_2, \ldots\|^2$ stands for the sum $\|u_1\|^2 + \|u_2\|^2 + \ldots$.

Our goal is to prove (with notations from [2])

Theorem 1. *Suppose the initial and boundary data satisfy conditions (8), (9), and $\rho_0 > 0, \theta_0 > 0$, and let the compatibility conditions be satisfied at $|x| = 1$:*

$$u_0(x) = 0, \quad \mathbf{v}_0(x) = \frac{x+1}{2}\mathbf{a}(0), \quad \sigma_{0x}(x) = 0, \quad \tau_{0x}(x) = \frac{x+1}{2}\mathbf{a}'(0),$$

and at $x = 0$:

$$[\sigma_{0x}] = [q_{0x} + \sigma_0 u_{0x} + \mu\rho_0|\mathbf{v}_{0x}|^2] = 0, \quad [\tau_{\mathbf{0x}}] = 0.$$

Then there exists a unique solution to the A-problem in the class

$$u, v_i, \theta \in C_{1+\frac{\alpha}{2}, 2+\alpha}(Q_\pm); \quad \rho, \rho_t, \rho_x \in C_{\frac{\alpha}{2},\alpha}(Q_\pm), \quad \rho > 0, \quad \theta > 0.$$

To prove the theorem, we define for $\varepsilon > 0$ a perturbation to the A-problem by replacing conditions (8) at $x = 0$ by

$$[u] = [e] = [\varepsilon u_t - \sigma] = [\varepsilon e_t - q] = 0, \quad [\mathbf{v}] = [\varepsilon \mathbf{v}_t - \tau] = 0. \qquad (10)$$

Problem (7),(10),(3), and (4) will be refered to as the A_ε-problem.

2. Estimates for solutions of the perturbated problem

In this section we obtain global estimates for solutions of the A_ε-problem uniformly in ε. Introduce the following notations for norms

$$\|u\|_p = \|u\|_{L^p(\Omega)}, \|u\|^2 = (u,u)_\Omega, \|u\|^2_{Q_t} = \int_0^t \|u(s)\|^2_\Omega ds, \Omega = \Omega_+ \cup \Omega_-, Q_t = \Omega \times (0,t),$$

where $(\cdot,\cdot)_\Omega$ is the scalar product in $L^2(\Omega)$. In what follows we denote by c different positive constants dependent on T and independent of ε.

The functions $U(t) = u(0,t), \mathbf{V}(t) = \mathbf{v}(0,t)$, and $E(t) = e(0,t)$ are well defined due to (10). To start on consecutive estimations, we observe that equation $(7)_4$, conditions (3),(5), and (10) imply that $\int_\Omega \rho^{-1}(x,t)dx = \int_\Omega \rho_0^{-1}(x)dx = 2$ for any $t > 0$.

Lemma 1. *The following bound is valid for any $t \in I$:*

$$\|v_i\|_\infty \le \max\{\|a_i\|_{L^\infty(I)}, \|v_{0i}\|_\infty\} \equiv c_0, \quad i \in \{1,2\}.$$

Proof. Given a non-negative convex function $F : R \to R$, we infer from equation $(7)_2$ and conditions (10) that

$$\frac{d}{dt}\|F(v_i)\|_1 + \varepsilon \frac{d}{dt} F(V_i) + \|\mu\rho v_{ix}^2 F''(v_i)\|_1 = \mu\rho \frac{\partial}{\partial x} F(v_i)|_{x=-1}^{x=1}.$$

The assertion of the lemma now follows if, for $\delta > 0$, we choose F such that $F(s) = 0$ for $|s| \le c_0 + \delta$ and $F(s) > 0$ for $|s| \ge c_0 + 2\delta$.

Lemma 2. *The following identities are valid in Q:*

$$\frac{d}{dt}\left(\|\frac{u^2}{2}+\frac{|\mathbf{v}|^2}{2}+e\|_1+\varepsilon(\frac{U^2}{2}+\frac{|\mathbf{V}|^2}{2}+E)\right)=\mathbf{ag}, \tag{11}$$

$$\frac{d}{dt}\left\{\|\frac{u^2}{2}+\frac{|\mathbf{v}|^2}{2}+\Psi(e)+\frac{b}{d}\Psi(\frac{1}{\rho})\|_1+\varepsilon\left(\frac{U^2}{2}+\frac{|\mathbf{V}|^2}{2}+\Psi(E)\right)\right\}+$$
$$\|\frac{\kappa\rho\theta_x^2}{d\theta^2}+\frac{\nu\rho u_x^2}{d\theta}+\frac{\mu\rho|\mathbf{v}_x|^2}{d\theta}\|_1=\mathbf{ag}-[\frac{b}{d}]U, \tag{12}$$

$$\frac{\rho_0(x)e^{-\varphi(x,t)}}{\rho(x,t)}=\left(1+\frac{b}{\nu}\rho_0(x)\int_0^t\frac{\theta(x,s)}{e^{\varphi(x,s)}}ds\right),\ \int_0^t U(s)ds=\int_{\Omega_-}(\frac{1}{\rho}-\frac{1}{\rho_0})dx, \tag{13}$$

where $\Psi(s)=s-\ln s-1$ *and*

$$2\mathbf{g}(t)=\mu^+\mathbf{a}-[\mu]\mathbf{V}+\left(\frac{1}{\rho},\int_x^1\mathbf{v}_t dy\right)_\Omega+\varepsilon\mathbf{V}'\int_{\Omega_-}\frac{1}{\rho}dx, \tag{14}$$

$$2\left(\nu\varphi(x,t)+\int_0^x u_0(y)dy\right)=\left(\rho_0^{-1},\int_0^x u_0(y)dy\right)_\Omega-\int_0^t\|u^2+b\theta\|_1 ds-[\nu]\int_0^t U ds$$
$$+\int_\Omega\rho^{-1}(y,t)\left(\int_y^x u(z,t)dz+\varepsilon(U-U_0)A(x,y)\right)dy-\varepsilon\int_0^t U(U-U_0)ds, \tag{15}$$
$$A(x,y)=sign(signx-signy).$$

