ON THE TIME DECAY OF DIFFUSION WAVES FOR A CLASS OF PARABOLIC SYSTEMS OF CONSERVATION LAWS

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Abstract

We derive time decay rates in L^2-norm for solutions to the Cauchy problem of the parabolic system

$$\boldsymbol{u}_t \;+\; (\,\boldsymbol{u}\,\varphi(|\boldsymbol{u}|)\,)_x \;=\; (\,B(\boldsymbol{u})\,\boldsymbol{u}_x\,)_x$$

and its N-dimensional analogue

$$\boldsymbol{u}_t \;+\; \sum_{k=1}^{N} \frac{\partial}{\partial x_k}\,(\,\boldsymbol{u}\,\varphi_k(|\boldsymbol{u}|)\,) \;=\; \sum_{k=1}^{N} \frac{\partial}{\partial x_k}\left(B^{[k]}(\boldsymbol{u})\,\frac{\partial \boldsymbol{u}}{\partial x_k} \right)$$

provided only that the initial state $\boldsymbol{u}(\cdot,0)$ is in $L^1 \cap L^2$, but otherwise arbitrary. Here, $\boldsymbol{u} = (u_1, \ldots, u_m)$ is the vector of unknowns, $\varphi, \varphi_1, \ldots, \varphi_k$ are given scalar functions, and $B, B^{[1]}, \ldots, B^{[N]}$ are uniformly positive definite matrices of order m whose off-diagonal elements $b_{ij}(\boldsymbol{u})$ are bounded in terms of u_i. The method is based on energy estimates and can be adapted to other problems. The decay rates obtained are optimal.

Resumo

São derivadas taxas de decaimento na norma L^2 para as soluções do problema de Cauchy do sistema parabólico

$$\boldsymbol{u}_t \;+\; (\,\boldsymbol{u}\,\varphi(|\boldsymbol{u}|)\,)_x \;=\; (\,B(\boldsymbol{u})\,\boldsymbol{u}_x\,)_x$$

e do problema correspondente em N dimensões

$$\boldsymbol{u}_t \;+\; \sum_{k=1}^{N} \frac{\partial}{\partial x_k}\,(\,\boldsymbol{u}\,\varphi_k(|\boldsymbol{u}|)\,) \;=\; \sum_{k=1}^{N} \frac{\partial}{\partial x_k}\left(B^{[k]}(\boldsymbol{u})\,\frac{\partial \boldsymbol{u}}{\partial x_k} \right)$$

*This work was supported by CNPq (PQ 301236/91, AVG 453835/96-1) and FAPERGS (APC 96/1696-6)

assumindo-se que o estado inicial $\boldsymbol{u}(\cdot,0)$ pertence ao espaço $L^1 \cap L^2$, sendo possivelmente grande. Nas equações acima, $\boldsymbol{u} = (u_1, ..., u_m)$, $\varphi, \varphi_1, \dots, \varphi_k$ são funções escalares dadas, e $B, B^{[1]}, ..., B^{[N]}$ denotam matrizes definidas positivas de ordem m cujos elementos $b_{ij}(\boldsymbol{u})$ fora da diagonal são limitados em termos de u_i. O método é baseado em estimativas de energia e pode ser adaptado para outros problemas. As taxas de decaimento obtidas são optimais.

1. Introduction

We will establish in this article the time-asymptotic decay in L^2-norm of solutions to the initial-value problem

$$\boldsymbol{u}_t + (\boldsymbol{u}\,\varphi(|\boldsymbol{u}|))_x = (B(\boldsymbol{u})\,\boldsymbol{u}_x)_x \tag{1a}$$

$$\boldsymbol{u}(\cdot,0) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R}) \tag{1b}$$

and some of its generalizations, where $\boldsymbol{u}(x,t) = (u_1(x,t), \dots, u_m(x,t))$ stands for the vector of unknown quantities, $|\boldsymbol{u}|$ is the Euclidean norm of $\boldsymbol{u}$, φ is a continuously differentiable scalar function and $B(\boldsymbol{u})$ is an $m \,\mathrm{x}\, m$ uniformly positive definite matrix, i.e.,

$$(\boldsymbol{\xi}, B(\boldsymbol{u})\boldsymbol{\xi}) \geq \mu\,|\boldsymbol{\xi}|^2 \qquad \forall \quad \boldsymbol{\xi} \in \mathbf{R}^m \tag{2}$$

for all $\boldsymbol{u}$ concerned, where μ is a positive constant and $(\cdot\,,\cdot)$ denotes the standard inner product in $\mathbf{R}^m$. We also assume that the off-diagonal entries $b_{ij}(\boldsymbol{u})$ of $B(\boldsymbol{u})$ satisfy, for all $\boldsymbol{u} = (u_1, \dots, u_m)$ concerned,

