NON LINEAR INTERACTION BETWEEN LONG WAVES IN THE EQUATORIAL WAVEGUIDE

C. V. Turner E. G. Tabak R. R. Rosales
A. J. Majda *

1. Introduction

The change of sign of the Coriolis force in the Equator produces an effective waveguide for a large class of long atmospheric waves. Among these waves, the longest and slowest are the nondispersive Kelvin and the dispersive Rossby and Yanai —or mixed Rossby-gravity— waves. These waves, which have been observed in the troposphere and lower stratosphere, are believed to play a significant role in the dynamics of the tropical atmosphere.

The most prominent waves observed are very long Kelvin waves [2], with wavenumbers $l = 1$ and $l = 2$ —i.e. inverse wavelength equal to the whole and a half of the circumference of the Earth— and Yanai waves [8] with $l = 4$. It has been proposed in [9] that these waves force the stratosphere from below, contributing to generate the quasi-biennial oscillation of the mean wind in the tropical middle atmosphere. The occurrence and persistence of these waves raise a number of important questions:

a) Why are these particular waves selected?

b) Why do they not dissipate away? The Kelvin wave, in particular, should rapidly generate shocks, which dissipate energy very efficiently [5]. In addition, one would expect a significant amount of energy transfer to shorter waves, particularly to other Kelvin, Yanai and Rossby modes, and to the faster Poincaré waves.

*This research was partially supported by Fundacion Antorchas-Conicet.

c) Do these various waves interact nonlinearly? And do they feel the equatorial topography and the distribution of land and sea?

In this work, we present a reduced model which sheds light on these questions. A more thorough description of this model together with various geophysical applications can be found in [4]. In its simplest version, the model has only two waves: an $l = 1$ Kelvin and an $l = 4$ Yanai, interacting through the $l = 5$ mode of the topography, which roughly corresponds to the distribution of the continents. However, the model can be straightforwardly extended to include a larger set of very long waves, which realistically should probably include at least two Kelvin modes ($l = 1$ and 2) and about four Yanai and Rossby modes.

For clarity, we present the model in the simplest possible scenario of the β-plane one-layer shallow-water equations with topography and no mean flow, where the dependent variables are required to be bounded as $y \to \pm\infty$. All these simplifying hypothesis can be relaxed, however, in order to build a more realistic model. We are presently extending the model to deal with more general vertically stratified flows, with a mean zonal wind component, with full planetary scales, and with waves interacting through a more general "topography", such as the one provided by convection. In addition, various applications of the model are being pursued, particularly to the Madden-Julian oscillation of tropical cloud clusters and to El Niño Southern Oscillation of the surface sea t emperature.

In our reduced model, the non linear interaction between Kelvin and Yanai waves through the topography is described by two coupled differential equations. The equation describing the evolution of the Kelvin wave is an inviscid Burgers equation forced by the Yanai wave, and the evolution of the latter is described by an ordinary differential equation forced by just one Fourier mode of the Kelvin wave. After scaling, the equations take the canonical form

$$\begin{aligned} K_\tau + \left(\frac{1}{2}K^2\right)_\theta &= Y(\tau)e^{i\theta} + \bar{Y}(\tau)e^{-i\theta} \\ Y_\tau &= -\hat{K}(1)\,, \end{aligned}$$

where $K(\theta, \tau)$ and $Y(\tau)$ are the amplitudes of the Kelvin and the $l = 4$ Yanai wave respectively. The independent variables are $\theta = (x - ct)$, where x measures length in the zonal direction, t is time and c is the speed of a linear Kelvin wave, and $\tau = \epsilon t$, a slow time variable scaled by ϵ, a measure of the strength of the nonlinearity. The reason why only the $l = 1$ mode of K appears in the second equation, is that this is the only mode which interacts with the topography to generate a Yanai wave, the $l = 4$ Yanai mode represented by Y.

This reduced model has a conserved energy, and a shock-free traveling wave solution for K with a corner, dominated by the $l = 1$ mode. Moreover, numerical experiments show that most big enough initial data converge to a shock-free wave close to this traveling solution. Thus the reduced model provides a simple explanation for questions a, b and c above: the $l = 1$ Kelvin and the $l = 4$ Yanai wave are observed because, interacting nonlinearly through the topography, they avoid shock formation, the main dissipative mechanism for Kelvin waves. A Kelvin wave with any initial shape generates the $l = 4$ Yanai, and this in turn feeds the $l = 1$ mode of the Kelvin. For a range of vertical scales comparable to the observations, no other Yanai or Rossby modes are generated nonlinearly. Fast Poincaré waves may be generated initially by large gradients in the Kelvin wave but, once this reaches its shock-free final state, such generation of fast waves is highly reduced.