Proof. First, we obtain from equations (7) that equalities (11) and (12) hold with $\mathbf{g}=\mu\rho\mathbf{v}_x|_{x=1}$. To prove the representation formula (14) for $\mathbf{g}$, it suffices to consider equation $(7)_2$ as a linear ordinary differential equation for $\mathbf{v}$ and apply integration with respect to x, taking into account the equality $\int_\Omega\rho^{-1}dx=2$ and boundary conditions.

Formula $(13)_2$ follows immediately from equation $(7)_2$. To prove $(13)_1$, we derive from the first two equations (7) that $\sigma=-\nu(\ln\rho)_t-p$. Then we consider this equality as an ordinary differential equation for ρ and integrate it to obtain $(13)_1$, with φ equal to $\nu^{-1}\int_0^t\sigma ds$. To justify the representation formula (15) for

φ, we define $\Phi(x,t)=\int_0^t\sigma(x,s)ds+\int_0^x u_0(y)dy$ and see that, due to equation (7), $\Phi_t=\sigma$, $\Phi_x=u$, and

$$\frac{\partial}{\partial t}\frac{\Phi}{\rho}=\nu\Phi_{xx}+(\Phi_x\Phi)_x-\Phi_x^2-b\theta, \tag{16}$$

$$\Phi(x,t)=\Phi(y,t)+\int\limits_y^x u(z,t)dz+A(x,y)[\Phi](t). \tag{17}$$

Next, we multiply (17) by $\rho^{-1}(y,t)$ and integrate with respect to y to obtain that

$$2\Phi(x,t)=(\frac{1}{\rho},\Phi)_\Omega+\int\limits_\Omega\rho^{-1}(y,t)\int\limits_y^x u(z,t)dzdy+\int_\Omega\rho^{-1}(y,t)A(x,y)[\Phi]dy. \tag{18}$$

Now, formula (15) follows immediately if we find the first term of the right-hand side of (18) by integrating equality (16) over Ω.

Lemma 3. *The following bounds are valid for any $t\in I$:*

$$\|u(t)\|^2+\varepsilon U^2(t)+\varepsilon\Psi(E(t))+\|\frac{\rho\theta_x^2}{\theta^2},\frac{\rho u_x^2}{\theta},\frac{\rho|\mathbf{v}_x|^2}{\theta}\|_{Q_t}^2\le c,\quad \|\rho(t)\|_\infty\le c.$$

Proof. Let $J_1(t)$ stands for the right-hand side of equality (12). Due to the representation formula (14),

$$2J_1(t)=\frac{d}{dt}\left\{\mathbf{a}\left(\frac{1}{\rho},\int\limits_x^1\mathbf{v}(y,t)dy\right)_\Omega+\varepsilon\mathbf{V}\cdot\mathbf{a}\int\limits_\Omega\frac{1}{\rho}dx\right\}-\mathbf{a}'\left(\frac{1}{\rho},\int\limits_x^1\mathbf{v}(y,t)dy\right)_\Omega-$$

$$(u,\mathbf{a}\cdot\mathbf{v})_\Omega-[\mu]\mathbf{V}\cdot\mathbf{a}+\mu^+|\mathbf{a}|^2-\varepsilon\mathbf{V}\cdot\mathbf{a}'\int\limits_\Omega\frac{1}{\rho}dx-\varepsilon U\mathbf{V}\cdot\mathbf{a}-2[\frac{b}{d}]U,$$

which gives

$$|\int\limits_0^t J_1(s)ds|\le c+c\int\limits_0^t\left(\varepsilon U^2(s)+\|u(s)\|^2\right)ds.$$

Now, Gronwall's lemma finishes the proof of the first estimate of the lemma.

If we again use the representation formula for J_1, we obtain from (11) that $\|\theta(t)\|_1\le c$ and $\varepsilon E(t)\le c$ for any $t\in I$. Now, we conclude, by formula (5), that $\|\varphi(t)\|_\infty \le c$ for any $t \in I$. Using the representation formula $(13)_1$, we obtain the second estimate of the lemma.

Lemma 4. *The estimate $\|\rho(t)\|_\infty \ge c > 0$ is valid for any $t \in I$.*

Proof. It follows from $(13)_1$, that

$$y(t) \le c + c\int_0^t \|\theta(s)\|_\infty ds, \quad y(t) = \|\frac{1}{\rho(t)}\|_\infty.$$

On the other hand, a simple calculation shows that (cf. [3])

$$\|\theta\|_\infty \le \|\theta\|_1(2 + \frac{J_2 y}{2}), \quad J_2 = \|\frac{\rho\theta_x^2}{\theta^2}\|_1.$$

By Lemma 3, $\|J_2(t)|L^1(I)\| \le c$. So, the Gronwall lemma can be applied to the inequality $y(t) \le c + c\int_0^t J_2 y ds$ to finish the proof of the lemma.