$$|\,b_{ij}(\boldsymbol{u})\,| \leq C\,|\,u_i\,| \qquad \forall \quad i,j = 1,\dots,m,\; i \neq j \tag{3}$$

for some constant $C > 0$. For basic properties of (1), we refer the reader to [11] and references therein. One example is given by the class of rotationally invariant systems considered in [6], [13], namely

$$\boldsymbol{u}_t + \left(\boldsymbol{u}\,|\boldsymbol{u}|^2\right)_x = \mu\,\boldsymbol{u}_{xx} \tag{4}$$

where μ is a positive constant. The inviscid form of equation (4) was considered in 1979 by Keyfitz and Kranzer in connection with the elastic string problem in elasticity [9]. This system has also been studied in one-dimensional multiphase flow [7], [12], magnetohydrodynamics [1], [5] and more generally in continuum mechanics as a basic model for the propagation of plane waves in isotropic, multidimensional systems [1], [2]. For these and all systems of the form (1)–(3), we will show in Section 2 that

$$\| \boldsymbol{u}(\cdot, t) \|_{L^2(\mathbf{R})} \leq C \, (1 + t)^{-1/4} \tag{5}$$

for all $t > 0$, where the constant C depends on the magnitude of $\| \boldsymbol{u}(\cdot, 0) \|_{L^1(\mathbf{R})}$ and $\| \boldsymbol{u}(\cdot, 0) \|_{L^2(\mathbf{R})}$, the dimension m and $\mu > 0$ given in (2) above. Thus, the solution $\boldsymbol{u} = \mathbf{0}$ of (1)–(3) is asymptotically stable under arbitrarily large disturbances, provided only that they belong to $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. The same decay behavior in the L^2-norm has been shown to hold for general systems of conservation laws

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = (B(\boldsymbol{u}) \, \boldsymbol{u}_x)_x \tag{6}$$

with an arbitrary viscosity matrix $B(\boldsymbol{u})$ and flux function $\boldsymbol{f}(\boldsymbol{u})$ such that the Jacobian matrix $\boldsymbol{f}'(\mathbf{0})$ is completely hyperbolic [3], [4], [8], under the further assumption that $\boldsymbol{u}(\cdot, 0)$ is small enough to satisfy

$$\int_{-\infty}^{+\infty} | \boldsymbol{u}(x,0) | \, (1 + |x|) \, dx + \int_{-\infty}^{+\infty} \Big(| \boldsymbol{u}(x,0) |^2 + | \boldsymbol{u}_x(x,0) |^2 \Big) \, dx \leq \delta \tag{7}$$

for $\delta \ll 1$. In the particular case of systems verifying (1)–(3), condition (7) is not necessary and we will show in Section 2 using a very simple argument that the estimate (5) holds for *any* initial state $\boldsymbol{u}(\cdot, 0)$ in $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, however large. If one replaces (1a) by the slightly more general equation

$$\boldsymbol{u}_t + (\boldsymbol{u} \, \varphi(\boldsymbol{u}))_x = (B(\boldsymbol{u}) \, \boldsymbol{u}_x)_x \tag{8}$$

then we can still derive (5) provided that $\| \boldsymbol{u}(\cdot, 0) \|_{L^1(\mathbf{R})}$ is sufficiently small, see Section 2. Finally, in Section 3 we will use a similar argument to investigate the corresponding behavior for the N-dimensional analogue of (1),

$$\boldsymbol{u}_t + \sum_{k=1}^{N} \frac{\partial}{\partial x_k} \Big(\boldsymbol{u} \, \varphi_k(|\boldsymbol{u}|) \Big) = \sum_{k=1}^{N} \frac{\partial}{\partial x_k} \left(B^{[k]}(\boldsymbol{u}) \, \frac{\partial \boldsymbol{u}}{\partial x_k} \right) \tag{9}$$

where $\varphi_1, \dots, \varphi_N$ are scalar functions and $B^{[1]}(\boldsymbol{u}), \dots, B^{[N]}(\boldsymbol{u})$ are positive definite matrices of order m such that

$$\Big(\boldsymbol{\xi}\,, B^{[k]}(\boldsymbol{u})\,\boldsymbol{\xi}\Big) \;\geq\; \mu\,|\,\boldsymbol{\xi}\,|^2 \qquad \forall \;\; \boldsymbol{\xi} \in \mathbf{R}^m \;\;,\;\; k = 1, \dots, N \tag{10}$$

and whose elements satisfy, for each $k = 1, \dots, N$,

$$|\,b_{ij}^{[k]}(\boldsymbol{u})\,| \;\leq\; C\,|\,u_i\,| \qquad \forall \;\; i, j = 1, \dots, m\,,\; i \neq j \tag{11}$$

for all values of $\boldsymbol{u}$ in the region concerned, where μ, C are positive constants. It is then shown that

