MULTIDIMENSIONAL KURAMOTO-SIVASHINSKY TYPE EQUATIONS: SINGULAR INITIAL DATA AND ANALYTIC REGULARITY

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Abstract

We consider the Cauchy problem for multidimensional Kuramoto-Sivashinsky type equations in $\mathbb{R}^n$ and in $\mathbb{T}^n$. The initial data can be singular, in particular, can belong to Sobolev spaces H_p^r, with negative r. We introduce weighted analytic-Gevrey type spaces which allow us to get new results both on the critical index of singularity of the initial data and on analytic regularith with respect to x when $t > 0$. We get also global well-posedness results in L^2 in the case of conservative nonlinearities.

Resumo

Consideramos o problema de Cauchy para a equações do tipo Kuramoto-Sivashinsky multidimensionais em $\mathbb{R}^n$ e em $\mathbb{T}^n$. Os dados iniciais podem ser singulares, em particular, podem pertencer a espaços de Sobolev H_p^r, com r negativo. Introduzimos espaços do tipo Gevrey-analíticos com peso, que nos permitem obter novos resultados tanto quanto ao índice crítico de singularidade do dado inicial, como resultados sobre regularidade analítica de soluções com respeito a x para $t > 0$. Obtemos também resultados de boa postura global em L^2 no caso da não linearidade ser conservativa.

1. Introduction

We consider the following initial value problem (IVP)

$$\partial_t u + \Delta^2 u + P(D)u + \nabla F(u) = 0, \qquad t > 0,\ x \in \Omega, \tag{1.1}$$

$$u(0, \cdot) = u^0, \tag{1.2}$$

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where $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$, $P(D) = \sum_{|\alpha|\leq 3} p_\alpha D^\alpha$, $p_\alpha \in \mathbb{C}$, $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \leq 3$, $D = (D_{x_1}, \ldots, D_{x_n})$, $D_{x_k} = i^{-1}\partial_{x_k}$, $F(u) = (F_1(u), \ldots, F_n(u))$, $F_\ell(u)$ are homogeneous polynomials of order $s \geq 2$, $\nabla F(u) = \sum_{\ell=1}^n \partial_{x_\ell}(F_\ell(u))$. In order to simplify the presentation of the main novelties we consider scalar equations in (1.1)

We recall that the "derived" 1-D Kuramoto-Sivashinsky (KS) equation cf. E. Tadmor [22] and the references therein, the equation $\partial_t\phi + \phi\partial_{x_1}\phi + \partial_{x_1}^2\phi + \Delta^2\phi = 0$, $x = (x_1, x_2)$, which describes the evolution of the disturbed surface of a film flowing down an infinite flat vertical wall cf. T. Shlang and G. I. Sivashinsky [21], and the "derived" Korteweg-de Vries-Kuramoto-Sivashinsky (KdV-KS) equation, studied by H. A. Biagioni, J. Bona, R. Iorio and M. Scialom [3], can be reduced to (1.1). Systems of the type (1.1) with $u^0 \in H^s(\Omega)$, $s > n/2$ have been investigated by B. Guo [13].

The present work has two aims. Firstly, we define the critical L^p index for the equation (1.1) as $p_{cr} = \frac{n(s-1)}{3}$ and show that if $p \geq \max\{1, p_{cr}\}$ we can find explicitly a nonnegative number $r_{cr}(p) \leq 0$ such that we can resolve (1.1), (1.2) with initial data modelled by $u^0 \in H_p^r(\Omega)$, $p > 1$ or $u^0 = |D|^{-r}\mu$, $\mu \in \mathcal{M}(\Omega)$ if $p = 1$, $0 \geq r \geq r_{cr}(p)$, $\mathcal{M}(\Omega)$ being the space of the finite Radon measures in Ω, $|D| = (-\Delta)^{\frac{1}{2}}$. The question whether r might reach $r_{cr}(p)$ is not easy to answer (for the analogous problem on the L^p critical index of singularity for semilinear heat equations we refer e.g. to H. Kozono and M. Yamazaki [16], F. Ribaud [18], [19], [20], D. Bekhiranov [2], D. Dix [10], H. Brezis and T. Cazenave [7], H. A. Biagioni, L. Cadeddu and T. Gramchev [5], [6], J. Arrieta and A. N. Carvalho [1], while for the complex Ginzburg-Landau equation see D. Levermore and M. Oliver [17]). In fact, one essential novelty of the present paper is the construction of weighted analytic-Gevrey type spaces which generalize the weighted spaces of Kato-Fujita type. These new spaces allow us not only to resolve the IVP (1.1), (1.2) for large class of singular initial data but also to obtain new results for analytic regularity with respect to the space variables x when $t > 0$. Such unified approach for studying simultaneously solutions of (1.1), (1.2) for strongly singular initial data and their analytic regularity in x