Lemma 5. *The following bounds are valid for any $t \in I$:*

$$\int_0^t \|\theta(s)\|_\infty ds \le c, \quad \int_0^t \left(\|\theta^2(s)\|_1 + \|\theta_x(s)\|_1\right) ds \le c, \quad \|u_x, \mathbf{v}_x\|_{Q_t}^2 \le c.$$

Proof. The first estimate is a consequence of Lemma 4. The second one results from the following inequalities

$$\int_0^t \|\theta^2(s)\|_1 ds \le \max_I \|\theta(t)\|_1 \int_0^t \|\theta(s)\|_1 ds,$$

$$\left(\int_0^t \|\theta_x(s)\|_1 ds\right)^2 \le \int_0^t \int_\Omega \frac{\rho\theta_x^2}{\theta^2} dx ds \int_0^t \int_\Omega \frac{\theta^2}{\rho} dx ds.$$

To obtain the last estimate, it suffices to multiply equations $(7)_1$ and $(7)_2$ by u and $\mathbf{v}$ respectively and use the lemmas above. The lemma is proved.

Let us denote

$$\mathbf{z} = \mathbf{v} - \frac{w\mathbf{a}}{2}, \quad w(x,t) = \int_{-1}^x \frac{1}{\rho(y,t)} dy.$$

Thus, we get in Q

$$2\mathbf{z}_t = 2\left(\mu\rho\mathbf{z}_x\right)_x - \mathbf{a}u - w\mathbf{a}', \quad [\mathbf{z}] = 0, \quad [\mu\rho\mathbf{z}_x]|_{x=0} = \varepsilon\mathbf{V}' - \frac{[\mu]\mathbf{a}}{2}, \tag{19}$$

and $\mathbf{z}$ vanishes at $|x| = 1$. Multiplying equation (19) by $|\mathbf{z}|^2\mathbf{z}$ and using the estimate $\|\mathbf{z}|L^\infty(Q)\| \le c$, we obtain in a straightforward manner that

$$\varepsilon \max_{t\in I} |\mathbf{Z}(t)|^4 + \sum_i \|z_i z_{ix}\|_Q^2 \le c, \quad \mathbf{z} = (z_1, z_2), \tag{20}$$

where $\mathbf{Z}(t) = \mathbf{z}(0,t)$.

Lemma 6. *The following bound is valid for any* $t \in I$:

$$\|\theta(t)\| + \|u(t)\|_4 + \|\theta_x, uu_x\|_{Q_t}^2 \le c.$$

Proof. Multiplying equation $(7)_1$ by u^3, we get

$$\frac{d}{dt}\left(\varepsilon U^4 + \|u\|_4^4\right) + 12\|\nu\rho u^2 u_x^2\|_1 = 12(p, u^2 u_x)_\Omega \equiv J_3(t). \tag{21}$$

Let us denote $m = e + \frac{u^2}{2} + \frac{\mathbf{z}^2}{2}$. Using the inequality $\|u\|_\infty \le \|u_x\|$ and the Young inequality, we find that $J_3 \le \delta\|\nu\rho u^2 u_x^2\|_1 + c\delta^{-1}\|u_x\|^2\|m\|^2$ for any small δ. Thus, it follows from (21) that

$$\frac{d}{dt}\left(\varepsilon U^4 + \|u\|_4^4\right) + 6\|\nu\rho u^2 u_x^2\|_1 \le c\|m\|^2. \tag{22}$$

Equations (7) are endowed with the equality

$$m_t = (\mu\rho\mathbf{z}_x\mathbf{z} + \sigma u + q)_x + \frac{\mu\mathbf{a}^2}{4\rho} + \mu\mathbf{a}\cdot\mathbf{z}_x - \frac{u\mathbf{a}\cdot\mathbf{z}}{2} - \frac{w\mathbf{a}'\mathbf{z}}{2}. \tag{23}$$

Denoting $M(t) = m(0,t)$, we multiply (23) by m to get

$$\frac{d}{dt}\overbrace{\left(\varepsilon M^2 + \|m\|^2\right)} + 2\underbrace{\|\mu\rho|\mathbf{z}_x\mathbf{z}|^2, \nu\rho|uu_x|^2, \kappa\rho\theta_x^2\|_1} = J_4(t), \tag{24}$$

where the function $J_4(t)$ meets the bound

$$J_4 \le \delta\underbrace{\cdots} + \frac{c}{\delta}\|\nu\rho u^2 u_x^2\|_1 + \frac{c}{\delta}\|m, u\theta, |\mathbf{z}\mathbf{z}_x|, |\mathbf{z}_x|\|^2 + \varepsilon cM^2 + cM.$$

Since $M \le \|\theta\|_1 + \|\theta_x\|_1 + \|u\|^2 + \|uu_x\|_1$ and $\|u\|_\infty \le \|u_x\|$, we get from (24) that

$$\frac{d}{dt}\underbrace{\cdots}+\overbrace{\cdots} \le c_*\|\nu\rho u^2 u_x^2\|_1 + J_5(t)\|m\|^2 + J_5(t), \tag{25}$$

where $\|J_5(t)|L^1(I)\| \le c$.