$$\|\,\boldsymbol{u}(\cdot, t)\,\|_{L^2(\mathbf{R}^N)} \;\leq\; C\,(\,1 + t\,)^{-N/4} \tag{12}$$

whenever $\boldsymbol{u}(\cdot, 0) \in L^1(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$, where C is a positive constant which depends on the magnitude of $\|\,\boldsymbol{u}(\cdot, 0)\,\|_{L^1(\mathbf{R}^N)}$ and $\|\,\boldsymbol{u}(\cdot, 0)\,\|_{L^2(\mathbf{R}^N)}$, the dimension parameters m, N and $\mu > 0$ given in (10). For all these problems, the decay rates given above are optimal, so that the method discussed here, in spite of its simplicity, gives sharp results. Moreover, it can be applied to other problems, see e.g. [14], [15].

As to the notation used, boldface characters will always denote vector quantities, while capital letters will be usually reserved for matrices, with the exception of letters C and K, which will be used for scalar constants. A symbol like $C_{\mathcal{A}}$ represents a constant whose value depends on a set of parameters specified by $\mathcal{A}$; we note that distinct references to the same constant symbol will not necessarily mean the same numerical value, so that we will write $2\,C_{\mathcal{A}}$ again as $C_{\mathcal{A}}$, and so on. Also, we will often use subscripted variables to indicate differentiation, as in $u_t = \dfrac{\partial u}{\partial t}$, $f(u)_x = \dfrac{\partial}{\partial x} f(u(x, t))$, and so forth. All other notation, when not standard, will be explained right after its first occurrence in the text.

2. One-dimensional systems

We will consider in this section the L^2 decay of solutions $\boldsymbol{u}(x, t)$ of the Cauchy problem (1)–(3) described above. The initial state $\boldsymbol{u}(\cdot, 0)$ is any Lebesgue measurable pulse with finite mass and energy, i.e., $\boldsymbol{u}(\cdot,0) \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, and we let $K > 0$ be sufficiently large so that

$$\| \boldsymbol{u}(\cdot,0) \|_{L^1(\mathbf{R})} \leq K \quad , \quad \| \boldsymbol{u}(\cdot,0) \|_{L^2(\mathbf{R})} \leq K \tag{13}$$

Under these conditions, we will show below that there exists a positive constant $C_{\mathcal{K}}$, depending on the set of parameters $\mathcal{K} = \{ m, K, \mu \}$, such that

$$\| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbf{R})} \leq C_{\mathcal{K}} \, (1 + t)^{-1/4} \qquad \forall \;\; t > 0 \tag{14}$$

We will prove this estimate in the following way. First, we note that the solution operator of (1) is L^1-*contractive*, i.e., we have

$$\| \boldsymbol{u}(\cdot,t) \|_{L^1(\mathbf{R})} \leq \| \boldsymbol{u}(\cdot,0) \|_{L^1(\mathbf{R})} \qquad \forall \;\; t > 0 \tag{15}$$

where

$$\| \boldsymbol{u}(\cdot,t) \|_{L^1(\mathbf{R})} = \| u_1(\cdot,t) \|_{L^1(\mathbf{R})} + \dots + \| u_m(\cdot,t) \|_{L^1(\mathbf{R})}$$

and similarly for $\| \boldsymbol{u}(\cdot,0) \|_{L^1(\mathbf{R})}$. In fact, more is true: one has, for each individual component $u_i(\cdot,t)$ of the solution vector $\boldsymbol{u}(\cdot,t)$, $i = 1, \dots, m$, the estimate

$$\| u_i(\cdot,t) \|_{L^1(\mathbf{R})} \leq \| u_i(\cdot,0) \|_{L^1(\mathbf{R})} \qquad \forall \;\; t > 0 \tag{15$'$}$$

This can be proved in a standard way as in [10], but for convenience of the reader we will briefly review the argument. Taking a regularized sign function L'_δ (see e.g. [10], [14]), we multiply the i^{th} component of equation (1a) by $L'_\delta(u_i(x,t))$ and integrate the result over $\mathbf{R} \times [\,0, T\,]$ to get, after a few integrations by parts,

$$\begin{aligned}
&\int_{-\infty}^{+\infty} L_\delta(\, u_i(x,T)\,)\, dx \;+\; \int_0^T \!\! \int_{-\infty}^{+\infty} L''_\delta(\, u_i(x,t)\,)\, b_{ii}(\, \boldsymbol{u}(x,t)\,) \left(\frac{\partial u_i}{\partial x} \right)^2 dx\, dt \;= \\
&= \int_{-\infty}^{+\infty} L_\delta(\, u_i(x,0)\,)\, dx \;+\; \int_0^T \!\! \int_{-\infty}^{+\infty} L''_\delta(\, u_i(x,t)\,)\, u_i(x,t) \frac{\partial u_i}{\partial x} \varphi(\, |\, \boldsymbol{u}(x,t)\,|\,)\, dx\, dt \\
&\qquad + \sum_{\substack{j=1 \\ j \neq i}}^{m} \int_0^T \!\! \int_{-\infty}^{+\infty} L''_\delta(\, u_i(x,t)\,)\, b_{ij}(\boldsymbol{u}(x,t)) \frac{\partial u_i}{\partial x} \frac{\partial u_j}{\partial x}\, dx\, dt
\end{aligned}$$