for $t > 0$ seems to be new in comparison with the methods used for answering the latter question cf. C. Foias and R. Temam [12], P. Collet, J.-P. Eckmann, H. Epstein and J. Stubbe [9], P. Takàč, P. Bollerman, A. Doelman, A. van Harten and E. Titi [23] and A. Ferrari and E. Titi [11]. We also mention that, as a consequence of our results, if $\Omega = \mathbb{R}^n$, we can solve (1.1) with initial data homogeneous distributions of order $-\frac{n}{p_{cr}}$ under suitable hypotheses, in the spirit of the paper of M. Cannone and F. Planchon [8] on self-similar solutions for the 3-D Navier-Stokes equation. However, since $\Delta^2 + P(D)$ is not homogeneous if $P(D) \not\equiv 0$, we do not obtain, as in [8], self-similar solutions. The value of the critical index p_{cr} could be determined by the usual scaling argument applied to (1.1) with $P(D) = 0$, namely to look for self-similar solutions of the form $u(t,x) = t^{-\frac{n}{p}} g(\frac{x}{\sqrt[4]{t}})$, which is possible only for $p = p_{cr}$.

Finally, we exhibit new results on global in time solutions in the case of L^2 conservative nonlinearities. More precisely, if $s < 1 + \frac{6}{n}$, which is equivalent to $p_{cr} < 2$, we show global well-posedness for (1.1), (1.2) with arbitrary $u^0 \in L^2(\Omega)$ while in the critical case $n = \frac{6}{s-1}$ i.e. $p_{cr} = 2$ we require smallness of $\|u^0\|_{L^2}$ and nonnegativity of $Re(\Delta^2 + P(D))$. Moreover, differences occur in the critical case between $\mathbb{R}^n$ and $\mathbb{T}^n$, namely we show stronger results if $\Omega = \mathbb{T}^n$ provided in addition the initial data have zero mean value on $\mathbb{T}^n$. For example, if $P(D) = c\Delta$, $c > 0$, $s = 2$ and $n = 6$, we are not able to show global existence for $\Omega = \mathbb{R}^6$ while for $\Omega = \mathbb{T}^6$ we get global results for all u^0 with small $L^2(\mathbb{T}^6)$ norm and zero mean value if $0 < c \leq 1$. In particular, we generalize the global well-posedness results in [22] for the one dimensional "derived" (KS) equation, results in [3] for the one dimensional "derived" (KdV-KS) equation, and results in the multidimensional case in [13], where $u^0 \in H^s(\mathbb{R}^n)$, $s > \frac{n}{2}$ and $s < 1 + \frac{6}{n}$.

Our methods are applicable for quite general systems of evolution equations with dissipative elliptic terms and this will be done in other works.

The weighted analytic-Gevrey type spaces are introduced in section 2 and the general results for the existence, the uniqueness and the analytic regularity of the solutions to (1.1) with singular initial data (1.2) are stated in section 3. Sections 4 and 5 deal with the proofs of these results. The last section 6 is devoted to the regularity of the weak solutions and the global well-posedness for L^2 initial data and conservative nonlinearities.