Combining inequalities (22) and (25) we have

$$\frac{d}{dt}\left(\varepsilon c_* U^4 + c_*\|u\|_4^4 + \overbrace{\cdots}\right) + \underbrace{\cdots} \le c\|m\|^2 + J_6,$$

where $J_6(t)$ is bounded on $L^1(I)$. Now, by Gronwall's lemma, we obtain the required estimates of the lemma.

As a consequence, we have the following bound

$$\|\rho_x(t)\| \le c, \quad t \in I. \tag{26}$$

Indeed, denoting $h = u + \nu(\ln\rho)_x$ and multiplying the equality $h_t = -p_x$ by h, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|h\|^2 + \|\frac{b}{\nu}\rho\theta h^2\|_1 = (h, J_7)_\Omega, \quad J_7 = \frac{b}{\nu}u\rho\theta + b\rho\theta_x.$$

Sine J_7 is bounded in $L^2(Q)$, estimate (26) follows.

Lemma 7. *There is a positive constant c independent of ε such that $\theta \ge c$ uniformly in $(x,t) \in Q$.*

Proof. Given a non-negative convex function $F : R^+ \to R$, we get from equation $(7)_3$ that

$$\frac{d}{dt}\left(\int_\Omega F(e^{-1})dx + \varepsilon F(E^{-1})\right) \le \int_\Omega F'(e^{-1})\rho dx.$$

By taking $F(s) = s^{2k-1}$ and sending k to ∞, we obtain the lemma.

The next lemmas assert that solutions of the A_ε-problem in Theorem 2, in fact, are more regular.

Lemma 8. *The bounds $y_1(t) \le c$ and $\int_I Y_1(t)dt \le c$ are valid in I, where*

$$y_1 = 1 + \|\sigma, \sigma_x, \theta, q, \mathbf{v}_x, \mathbf{z}_t\|^2 + \varepsilon|\mathbf{Z}'|^2 + \varepsilon|U'|^2, \quad Y_1 = \varepsilon|E'|^2 + \|q_x, \sigma_{xx}, \mathbf{z}_{xt}, \tau_x\|^2.$$

Proof. Because of the embedding-like inequalities $\|f\|_\infty^2 \le \|f\|^2 + 2\|f\|\|f_x\|$ and $\|\rho f\|_\infty \le \|\rho f\|_1 + \|(\rho f)_x\|_1$, the relationships

$$\|u_x\|_\infty^2 \le c\|\sigma, \sigma_x, \theta, q\|^2, \quad \|\mathbf{v}_x\|_\infty^2 \le c + c\|\mathbf{v}_x, \mathbf{z}_t\|^2, \quad \|\theta\|_\infty^2 \le c\|\theta, q\|^2, \tag{27}$$

hold.

Let us multiply equations $(7)_1, (7)_2$, and $(7)_3$ by σ_x, τ_x, and q_x respectively. As a result, we get

$$\frac{1}{2}\frac{d}{dt}\|\frac{\sigma^2}{\nu\rho}\|_1 + \varepsilon|U^{'}|^2 + \|\sigma_x\|^2 = J_8(t), \tag{28}$$

$$\frac{1}{2}\frac{d}{dt}\|(\mu\rho)^{\frac{1}{2}}\mathbf{v}_x\|^2 + \varepsilon|\mathbf{V}^{'}|^2 + \|\frac{\tau_x}{\mu^{\frac{1}{2}}}\|^2 = \frac{1}{2}\frac{d}{dt}\left\{\mathbf{a}(\frac{1}{\rho}, \int_x^1 \mathbf{v}dy)_\Omega\right\} + J_9(t), \tag{29}$$

$$\frac{1}{2}\frac{d}{dt}\|\frac{dq^2}{\kappa\rho}\|_1 + \varepsilon|E^{'}|^2 + \|q_x\|^2 = J_{10}(t). \tag{30}$$

Then we differentiate equations $(7)_1$ and $(19)_1$ with respect to t and multiply the resulting equations by u_t and $\mathbf{z}_t$ respectively. It gives

$$\frac{1}{2}\frac{d}{dt}\left(\|\sigma_x\|^2 + \varepsilon|U^{'}|^2\right) + \|(\nu\rho)^{\frac{1}{2}}\sigma_{xx}\|^2 = J_{11}(t), \tag{31}$$

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{z}_t\|^2 + \varepsilon|\mathbf{Z}^{'}|^2\right) + \|(\mu\rho)^{\frac{1}{2}}\mathbf{z}_{xt}\|^2 = \frac{[\mu]}{2}\frac{d}{dt}\left(\mathbf{Z}\mathbf{a}^{'}\right) + J_{12}(t). \tag{32}$$