Since $L''_\delta(u_i)$ and $b_{ii}(\boldsymbol{u})$ are nonnegative, we then obtain, using (3) above,

$$\int_{-\infty}^{+\infty} L_\delta(\, u_i(x,T)\,)\, dx \;\leq\; \int_{-\infty}^{+\infty} L_\delta(\, u_i(x,0)\,)\, dx \; +$$

$$+ \int_0^T \int_{-\infty}^{+\infty} L''_\delta(\, u_i(x,t)\,)\, u_i(x,t)\, \frac{\partial u_i}{\partial x}\, \varphi(\,|\,\boldsymbol{u}(x,t)\,|\,)\, dx\, dt$$

$$+ \; C \sum_{\substack{j=1 \\ j \neq i}}^{m} \int_0^T \int_{-\infty}^{+\infty} L''_\delta(\, u_i(x,t)\,)\, |\, u_i(x,t)\,| \left| \frac{\partial u_i}{\partial x} \right| \left| \frac{\partial u_j}{\partial x} \right| \, dx\, dt$$

where C is the constant given in (3). Letting $\delta \to 0$, we get (15)$'$, since

$$\int_0^T \int_{-\infty}^{+\infty} L''_\delta(\, u_i(x,t)\,)\, u_i(x,t)\, \frac{\partial u_i}{\partial x}\, \varphi(\,|\,\boldsymbol{u}(x,t)\,|\,)\, dx\, dt \;\to\; 0$$

and

$$\int_0^T \int_{-\infty}^{+\infty} L''_\delta(\, u_i(x,t)\,)\, |\, u_i(x,t)\,| \left| \frac{\partial u_i}{\partial x} \right| \left| \frac{\partial u_j}{\partial x} \right| \, dx\, dt \;\to\; 0$$

by Lebesgue's Dominated Convergence Theorem. Another property which can be easily derived is the following energy estimate,

$$\|\, \boldsymbol{u}(\cdot,T)\,\|^2_{L^2(\mathbf{R})} \; + \; 2\,\mu \int_0^T \|\, D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbf{R})}\, dt \;\; \leq \;\; \|\, \boldsymbol{u}(\cdot,0)\,\|^2_{L^2(\mathbf{R})} \tag{16}$$

where

$$\|\, \boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbf{R})} \;=\; \sum_{i=1}^{m} \|\, u_i(\cdot,t)\,\|^2_{L^2(\mathbf{R})}$$

and

$$\|\, D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbf{R})} \;=\; \sum_{i=1}^{m} \|\, \frac{\partial u_i}{\partial x}(\cdot,t)\,\|^2_{L^2(\mathbf{R})}$$

In fact, multiplying the i^{th} component of equation (1a) by $u_i(x,t)$, integrating the result over $\mathbf{R} \times [\,0,T\,]$ and summing from $i=1$ to m, we get, after a few similar computations,

$$\int_{-\infty}^{+\infty} |\, \boldsymbol{u}(x,T)\,|^2\, dx \; + \; 2 \int_0^T \int_{-\infty}^{+\infty} (\, \boldsymbol{u}_x, B(\boldsymbol{u})\, \boldsymbol{u}_x\,)\; dx\, dt \;=$$

$$= \int_{-\infty}^{+\infty} |\, \boldsymbol{u}(x,0)\,|^2\, dx \; + \; \int_0^T \int_{-\infty}^{+\infty} \varphi(\,|\,\boldsymbol{u}(x,t)\,|\,)\, \frac{\partial}{\partial x}\, |\, \boldsymbol{u}(x,t)\,|^2\, dx\, dt$$

from which we immediately get (16), in view of assumption (2) and

$$\int_{-\infty}^{+\infty} \varphi(\,|\,\boldsymbol{u}(x,t)\,|\,)\,\frac{\partial}{\partial x}\,|\,\boldsymbol{u}(x,t)\,|^2\,dx \;=\; 0 \tag{17}$$