Phenomena very similar to those described in this report have been found in purely hyperbolic systems, particularly in gas dynamics [3] [1] [6] [10]. In this latter context, the role of the topography is played by a variable entropy, which acts as a bridge for energy exchange between right and left-going sound waves.

2. The Reduced Model

We start with the non-dimensionalized equations for long waves in a single layer of fluid of constant density in the equatorial waveguide:

$$\eta_t + [(1+\eta-h)u]_x + [(1+\eta-h)v]_y = 0 \tag{1}$$

$$u_t + uu_x + vu_y + \eta_x - yv = 0 \tag{2}$$

$$v_t + uv_x + vv_y + \eta_y + yu = 0, \tag{3}$$

where the total depth of the fluid is given by

$$\frac{c^2}{g}(1+\eta-h).$$

Here c is the characteristic speed of the linear waves, g is the acceleration of gravity, η is the non-dimensional perturbation of the free surface, and h the non-dimensional bottom topography. The x and y velocities u and v are non-dimensionalized by the characteristic speed c, and the spatial variables by the scale

$$L = \sqrt{\frac{c}{\beta}},$$

where β is the linear variation of the Coriolis parameter with latitude, given by

$$\beta = \frac{2\Omega}{R}.$$

Here $\Omega = \frac{2\pi}{24hs}$ is the angular velocity of the Earth, and $R = 6378$ km its radius. The time-scale T is given by

$$T = \frac{L}{c}.$$

We shall propose an ansatz for the solution to (1,2,3) in which the dependent variables η, u and v and the topography h are all small and of the same order of magnitude ϵ.

There are two classes of solutions to the linearization of the equations above: a nondispersive Kelvin wave and an infinite set of dispersive waves, the Rossby, Yanai and Poincaré waves. The Kelvin wave is given by

$$\eta = K(x-t)e^{-\frac{y^2}{2}} \tag{4}$$

$$u = K(x-t)e^{-\frac{y^2}{2}} \tag{5}$$

$$v = 0, \tag{6}$$

where K is an arbitrary function. The dispersive waves have the general form

$$\eta = \left[\frac{y}{k-\omega}H_n(y) + \frac{\omega}{\omega^2-k^2}H_n'(y)\right] e^{i(kx-\omega t)}e^{-\frac{y^2}{2}} \tag{7}$$

$$u = \left[\frac{y}{k-\omega}H_n(y) + \frac{k}{\omega^2-k^2}H_n'(y)\right] e^{i(kx-\omega t)}e^{-\frac{y^2}{2}} \tag{8}$$

$$v = iH_n(y)e^{i(kx-\omega t)}e^{-\frac{y^2}{2}}, \tag{9}$$

where $H_n(y)$ is the Hermite polynomial of order n, and $\omega = W(k)$ satisfies the dispersion relation

$$-2n + \frac{(\omega+k)\,(\omega^2 - k\omega - 1)}{\omega} = 0. \tag{10}$$

The solutions with positive ω to this cubic equation are displayed in figure 1. The case with $n = 0$ has only two solutions, corresponding to the Yanai (or mixed Rossby-gravity) wave. The third solution ($\omega = -k$) to the cubic is spurious; it corresponds to a solution that grows exponentially away from the Equator. For $n \geq 1$, the solutions are one Rossby and two Poincaré waves, characterized respectively by the inequalities

$$\omega^2 \leq \frac{1+2n}{2} - \sqrt{n(n+1)}$$

and

$$\omega^2 \geq \frac{1+2n}{2} + \sqrt{n(n+1)}.$$

Notice that there is a wide scale separation between the Rossby and the Poincaré waves.

We shall concentrate on very long waves, with inverse wavelength comparable to the circumference of the Earth $P = 40000$ km. In order to see what this means for the wavenumbers k, we need to express P in the spatial unit of our nondimensionalization, i.e. L, which depends on the characteristic speed c.

A Kelvin wave $K(x-t)$, for instance, can be written as a Fourier series

$$K(x-t) = \sum_{l=1}^{\infty} \hat{K}(l)e^{i\alpha l(x-t)},$$

where $\alpha = \frac{2\pi L}{P}$. A wave with $l = 1$, i.e., with period equal to the circumference of the Earth, has a wavenumber $k = \alpha$, and all other wavenumbers are integer multiples of this. Similarly, each mode D_j of the dispersive waves has the form

$$D(x,t) = D_j \, e^{i(j\alpha)x - \omega_j t} \, ,$$

where j is an integer and $\omega_j = W(j\alpha)$. The actual value of α depends on the vertical structure of the waves under consideration. Throughout this communication, we shall choose a value of $c = 187$ m/s, with corresponding $L = 3000$ km, $T = 4$ h 15 min and $\alpha = 0.45$. Other realistic values for c can be found in Table 1 of [4]. The main effect of c on this model is to select sets of resonant waves. In [4] , we show how different choices affect this selection, without changing, however, the qualitative properties of the resulting model. Notice that the grid underlying the dispersion relation in figure 1 has been drawn using this value of $\alpha = 0.45$ as a grid-size.