We omit description of the functions $J_8 - J_{12}$ since it is clear how to reproduce them. It should only be noticed that, as far as equations (29) and (32) are concerned, we used the representation formula (14) for $\mu\rho\mathbf{v}_x$ at $x = 1$. Using the Yung inequality and inequalities (27), we can estimate the functions $J_8 - J_{12}$ as follows

$$J_8 \le \delta\|q_x\|^2 + \frac{c}{\delta}y_1\|\sigma, \mathbf{v}_x\|^2,$$

$$J_9 \le \varepsilon c|\mathbf{V}^{'}| + cy_1\|u_x, \mathbf{v}_x\|^2,$$

$$J_{10} \le \delta\|q_x\|^2 + \frac{c}{\delta}y_1\|\sigma, \theta_x\mathbf{v}_x\|^2,$$

$$J_{11} \le \frac{1}{2}\|(\nu\rho)^{\frac{1}{2}}\sigma_{xx}\|^2 + c\|q_x\|^2 + cy_1\|u_x, \sigma, \mathbf{v}_x\|^2,$$

$$J_{12} \leq c|\mathbf{Z}| + c\varepsilon|\mathbf{Z}^{'}|^2 + c\varepsilon|U^{'}|^2 + cy_1\left(1 + \|\mathbf{v}_x\|^2\right)$$

Now, combining equalities (28)-(32), choosing δ small enough, and taking into account that $\|\sigma, \theta_x, u_x, \mathbf{v}_x\|^2$ are bounded in $L^1(I)$, we apply the Gronwall lemma to prove Lemma 7.

Remark 1. In order the proof above be absolutely strict, we should use a regularization with respect to t before differentiating equations (7). For illustration, let us consider the linear diffraction problem for the domain $\Omega = \Omega_- \cup \Omega_+$:

$$u_t = (au_x)_x + F, \quad [u]|_{x=0} = 0, \quad \varepsilon U_t = [au_x], \quad u|_{|x|=1} = 0, \quad U = u(0,t).$$

Denoting $u_h(x,t) = h^{-1}\int_t^{t+h} u(x,s)ds$, we see that for $t \in (0, T-h)$

$$\frac{1}{2}\frac{d}{dt}\left(\|u_{ht}\|^2 + U_{ht}^2\right) + \|a(x,t+h)^{1/2}u_{htx}\|^2 = (u_{htx}, a_{ht}u_x)_\Omega + (u_{ht}, f_{ht})_\Omega\,.$$

Thus, the differentiation with respect to t is justified, provided $a_t, f_t \in L^2(Q)$.

Lemma 9. *The following bound is valid for any $t \in I$:*

$$\|e_t\|^2 + \varepsilon|E^{'}|^2 + \|\theta_{xt}\|^2_{Q_t} \leq c.$$

Proof. Let us differentiate equation $(7)_3$ with respect to t and multiply the resulting equation by e_t. On this way we obtain that

$$\frac{1}{2}\frac{d}{dt}\left(\|e_t\|^2 + \varepsilon|E^{'}|^2\right) + \|\kappa d\rho\theta_{xt}^2\|_1 = J_{13},$$

$$J_{13} \leq \frac{1}{2}\|\kappa d\rho\theta_{xt}^2\|_1 + \left(1 + \|e_t, \sigma, q\|^2\right)\left(1 + \|\sigma, \sigma_x, \sigma_{xx}, \theta, q, \mathbf{z}_x, \mathbf{z}_t, \mathbf{z}_{xt}\|^2\right).$$

It implies the assertion of the lemma.

The next lemma is a consequence of the above estimates.

Lemma 10. *Solutions of the A_ε-problem satisfy the following estimates which are uniform in $\varepsilon \in (0, \varepsilon_0]$:*

$$\sup_{t\in I}\left(\|u, \mathbf{v}, \theta\|^2_{W^{2,2}(\Omega_{\pm})} + \|u_t, \mathbf{v}_t, \theta_t, \rho_x, \rho_t\|^2\right) + \varepsilon\|U, \mathbf{V}, E\|^2_{C^1(I)} \leq c,$$

$$\|u_{xt}, \mathbf{v}_{xt}, \theta_{xt}\|_Q^2 \le c, \quad 0 < c^{-1} \le \rho \le c, \quad \theta \ge c.$$

This lemma provides conditions of Lemma 3.1 from [2], therefore the functions $u, \mathbf{v}, \theta$, $u_x, \mathbf{v}_x, \theta_x$ are bounded in $C_{\gamma/2,\gamma}(Q_{\pm})$ for some $\gamma \in (0, \alpha]$ uniformly in ε. Then it follows from the representation formula $(13)_1$ that ρ, ρ_x are also bounded in $C_{\gamma/2,\gamma}(Q_{\pm})$ uniformly in ε. Thus, there is a positive constant c independent of ε such that

$$\|u, \mathbf{v}, \theta, \rho, u_x, \mathbf{v}_x, \theta_x, \rho_x, \rho_t\|_{C_{\gamma/2,\gamma}(Q_{\pm})} \le c, \quad \rho \ge c, \quad \theta \ge c. \tag{33}$$

It now follows from the jump conditions (10) that $\|U, \mathbf{V}, E\|_{C^{1+\gamma/2}(I)} \le c(\varepsilon)$. Let us treat the first three equations (7) as linear parabolic. Due to the Schauder estimates [2] ,we have $\|u, \mathbf{v}, \theta\|_{C_{1+\gamma_1/2,2+\gamma_1}(Q_{\pm})} \le c(\varepsilon)$ where $\gamma_1 = \min\{\gamma, \alpha\}$. Returning to formula $(13)_1$ for ρ and using the parabolicity property again, we see that

$$\|u, \mathbf{v}, \theta\|_{C_{1+\alpha/2,2+\alpha}(Q_{\pm})} \le c(\varepsilon), \quad \|\rho_x, \rho_t\|_{C_{\alpha/2,\alpha}(Q_{\pm})} \le c(\varepsilon). \tag{34}$$

We emphasize that estimate (34) depends on ε.