In order to obtain a decay rate for $\|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbf{R})}$, however, we need to do a more careful analysis. To this end, we multiply the i^{th} component of equation (1a) by $(1+t)\,u_i(x,t)$ and integrate the result over $\mathbf{R}\,\mathrm{x}\,[\,0,T\,]$, which gives, summing from $i=1$ to m,

$$\begin{aligned}
&(\,1\,+\,T\,)\int_{-\infty}^{+\infty}|\,\boldsymbol{u}(x,T)\,|^2\,dx\,+\\
&+\,2\int_0^T(\,1\,+\,t\,)\int_{-\infty}^{+\infty}(\,\boldsymbol{u}_x,B(\boldsymbol{u})\,\boldsymbol{u}_x\,)\;dx\,dt\;=\\
&=\;\int_{-\infty}^{+\infty}|\,\boldsymbol{u}(x,0)\,|^2\,dx\;+\;\int_0^T\int_{-\infty}^{+\infty}|\,\boldsymbol{u}(x,t)\,|^2\,dx\,dt\,+\\
&+\int_0^T(\,1\,+\,t\,)\int_{-\infty}^{+\infty}\varphi(\,|\,\boldsymbol{u}(x,t)\,|\,)\,\frac{\partial}{\partial x}\,|\,\boldsymbol{u}(x,t)\,|^2\,dx\,dt
\end{aligned}$$

so that, recalling (2) and (17), we obtain

$$\begin{aligned}
&(\,1\,+\,T\,)\,\|\,\boldsymbol{u}(\cdot,T)\,\|_{L^2(\mathbf{R})}^2\;+\;2\,\mu\int_0^T(\,1\,+\,t\,)\,\|\,D\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbf{R})}^2\,dt\\
&\le\;\|\,\boldsymbol{u}(\cdot,0)\,\|_{L^2(\mathbf{R})}^2\;+\;\int_0^T\|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbf{R})}^2\,dt
\end{aligned} \tag{18}$$

Using the elementary Sobolev inequality

$$\|\,u_i(\cdot,t)\,\|_{L^2(\mathbf{R})}\;\le\;C\,\|\,u_i(\cdot,t)\,\|_{L^1(\mathbf{R})}^{2/3}\,\|\,\frac{\partial\,u_i}{\partial\,x}(\cdot,t)\,\|_{L^2(\mathbf{R})}^{1/3} \tag{19}$$

we get, from (13), (15),

$$\|\,\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbf{R})}\;\le\;C_{m,K}\,\|\,D\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbf{R})}^{1/3}$$

for each $t>0$, where $C_{m,K}$ denotes a constant which depends on m, K. Hence, (18) yields

$$\begin{aligned}
&(\,1\,+\,T)\,\|\,\boldsymbol{u}(\cdot,T)\,\|_{L^2(\mathbf{R})}^2\;+\;2\,\mu\int_0^T(1+t)\,\|\,D\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbf{R})}^2\,dt\\
&\le\;\|\,\boldsymbol{u}(\cdot,0)\,\|_{L^2(\mathbf{R})}^2\;+\;C_{m,K}\int_0^T\|\,D\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbf{R})}^{2/3}\,dt
\end{aligned} \tag{20}$$

Since, by Hölder's inequality, we have

$$\int_0^T \| D\boldsymbol{u}(\cdot,t)\|^{2/3}_{L^2(\mathbf{R})}\, dt \;\leq\; 2^{2/3} \left\{ (\,1\,+\,T\,) \int_0^T (\,1\,+\,t\,)\, \| D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbf{R})}\, dt \right\}^{1/3}$$

we see that, setting

$$E(T) \;\equiv\; (\,1\,+\,T\,)\, \| \boldsymbol{u}(\cdot,T)\|^2_{L^2(\mathbf{R})} \;+\; \int_0^T (\,1\,+\,t\,)\, \| D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbf{R})}\, dt \qquad (21)$$

we get, from (20),

$$E(T) \;\leq\; C_{\mathcal{K}} \left\{ 1 \,+\, (\,1\,+\,T\,)^{1/3}\, E(T)^{1/3} \right\}$$

for some constant $C_{\mathcal{K}}$ which depends on $\mathcal{K} = \{m, K, \mu\}$. This immediately gives

$$E(T) \;\leq\; C_{\mathcal{K}}\, (\,1\,+\,T\,)^{1/2} \qquad (22)$$

for some suitable constant $C_{\mathcal{K}}$ which, again, depends on $\mathcal{K} = \{\, m, K, \mu\,\}$. Recalling (21), we then have the following result.

Theorem 1. *Let $\boldsymbol{u}(x,t)$ be the solution of* (1)–(3) *corresponding to an initial profile $\boldsymbol{u}(\cdot,0)$ in $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Then, there exists a constant $C_{\mathcal{K}}$ (depending on the set of parameters $\mathcal{K} = \{\, m, K, \mu\,\}$ given in* (2), (13) *) such that*

$$(\,1\,+\,T\,)\, \| \boldsymbol{u}(\cdot,T)\|^2_{L^2(\mathbf{R})} \;+\; \int_0^T (\,1\,+\,t\,)\, \| D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbf{R})}\, dt \;\leq\; C_{\mathcal{K}}\, (\,1\,+\,T\,)^{1/2}$$

for every $T > 0$.