2. Weighted spaces and critical indices

For given $q \in [1,+\infty]$, $\gamma \geq 0$, $\theta \geq 0$ and $T \in]0,+\infty]$ we define the Gevrey Banach space $A^{\gamma}_{\theta,q}(T) = A^{\gamma}_{\theta,q}(\Omega;T) := \{u \in C(]0,T[: L^q(\Omega)) \bigcap C([0,T[: \mathcal{S}'(\Omega)) : \|u\|_{A^{\gamma}_{\theta,q}(T)} < +\infty\}$, where

$$\|u\|_{A^{\gamma}_{\theta,q}(T)} \ := \sum_{\alpha \in \mathbb{Z}^n_+} \frac{\gamma^{|\alpha|}}{\alpha!} \sup_{0<t<T} (t^{\frac{|\alpha|}{4}+\theta} \|\partial^{\alpha} u(t)\|_{L^q}). \tag{2.1}$$

If $\gamma = 0$, with the convention $0^0 = 1$, we obtain that $A^{0}_{\theta,q}(T) = C_{\theta}(L^q;T)$, with $C_{\theta}(L^q;T) = C_{\theta}(L^q(\Omega);T)$ being the Kato-Fujita weighted space with norm $\|u\|_{C_{\theta}(L^q;T)} = \sup\limits_{0<t<T} (t^{\theta}\|u(t)\|_{L^q})$. Here $\mathcal{S}'(\mathbb{R}^n)$ (respectively $\mathcal{S}'(\mathbb{T}^n) = \mathcal{D}'(\mathbb{T}^n)$) stands for the space of all tempered (respectively periodic) distributions in $\mathbb{R}^n$ (respectively $\mathbb{T}^n$) while $\|f\|_{L^q}$ stands for the $L^q(\Omega)$ norm of f. If $T = +\infty$ we set $A^{\gamma}_{\theta,q} := A^{\gamma}_{\theta,q}(+\infty)$, $C_{\theta}(L^q(\Omega)) := C_{\theta}(L^q(\Omega);+\infty)$.

The Sobolev embedding theorems and the Cauchy formula for the radius of convergence of power series imply, for $\gamma > 0$, that if $u \in A^{\gamma}_{\theta,q}(T)$ then $u(t,x)$ is holomorphic in the strip $\Gamma_{\rho} := \{x \in \mathbb{C}^n : |Im(x)| < \rho\}$, $\rho = \gamma t^{\frac{1}{4}}$, $t \in]0,T[$. Given $u \in C(]0,T[: L^1_{loc}(\Omega))$ and $t \in]0,T[$, we define $\rho_{[u]}(t) = \sup\{\rho > 0 : u(t,\cdot) \in \mathcal{O}(\Gamma_{\rho})\}$ with $\rho_{[u]}(t) := 0$ if it cannot be extended to a function in $\mathcal{O}(\Gamma_{\rho})$ for any $\rho > 0$. Here $\mathcal{O}(\Gamma)$ stands for the space of all holomorphic functions in an open set $\Gamma \subset \mathbb{C}^n$. Clearly $\rho_{[u]}(t) \geq \gamma t^{\frac{1}{4}}$, $t \in]0,T]$ provided $u \in A^{\gamma}_{\theta,q}(T)$.

Typically for perturbative methods dealing with (1.1), (1.2) we want to find the space of all $u^0 \in \mathcal{S}'(\Omega)$ such that $E^{\Omega}[u^0] \in A^{\gamma}_{\frac{\theta}{4},q}(T)$ for some (all) $T \in]0,+\infty[$, where $E^{\Omega}[f](t) := E^{\Omega}(t) * f$, $E^{\Omega}(t) = \mathcal{F}^{-1}_{\xi \to x}(e^{-t(|\xi|^4+P(\xi))})$ is the fundamental solution of $\partial_t + \Delta^2 + P(D)$. If $\gamma = 0$ we refer to [15], [4], [18], [5] for such approach in studying semilinear heat equations with singular initial data. The norm of such Banach space depends on the $A^{\gamma}_{\frac{\theta}{4},q}(T)$ norm while as a set it depends only on Ω, θ, q but not on $\gamma \geq 0$ and $P(D)$. We denote it by $\mathcal{B}^{-\theta}_q(\Omega)$. Next we define $\dot{\mathcal{B}}^{-\theta}_q(\Omega)$ as the space of all $u \in \mathcal{S}'(\Omega)$ such that