3. Unique solvability of the perturbated problem

We follow basically the same line of arguments developed in [4] for the flow without a shear component. To make the presentation self-contained, we repeat the argumentation briefly.

Theorem 2. *Let all the conditions of Theorem 1 be satisfied. Then A_ε-problem has a unique solution in the domain Q in the Hölder class described in Theorem 1.*

Proof. First, we discuss uniqueness. Given two solutions $(u_i, \mathbf{v}_i, \theta_i, \rho_i)$, $i \in \{1, 2\}$, the functions $u = u_1 - u_2, \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$, $\theta = \theta_1 - \theta_2$, $\rho = \rho_1 - \rho_2$ satisfy the following equations in the domain Q

$$u_t = \sigma_x, \quad \mathbf{v}_t = \tau_x, \quad \rho = -\rho_1\rho_2 \int\limits_0^t u_x ds, \tag{35}$$

$$e_t = q_x + \sigma_1 u_{1x} - \sigma_2 u_{2x} + \mu\rho_1 |\mathbf{v}_{1x}|^2 - \mu\rho_2 |\mathbf{v}_{2x}|^2,$$

and the jump conditions at $x = 0$

$$[u] = [e] = 0, \quad [\mathbf{v}] = 0, \quad \varepsilon U_t = [\sigma], \quad \varepsilon \mathbf{V}_t = [\tau], \quad \varepsilon E_t = [q],$$

where $\sigma = \sigma_1 - \sigma_2$, $\tau = \tau_1 - \tau_2$, and $q = q_1 - q_2$. Due to (34) and the regularity of the solutions, the function

$$y(t) = \frac{1}{2}\|u, \mathbf{v}, e\|^2 + \frac{\varepsilon}{2}\left(U^2 + E^2 + |\mathbf{V}|^2\right) + \int\limits_0^t \int\limits_\Omega \rho_2 \left(\nu u_x^2 + \mu|\mathbf{v}_x|^2 + \frac{\kappa}{d} e_x^2\right) dxds$$

satisfies the inequality $y' \leq c(t)y$, with a non-negative function $c(t)$ from $L^1(I)$. Thus, the uniqueness is proved.

Remark 2. The proof of uniqueness for the A-problem is the same.

To prove existence, we apply the Leray-Schauder fixed point theorem.

For $\lambda \in [0, 1]$, we define an operator $A_\lambda : C(h) \to C(h)$, $C(h) = C_{1+\alpha/2}(0, h)^4$, as follows. Given a vector-function $\mathbf{S}^\#(t) = \left(U^\#(t), \mathbf{V}^\#(t), E^\#(t)\right)$, we first solve two initial boundary value problems for equations (7) in the domains Q_- and Q_+ respectively, with the boundary conditions

$$(u, \mathbf{v}, q)|_{x=-1} = 0, \quad (u, \mathbf{v}, e)|_{x=0} = \left(U^\#(t), \mathbf{V}^\#(t), E^\#(t)\right), \tag{36}$$

for the first problem and

$$(u, \mathbf{v}, q)|_{x=1} = (0, \mathbf{a}, 0), \quad (u, \mathbf{v}, e)|_{x=0} = \left(U^\#(t), \mathbf{V}^\#(t), E^\#(t)\right), \tag{37}$$

for the second one. The initial conditions for both the problems are

$$(u, \mathbf{v}, \theta, \rho)|_{t=0} = (u_0, \mathbf{v}_0, \theta_0, \rho_0)|_{x\in\Omega_-^+}, \tag{38}$$

with Ω_- standing for the first problem and Ω_+ for the second one. The data $(u_0, \mathbf{v}_0, \theta_0, \rho_0)$ in (38) are those of the A_ε-problem. We note that in these problems the constants ν, μ, κ, b, and d in (7) are different.

The problems described are locally solvable in the following sense (see [3-7]). For simplicity, we formulate a result for the second problem in the domain Q_+ only.

Lemma 11. *Given positive numbers M_c and M_b, let the initial and boundary data for the domain Q_+ satisfy the conditions*

$$\|u_0, v_{01}, v_{02}, \theta_0\|^2_{C_{2+\alpha,\Omega_+}} + \|\rho_0\|^2_{C_{1+\alpha,\Omega_+}} \le M_c, \quad \inf_x \rho_0 > 0, \quad \inf_x \theta_0 > 0, \quad (A1)$$

$$\|U^\#, V_1^\#, V_2^\#, a_1, a_2\|^2_{C_{1+\alpha/2,I}} \le M_b, \quad (A2)$$

$$\left(U_t^\#, \mathbf{V}_t^\#, E_t^\#\right)|_{t=0} = \left(\sigma_{0x}, \tau_{0x}, q_{0x} + \sigma_0 u_{0x} + \mu\rho_0|\mathbf{v}_{0x}|^2\right)|_{x=0}, \quad (A3)$$

$$(0, \mathbf{a}_t)|_{t=0} = (\sigma_{0x}, \tau_{0x})|_{x=1}, \quad (A4)$$

$$\left(U^\#, \mathbf{V}^\#, E^\#\right)|_{t=0} = (u_0, \mathbf{v}_0, e_0)|_{x=0} \equiv \mathbf{S}_*, \quad (0, \mathbf{a})|_{t=0} = (u_0, \mathbf{v}_0)|_{x=1}. \quad (A5)$$

Then there is a unique solution of problem (7),(37), and (38) in the domain $Q_h^+ = \Omega_+ \times (0, h)$ such that

$$\|u, v_1, v_2, \theta\|^2_{C_{1+\alpha/2,2+\alpha}(Q_h^+)} + \|\rho, \rho_x, \rho_t\|^2_{C_{\alpha/2,\alpha}(Q_h^+)} \le N; \quad \rho > 0, \quad \theta > 0, \quad (39)$$

with h and N depending on M_c and M_b.