In particular, this shows (5). As mentioned in the previous section, we can extend the above analysis to the slightly more general equation

$$\boldsymbol{u}_t \;+\; (\,\boldsymbol{u}\, \varphi(\boldsymbol{u})\,)_x \;=\; (\,B(u)\, \boldsymbol{u}_x\,)_x \qquad (23)$$

where $B(\boldsymbol{u})$ satisfies (2), (3) and φ now depends more generally on the conserved variables $\boldsymbol{u}$ instead of their magnitude $|\,\boldsymbol{u}\,|$ only. Observe that, changing the variable x to $\xi = x - \varphi(\mathbf{0})\, t$ if necessary, we may assume without any loss of generality that $\varphi(\mathbf{0}) = 0$. Thus, we will make the assumption that, for all $\boldsymbol{u}$ concerned, we have

$$|\,\varphi(\boldsymbol{u})\,| \;\leq\; \Phi\,|\,\boldsymbol{u}\,| \tag{24}$$

for some constant $\Phi > 0$. As before, the initial state $\boldsymbol{u}(\cdot,0)$ is taken in the space $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, and we let $K > 0$ be large enough so that

$$\|\,\boldsymbol{u}(\cdot,0)\,\|_{L^2(\mathbf{R})} \;\leq\; K \tag{25}$$

Multiplying the i^{th} component of equation (23) by $(1+t)\,u_i(x,t)$ and integrating the result over $\mathbf{R}\,\mathrm{x}\,[\,0,T\,]$, we then obtain, summing from $i = 1$ to m,

$$\begin{aligned}(\,1\,+\,T\,)\,\|\,\boldsymbol{u}(\cdot,T)\,\|^2_{L^2(\mathbf{R})} \;+\; 2\,\mu\int_0^T(\,1\,+\,t\,)\,\|\,D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbf{R})}\;dt \\ \leq\; \|\,\boldsymbol{u}(\cdot,0)\,\|^2_{L^2(\mathbf{R})} \;+\; \int_0^T \|\,\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbf{R})}\;dt \\ +\,2\int_0^T(\,1\,+\,t\,)\int_{-\infty}^{+\infty}(\,\boldsymbol{u}\,,\boldsymbol{u}_x\,)\;\varphi(\boldsymbol{u})\;dx\,dt\end{aligned} \tag{26}$$

where we have used (2). Since, for μ, Φ given in (2), (24) above, we have

$$\int_{-\infty}^{+\infty}|\,(\,\boldsymbol{u},\boldsymbol{u}_x\,)\,|\;\;|\,\varphi(\boldsymbol{u})\,|\;dx \;\;\leq$$

$$\leq\;\; \frac{\Phi^2}{2\,\mu}\,\|\,\boldsymbol{u}(\cdot,t)\,\|^4_{L^4(\mathbf{R})} \;+\; \frac{\mu}{2}\,\|\,D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbf{R})}$$

we see that (26) yields

$$\begin{aligned}(\,1\,+\,T\,)\,\|\,\boldsymbol{u}(\cdot,T)\,\|^2_{L^2(\mathbf{R})} \;+\; \mu\int_0^T(\,1\,+\,t\,)\,\|\,D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbf{R})}\;dt \\ \leq\; \|\,\boldsymbol{u}(\cdot,0)\,\|^2_{L^2(\mathbf{R})} \;+\; \int_0^T \|\,\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbf{R})}\;dt \\ +\,\tfrac{\Phi^2}{\mu}\int_0^T(\,1\,+\,t\,)\,\|\,\boldsymbol{u}(\cdot,t)\,\|^4_{L^4(\mathbf{R})}\;dt\end{aligned} \tag{26$'$}$$

Using the Sobolev inequality

$$\|\,\boldsymbol{u}(\cdot,t)\,\|^2_{L^4(\mathbf{R})} \;\leq\; C_m\,\|\,\boldsymbol{u}(\cdot,t)\,\|_{L^1(\mathbf{R})}\,\|\,D\boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbf{R})} \tag{27}$$

we then obtain, from $(26)'$,

$$\begin{aligned} &(1+T)\,\|\boldsymbol{u}(\cdot,T)\|^2_{L^2(\mathbf{R})} + \mu\int_0^T (1+t)\,\|D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbf{R})}\,dt \\ &\le \|\boldsymbol{u}(\cdot,0)\|^2_{L^2(\mathbf{R})} + \int_0^T \|\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbf{R})}\,dt \\ &+ \tfrac{\Phi^2}{\mu}\,C_m^2\int_0^T (1+t)\,\|\boldsymbol{u}(\cdot,t)\|^2_{L^1(\mathbf{R})}\,\|D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbf{R})}\,dt \end{aligned} \tag{28}$$