As a Kelvin wave interacts with the topography, it can generate other waves through three-mode resonance. If we denote by k_K, k_T and k^* the wavenumbers of the Kelvin wave, the topography and the dispersive wave, the conditions for resonance are

$$k_K + k_T = k^* \tag{11}$$

$$k_K = \omega^* = W(k^*), \tag{12}$$

where we have used the facts that for the Kelvin wave $\omega = k$, and that the topography is time-independent. Since all the wavenumbers have to be multiples of $\alpha = 0.45$, only a discrete set of resonances can take place. In particular, since the Kelvin wave cannot have a positive wavenumber k_K smaller than $\alpha = 0.45$, we conclude from inspection of the dispersion relation in figure 1 that no Rossby waves can result from the interaction of a Kelvin wave with topography, and the only wave that can be generated by the first two modes of a Kelvin wave is the $k = -1.8$ ($l = -4$) Yanai mode, generated by the $l = 1$ Kelvin wave and the $l = 5$ mode of the topography.

Based on the argument above, we shall propose an ansatz with only two waves, a Kelvin wave and a Yanai mode with $l = -4$. The corresponding asymptotic expansion is carried out in [4]. The resulting equations are

$$K_\tau + \left(\frac{1}{2}K^2\right)_\theta = Y(\tau)e^{i\theta} + \bar{Y}(\tau)e^{-i\theta} \tag{13}$$

$$Y_\tau = -\hat{K}(1,\tau) \tag{14}$$

where $K(\theta,\tau)$ and $Y(\tau)$ stand for the Kelvin wave and the $l = -4$ mode of Yanai respectively, $\theta = (x - ct)/P$ represents the linear phase of the Kelvin wave, normalized so that it has period 2π, and $\tau = \epsilon t$ represents the slow nonlinear time. The dependent variables K and Y and the slow time τ have been further rescaled so as to normalize to one the coefficients of the nonlinear interaction on the right-hand side, which depend on the projection of the (zonal) mode $l = 5$ of the topography on the (longitudinal) first Hermite polynomial.

Equation (13) for the evolution of the Kelvin wave is a Hopf –or inviscid Burgers– equation, forced on the right-hand side by the interaction of the Yanai wave with the topography. Equation (14), on the other hand, is an ordinary differential equation for the evolution of the Yanai wave, forced by the interaction of the Kelvin wave with the topography. This system of equations has very distinctive properties, some of which are treated below.

3. Properties of the Model

The equations (13, 14) have two main conserved quantities, the mean of K and the total energy, which take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} K(\theta,\tau)\,\mathrm{d}\theta = 0 \tag{15}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(2\pi|Y|^2 + \int_0^{2\pi}\frac{K^2}{2}\,\mathrm{d}\theta\right) = 0 \tag{16}$$

respectively. The energy, however, is only conserved while the solution remains smooth; when there are shocks, it decays at a rate proportional to the cube of the shock strength.

Equations (13) and (14) have a family of exact solutions where K is a traveling wave of the form

$$K(\theta,\tau) = F(\theta - s\tau)$$

where

$$F(z) = s \pm 2\sqrt{-\frac{\hat{F}_1}{s}}\sqrt{C + \cos(z)}. \tag{17}$$

Here C is a constant of integration, and s, the nonlinear correction to the Kelvin wave speed, is a function of C. Notice that, if C is strictly larger than one, the solution is smooth but, when $C = 1$, it develops a corner. In the latter case, the solution is

$$F(z) = s \pm 2\sqrt{-\frac{2\hat{F}_1}{s}}\,|\cos(z/2), \tag{18}$$

with

$$s = \mp\frac{4}{\pi}\sqrt{\frac{2}{3}} \sim 1.04\,.$$

Thus the wave with a downward peak moves more slowly than a linear Kelvin wave, and the one with the upward peak moves faster. The total energy corresponding to this exact solution is readily computed; its value is

$$E = \frac{14\pi}{9} - \frac{32}{3\pi} \sim 1.5\,.$$

We may wonder about the significance of these traveling wave solutions: do they act as attractors for a large enough set of initial data? Next we shall address this question through a numerical experiment; more experiments can be found in [4].

Solving numerically equations (13,14) is a relatively straightforward task. We have used a fractional step procedure, solving in one step the Hopf equation

$$K_\tau + \left(\frac{K^2}{2}\right)_\theta = 0$$

and in the other the system of integro–differential equations

$$\begin{aligned} K_\tau &= Y(\tau)e^{i\theta} + \bar{Y}(\tau)e^{-i\theta} \\ Y_\tau &= \hat{K}(1,\tau)\,. \end{aligned}$$

For the Hopf equation, we use a second order Godunov method (we need a conservative algorithm, since there is an initial development of shocks), and we solve the system of integro–differential equations with a second order Runge-Kutta method coupled with a Fast Fourier Transform. Finally, we put the two fractional steps together using the second order procedure due to Strang [7].