Next, given a positive number K, we define a set $M(h, K) \subseteq C(h)$. We say that $\mathbf{S}^\# \in M(h, K)$ if $\|\mathbf{S}^\# - \mathbf{S}_*\|_{C(h)} \le K$ and the vector-function $\mathbf{S}^\#$ satisfies conditions (A3) and (A5) of Lemma 11. Clearly $M(h, K)$ is convex and closed in $C(h)$.

Now, we define an operator $A_\lambda : M(h, K) \to C(h)$ by setting $\mathbf{S} = A_\lambda(\mathbf{S}^\#)$, where $\mathbf{S}(t) = (U(t), \mathbf{V}(t), E(t))$ and

$$\varepsilon U_t = \lambda[\sigma], \quad \varepsilon \mathbf{V}_t = \lambda[\tau], \quad \varepsilon E_t = \lambda[q], \quad \mathbf{S}(0) = \mathbf{S}^\#(0) = \mathbf{S}_*.$$

Here σ, τ and q are found by solving initial boundary value problems (36),(38) and (37),(38) for equations (7) in the domains $Q_{\pm}$ with $\mathbf{S}^\#(t)$ in boundary conditions (37) and (38). Clearly, the value of $\mathbf{S}(t)$ at $t = 0$ does not depend on $\mathbf{S}(t) \in M(h, K)$.

By Lemma 11, there is a positive number $h_1(K) \in I$ such that the operator $A_\lambda : M(h, K) \to C(h)$ is well defined for any $h \in (0, h_1(K))$.

It is shown in [8], by using the Schauder estimates for linear parabolic equations and estimates (39), that the operator A_λ improves smoothness, more exactly there are small δ and $h_2(K)$ such that the operator $A_\lambda : M(h, K) \to C_{1+\frac{\alpha+\delta}{2}}(0, h)^4$ is continuous for any $h \in (0, h_2(K))$. It means that the operator $A_\lambda : M(h, K) \to C(h)$ is compact for any $h \in (0, h_2(K))$.

It follows from the equalities

$$A_{\lambda_1}(\mathbf{S}^\#) - A_{\lambda_2}(\mathbf{S}^\#) = \frac{\lambda_1 - \lambda_2}{\varepsilon} \int\limits_0^t ([\sigma], [\tau], [E])\, ds \tag{40}$$

and from estimates (39) that the operator $A_\lambda : M(h, K) \to C(h)$ is continuous with respect to $\lambda \in [0, 1]$ uniformly in $M(h, K)$. Obviously, the equation $\mathbf{S} = A_0(\mathbf{S})$ has a unique solution $\mathbf{S} = \mathbf{S}_*$ and the map $\mathbf{S} \to \mathbf{S} - A_0(\mathbf{S})$ is bijective.

Lemma 12. *Let $\mathbf{a}(t) \in C^2(I)$, then there is a positive number K such that, for some $h_3 \in (0, h_2(K))$, each fixed point $\mathbf{S}_\lambda(t)$ of the operator $A_\lambda : M(h, K) \to C(h)$ satisfies the bound $\|\mathbf{S}_\lambda(t) - \mathbf{S}_*\|_{C(h)} < K$ for any $h \in (0, h_3(K))$.*

Proof. Taking $K = 1$, we have that the operators $A_\lambda : M(h, 1) \to C(h)$ are compact for any $h \in (0, h_2(1))$ and $\lambda \in [0, 1]$. Putting $\lambda_2 = 0$ in (40) and using estimate (39), we conclude that $\|A_\lambda(\mathbf{S}) - \mathbf{S}_*\|_{C(h)} \le \lambda N_1$, with N_1 independent of $h \in (0, h_2(1))$. Thus, $\|A_\lambda(\mathbf{S}) - \mathbf{S}_*\|_{C(h)} \le 1$ for $\mathbf{S} \in M(h, 1)$ if $\lambda \in [0, N_1^{-1}]$ and $h \in (0, h_2(1))$. By the Schauder fixed point theorem, each of the operators $A_\lambda : M(h, 1) \to C(h)$, $h \in (0, h_2(1))$, $\lambda \in [0, N_1^{-1}]$, has a unique fixed point $\mathbf{S}_\lambda$ (uniqueness is proved above) and $\|\mathbf{S}_\lambda(t) - \mathbf{S}_*\|_{C(h)} \le 1$.