Since (15) remains valid for equation (23), we then obtain, from (13), (25), (28),

$$\begin{aligned} &(1+T)\,\|\boldsymbol{u}(\cdot,T)\|^2_{L^2(\mathbf{R})} + \int_0^T (1+t)\,\|D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbf{R})}\,dt \\ &\le C_{\mu,K}\left\{1 + \int_0^T \|\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbf{R})}\,dt\right\} \end{aligned} \tag{29}$$

provided we assume that $\|\boldsymbol{u}(\cdot,0)\|_{L^1(\mathbf{R})}$ is small enough to satisfy

$$\frac{\Phi}{\mu}\,C_m\,\|\boldsymbol{u}(\cdot,0)\|_{L^1(\mathbf{R})} \le \frac{1}{2} \tag{30}$$

where $\mu, \Phi, C_m > 0$ are the constants given in (2), (24), (27). Proceeding with (29) as we did with (20) above, we then get the following result.

Theorem 2. *Let $\boldsymbol{u}(x,t)$ be the solution of (23) corresponding to an initial profile $\boldsymbol{u}(\cdot,0)$ in the space $L^1(\mathbf{R})\cap L^2(\mathbf{R})$. Then, there exists $\delta > 0$ (depending on μ, Φ, C_m given in (2), (24), (27)) such that, whenever $\|\boldsymbol{u}(\cdot,0)\|_{L^1(\mathbf{R})} \le \delta$, one has*

$$(1+T)\,\|\boldsymbol{u}(\cdot,T)\|^2_{L^2(\mathbf{R})} + \int_0^T (1+t)\,\|D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbf{R})}\,dt \le C_{\mathcal{K}}\,(1+T)^{1/2}$$

for every $T > 0$, where $C_{\mathcal{K}}$ is a positive constant which depends on the set of parameters $\mathcal{K} = \{m, K, \mu\}$ given in (2), (25).

3. Multidimensional systems

In this section, we will extend the analysis above to multidimensional systems of the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{k=1}^{N} \frac{\partial}{\partial x_k}\Big(\boldsymbol{u}\,\varphi_k(|\boldsymbol{u}|)\Big) = \sum_{k=1}^{N} \frac{\partial}{\partial x_k}\left(B^{[k]}(\boldsymbol{u})\,\frac{\partial \boldsymbol{u}}{\partial x_k}\right) \tag{31}$$

where $\varphi_1, \dots, \varphi_N$ are given scalar functions, $\boldsymbol{u}(\boldsymbol{x},t) = (\, u_1(\boldsymbol{x},t), \dots, u_m(\boldsymbol{x},t)\,)$ is the vector of unknowns, $\boldsymbol{x} = (x_1, \dots, x_N)$, and $B^{[k]}(\boldsymbol{u})$, $k = 1, \dots, N$, are positive definite matrices of order m satisfying (10), (11) for all $\boldsymbol{u}$ concerned, i.e.,

$$\Big(\boldsymbol{\xi}\,, B^{[k]}(\boldsymbol{u})\,\boldsymbol{\xi} \Big) \;\geq\; \mu\,|\,\boldsymbol{\xi}\,|^2 \qquad \forall \quad \boldsymbol{\xi} \in \mathbf{R}^m \;\;,\;\; k \,=\, 1, \dots, N \tag{32}$$

and

$$|\, b^{[k]}_{ij}(\boldsymbol{u})\,| \;\leq\; C\,|\,u_i\,| \qquad \forall \quad i,j \,=\, 1, \dots, m\,,\; i \neq j\;,\;\; k \,=\, 1, \dots, N \tag{33}$$

for some positive constants μ, C, where $b^{[k]}_{ij}(\boldsymbol{u})$, $i,j = 1, \dots, m$, denote the elements of $B^{[k]}(\boldsymbol{u})$, $k = 1, \dots, N$. One then obtain

$$\|\, \boldsymbol{u}(\cdot,t)\,\|_{L^1(\mathbf{R}^N)} \;\leq\; \|\, \boldsymbol{u}(\cdot,0)\,\|_{L^1(\mathbf{R}^N)} \qquad \forall \quad t > 0 \tag{34}$$

which can be shown in the same way as (15) above. In fact, it is straightforward to extend the one-dimensional analysis of the previous section to the N-dimensional equation (31), so that we will only give here a brief sketch of the argument in this case. To get the appropriate decay rates, we multiply the i^{th} component of (31) by $(\,1\,+\,t\,)^N u_i(\boldsymbol{x},t)$ and integrate the result over $\mathbf{R}^N \mathrm{x}\,[\,0, T\,]$ to get, summing from $i = 1$ to m,