We would like to point out that the reduction of complexity in going from the full system (1,2,3) to the reduced model (13,14) is so big, that we were able to program the model in the interpreter language Matlab, with a typical run taking only a few minutes (such runs represent, though in a very idealized sense, a few years of long wave propagation in the equatorial waveguide.) Next we present the results of a typical experiment.

This experiment illustrates quite dramatically the "attractive" nature of the traveling wave solution (18). Figure 2 shows the initial value of K, given by a randomly chosen periodic function; in this case,

$$K(\theta, 0) = 0.5\,(3\cos(\theta) + \sin(2\theta) - \sin(4\theta))\ ;$$

the initial value assigned to $Y(0)$ is

$$Y(0) = 0.5(1 + i)\,.$$

The total energy of this solution is $E = 7.5$, larger than the threshold value 1.5, so we should expect fast initial decay (the solution with a corner has the largest energy among traveling waves, and probably among all unsteady but energy–preserving solutions too.) In figure 3, we see the solution K at $\tau = 0.5$, with a freshly created strong shock. By the time $\tau = 5$, displayed in figure 4, most of the extra energy of the initial data has dissipated at this shock, which has also eliminated all but the longest modes of the solution. Finally, by the time $\tau \sim 50$ of figure 5, we have reached a steady state. Surprisingly, this state agrees nearly exactly with the exact traveling wave with a corner (18), which is also displayed. Figure 6 shows the time evolution of the real and imaginary part of $Y(\tau)$, the amplitude of the Yanai wave. Notice that, since the speed s of the traveling Kelvin wave is close to one, and this equals the frequency of the

Yanai wave, the period of oscillation of the latter is close to 2π. Finally, figure 7 has the total energy as a function of time, showing the fast initial dissipation, followed by stabilization at nearly the exact critical energy corresponding to the wave with a sharp corner.

This numerical experiment thus illustrates what we have found to be a general fact: nearly all initial data with large enough energy converge very rapidly to the traveling wave solution with a corner (18), through the dissipation of their extra energy at shocks (the exceptions are highly symmetric initial data, with a symmetry that inhibits any preference for left or right-going waves.) A mathematical description of this observed behavior, and the only slightly more complicated one for initial data which are not sufficiently energetic (here the final solution is quasi-periodic), is a challenging problem in the theory of partial differential equations and dynamical systems. Similar behavior for waves in gases has been reported in ([6]).

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

References

[1] Celentano, C.,*Finite Amplitude Resonant Acoustic Waves Without Shocks*, Ph. D. Thesis, M.I.T., (1995).

[2] Kousky, W., *Observational evidence of Kelvin waves in the tropical stratosphere*, J. Atmos. Sci. 25, (1968), 900-907.

[3] Majda, A. J., Rosales, R. R. and Schonbek, M., *A canonical system of integrodifferential equations arising in resonant nonlinear acoustics*, Studies Appl. Math. 79 (1988), 205-262.

[4] Majda, A. J., Rosales, R. R., Tabak, E. G., Turner, C. V., *Interaction of large scale Equatorial waves and dispersion of Kelvin waves through topographic resonances*, Submitted to J. Atmos. Sci.

[5] Ripa, P., *On the theory of non linear wave-wave interactions among geophysical waves*, J. Fluid Mech. 103 (1981), 87-115.

[6] Shefter, M., *Never Breaking Quasi-Periodic Solutions of Weakly Nonlinear Gas Dynamics*, Ph. D. Thesis, M.I.T., (1997).

[7] Strang, G., *On the Construction and Comparison of Difference Schemes*, SIAM J. Num. Anal., 5 (1968), 506-517.

[8] Maruyama, Y., *Stratospheric wave disturbances propagating over the equatorial Pacific*, J. Meteorol. Soc. Jpn. 44 (1966), 291-294.

[9] Yoden, S. and Holton, J. R., *A new look at Equatorial Quasi-Biennial Oscillation Models*, J. Atmos. Sci. 45 (1988), 2703-2717.

[10] Vaynblat, D., *The Strongly Attracting Character of Large Amplitude Nonlinear Resonant Acoustic Waves Without Shocks. A Numerical Study*, Ph. D. Thesis, M.I.T., (1996).

Andrew Majda
Courant Institute of Mathematical Sciences
New York, USA

Cristina Turner
FAMA - UNC
Cordoba, Argentina

Esteban Tabak
Courant Institute of Mathematical Sciences
New York, USA

Ruben Rosales
MIT
Cambridge, USA