Now, we repeat the considerations of section 2 to derive that any fixed point $\mathbf{S}_\lambda$, $\lambda \in [N_1^{-1}, 1]$, satisfies a uniform bound $\|\mathbf{S}_\lambda(t) - \mathbf{S}_*\|_{C(h)} \le q$. To this end it is sufficiently to put $\varepsilon := \varepsilon\lambda^{-1}$ and $T = h_2(1)$. (At this step we requir that $\mathbf{a}(t) \in C^2(I)$.)

Let us define $K_* = 1 + \max\{1, q\}$. Clearly, there is a positive number $h_2(K_*)$ such that the operators $A_\lambda : M(h, K_*) \to C(h)$ are compact for any $h \in (0, h_2(K_*))$. Now, we set $h_* = \min\{h_2(1), h_2(K_*)\}$ to derive the assertion of the lemma. Indeed, by the uniqueness property, the operator $A_\lambda : M(h_*, K_*) \to C(h_*)$ may have only one fixed point. Since $M(h_*, 1) \subseteq M(h_*, K_*)$ and the set $M(h_*, 1)$ contains a fixed point of A_λ, $\lambda \in [0, N_1^{-1}]$, we have that a fixed point $\mathbf{S}_\lambda$ of $A_\lambda : M(h_*, K_*) \to C(h_*)$ satisfies the bound $\|\mathbf{S}_\lambda(t) - \mathbf{S}_*\|_{C(h_*)} \leq 1 < K_*$ for $\lambda \in [0, N_1^{-1}]$. But for $\lambda \in [N_1^{-1}, 1]$ the bound $\|\mathbf{S}_\lambda(t) - \mathbf{S}_*\|_{C(h_*)} \leq q < K_*$ is shown above. Thus, the lemma is proved.

Taking into account all the properties of the operators $A_\lambda : M(h, K) \to C(h)$, we apply the Leray-Schauder fixed point theorem to conclude that there is a unique local in time solution of the A_ε-problem in the Hölder class desribed in Theorem 1. By *a priori* estimates (34), this solution is, in fact, global. Thus Theorem 2 is proved.

4. Proof of Theorem 1

By Lemma 10, we may send ε to zero to obtain a weak solution $(u, \mathbf{v}, \theta, \rho)$ of the A-problem satisfying equations (7) a.e. in Q , the bounds (33), and the bounds of Lemma 10.

Let us show that this solution is, in fact, Hölder continuous in the sense of Theorem 1 . The second equation in (7) for $\mathbf{v}$ can be treated as a linear parabolic one in the domains Q_+ and Q_- with the diffraction conditions $[\mathbf{v}] = 0$ and $[\mu\rho\mathbf{v}_x] = 0$ at their common boundary $x = 0$. Since $\rho, \rho_x \in C_{\gamma/2,\gamma}(Q_{\pm})$, $\mathbf{a}(t) \in C^2(I)$, and $\mathbf{v}_0 \in C_{2+\alpha}(\Omega^{\pm})$, we have, by [9], that $\mathbf{v} \in C_{1+\gamma_1/2,2+\gamma_1}(Q_{\pm})$ with $\gamma_1 = \min\{\gamma, \alpha\}$. By the same reason, we conclude from the third equation in (7) for θ that $\theta \in C_{1+\gamma_1/2,2+\gamma_1}(Q_{\pm})$. Then we treat the first equation in (7) for u as a parabolic one with the diffraction conditions $[u] = 0$ and $[\nu\rho u_x] = [p] \in C_{1+\gamma_1/2}(I)$. So, by [9], $u \in C_{1+\gamma_1/2,2+\gamma_1}(Q_{\pm})$. The solution $(u, \mathbf{v}, \theta, \rho)$ satisfies formula (13) with $\varepsilon = 0$. Due to it, ρ and ρ_x belong to $C_{\alpha/2,\alpha}(Q_{\pm})$. Hence, by the second equation in (7), $\mathbf{v} \in C_{1+\alpha/2,2+\alpha}(Q_{\pm})$. The inclusion $u \in C_{1+\gamma_1/2,2+\gamma_1}(Q_{\pm})$ implies [2] that $u_x \in C_{\gamma_2/2,\gamma_2}(Q_{\pm})$, $\gamma_2 = \frac{1+\gamma_1}{2}$. Let us denote $\gamma_3 = \min\{\alpha, \gamma_2\}$, then it follows from the equation for θ that $\theta \in C_{1+\gamma_3/2,2+\gamma_3}(Q_{\pm})$. Now, due to the equation for u, we have $u \in C_{1+\gamma_3/2,2+\gamma_3}(Q_{\pm})$ and $u_x \in C_{\gamma_4/2,\gamma_4}(Q_{\pm})$, where $\gamma_4 = \frac{1+\gamma_3}{2}$. In case $\gamma_4 < \alpha$ we repeat the procedure. Clearly, in finite number of steps, we will arrive at the conclusion of Theorem 1.

Remark 3. Given a solution $(u(x,t), \mathbf{v}(x,t), \theta(x,t), \rho(x,t))$ of the A-problem, we obtain a solution of the free-boundary value problem (1)-(5) in the Eulerian coordinates by the change of variables $(x,t) \to (\xi, t)$:

$$\xi = E(x,t) \equiv \int_{-1}^{x} \frac{1}{\rho}(y,t)dy - 1, \tag{41}$$

with the function $\Gamma(t) = E(0,t)$ being a free boundary. Due to formula (41), one can easily reformulate Theorem 1 as a result on unique global solvability of problem (1)-(5) in a Hölder class.

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