$$\begin{aligned} &(\,1\,+\,T\,)^N\,\|\, \boldsymbol{u}(\cdot,T)\,\|^2_{L^2(\mathbf{R}^N)} \;+\; 2\,\mu \int_0^T (\,1\,+\,t\,)^N\,\|\, D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbf{R}^N)}\;dt \\ &\qquad \leq\; \|\, \boldsymbol{u}(\cdot,0)\,\|^2_{L^2(\mathbf{R}^N)} \;+\; N \int_0^T (\,1\,+\,t\,)^{N-1}\,\|\, \boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbf{R}^N)}\;dt \end{aligned} \tag{35}$$

where

$$\|\, D\boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbf{R}^N)} \;=\; \sum_{i=1}^m \sum_{j=1}^N \,\|\, \frac{\partial\, u_i}{\partial\, x_j}\,(\cdot,t)\,\|^2_{L^2(\mathbf{R}^N)}$$

Using (34) and the Sobolev inequality

$$\|\, \boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbf{R}^N)} \;\leq\; C_{m,N}\,\|\, \boldsymbol{u}(\cdot,t)\|^{\frac{2}{N+2}}_{L^1(\mathbf{R}^N)}\,\|\, D\boldsymbol{u}(\cdot,t)\,\|^{\frac{N}{N+2}}_{L^2(\mathbf{R}^N)} \tag{36}$$

we get

$$\|\, \boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbf{R}^N)} \;\leq\; C_{m,N,K}\,\|\, D\boldsymbol{u}(\cdot,t)\|^{\frac{N}{N+2}}_{L^2(\mathbf{R}^N)}$$

where $C_{m,N,K}$ denotes a constant which depends on m, N and $K > 0$ such that

$$\| \boldsymbol{u}(\cdot,0) \|_{L^1(\mathbf{R}^N)} \leq K \quad , \quad \| \boldsymbol{u}(\cdot,0) \|_{L^2(\mathbf{R}^N)} \leq K \tag{37}$$

Hence, (35) gives

$$\begin{aligned} &(1+T)^N \|\boldsymbol{u}(\cdot,T)\|^2_{L^2(\mathbf{R}^N)} + 2\mu \int_0^T (1+t)^N \| D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbf{R}^N)}\, dt \\ &\quad \leq \|\boldsymbol{u}(\cdot,0)\|^2_{L^2(\mathbf{R}^N)} + C_{m,N,K} \int_0^T (1+t)^{N-1} \| D\boldsymbol{u}(\cdot,t)\|^{\frac{2N}{N+2}}_{L^2(\mathbf{R}^N)}\, dt \end{aligned} \tag{38}$$

where $C_{m,N,K}$ denotes a constant which depends on m, N, K. Since

$$\begin{aligned} &\int_0^T (1+t)^{N-1} \| D\boldsymbol{u}(\cdot,t)\|^{\frac{2N}{N+2}}_{L^2(\mathbf{R}^N)}\, dt \leq \\ &\quad \leq \left(\tfrac{2}{N}\right)^{\frac{2}{N+2}} \left\{ (1+T) \int_0^T (1+t)^N \| D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbf{R}^N)}\, dt \right\}^{\frac{N}{N+2}} \end{aligned} \tag{39}$$

we then get, from (38),

$$\begin{aligned} &(1+T)^N \|\boldsymbol{u}(\cdot,T)\|^2_{L^2(\mathbf{R}^N)} + \int_0^T (1+t)^N \| D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbf{R}^N)}\, dt \\ &\leq C_{\mathcal{K}} \left\{ 1 + (1+T)^{\frac{N}{N+2}} \left(\int_0^T (1+t)^N \| D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbf{R}^N)} dt \right)^{\frac{N}{N+2}} \right\} \end{aligned} \tag{40}$$

where $C_{\mathcal{K}}$ is some positive constant which depends on the set of parameters $\mathcal{K} = \{ m, N, K, \mu \}$. Proceeding as in (20)–(22), we then immediately obtain the following result.

Theorem 3. *Let $\boldsymbol{u}(\boldsymbol{x},t)$ be the solution of equation (31)–(33) corresponding to an initial state $\boldsymbol{u}(\cdot,0)$ in the space $L^1(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$. Then, there exists a positive constant $C_{\mathcal{K}}$ (depending on $\mathcal{K} = \{ m, N, K, \mu \}$ given in (32), (37)) such that*

$$(1+T)^N \| \boldsymbol{u}(\cdot,T)\|^2_{L^2(\mathbf{R})} + \int_0^T (1+t)^N \| D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbf{R})}\, dt \leq C_{\mathcal{K}} (1+T)^{N/2}$$

for every $T > 0$.

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