$e^{-t\Delta^2}u \in A^{\gamma}_{\frac{\theta}{4},q}$, for some (all) $\gamma \geq 0$. We stress that $\dot{\mathcal{B}}^{-\theta}_q(\Omega)$ are related to the homogeneous Besov spaces with negative indices cf. H. Triebel [24] but we do not investigate these aspects here. Roughly speaking, as for the usual Besov spaces, if θ increases then $\mathcal{B}^{-\theta}_q(\Omega)$ contains distributions with stronger singularities. Let $H^r_p(\Omega)$, $1 < p < +\infty$, $r \in \mathbb{R}$ be the L^p based Sobolev space with norm $\|f\|_{H^r_p(\Omega)} = \|(1-\Delta)^{r/2}f\|_{L^p(\Omega)}$ while for $r \in \mathbb{Z}_+$, $1 \leq p \leq \infty$ we can use the norm $\|f\|_{H^r_p(\Omega)} = \max_{|\alpha|\leq r}\|\partial^\alpha f\|_{L^p(\Omega)}$. We set $H^r(\Omega) := H^r_2(\Omega)$. Given $r \leq 0$ we have $H^r_p(\Omega) \subset \mathcal{B}^{-\theta}_q(\Omega)$ if $p > 1$ and $|D|^{-r}\mathcal{M}(\Omega) \subset \mathcal{B}^{-\theta}_q(\Omega)$ if $p = 1$, for $q > p$, $\theta = -r + n(\frac{1}{p} - \frac{1}{q})$. This will follow from the estimates on $E = E^\Omega$.

Set $\theta(q) = \dfrac{n}{p_{cr}} - \dfrac{n}{q}$;

$$\rho(q,\theta) = \frac{3}{4} - \frac{\theta(s-1)}{4} - \frac{n(s-1)}{4q}. \tag{2.2}$$

Next we define $\Theta(n) = \{(q,\theta) : \ s \leq q \leq +\infty, 0 \leq s\theta < 4,\ \rho(q,\theta) \geq 0,\ \rho(q,0) > 0\}$; $\partial\Theta(n) := \{(q,\theta(q)) \in \Theta(n)\}$; $\dot{\Theta}(n) := \Theta(n) \setminus \partial\Theta(n)$ ($\Theta(n)$ will be the set of admissible pairs (θ, q) such that, if $u^0 \in \mathcal{B}^{-\theta}_q(\Omega)$, we can resolve (1.1), (1.2)); and set

$$\mathcal{P}_c = \inf_{\xi\in\mathbb{R}^n}(|\xi|^4 + \mathrm{Re}P(\xi)) \tag{2.3}$$
$$\mathcal{P}_d = \inf_{\xi\in\mathbb{Z}^n\setminus 0}(|\xi|^4 + \mathrm{Re}P(\xi)). \tag{2.4}$$

Clearly $\mathcal{P}_d \geq \mathcal{P}_c$; for example for $P = \Delta$ we have $\mathcal{P}_c = -\frac{1}{4}$ while $\mathcal{P}_d = 0$. If $\mathcal{P}_c > 0$, the fundamental solution $E(t)$ decays exponentially for $t \to +\infty$.

Given $p \geq 1$ we define the L^p critical index $r_{cr}(p)$ for (1.1) by

$$r_{cr}(p) = \max\{\frac{n}{p} - \frac{n}{p_{cr}}, \frac{n}{p} - \frac{4}{s} - \frac{n}{\max\{p,s\}}\}. \tag{2.5}$$

The next proposition is readily obtained from the L^p estimates on the fundamental solution E in section 4 and the definition of the set $\Theta(n)$.

Proposition 2.1. *Set $I_{r,p} := \{q \geq p : (q, -r + n(\frac{1}{p} - \frac{1}{q})) \in \Theta(n)\}$ for $p \geq \max\{1, p_{cr}\}$, $r \leq 0$. Then we claim that $I_{r,p}$ is nonempty iff $p \geq p_{cr}$, $r \geq r_{cr}(p)$ with r allowed to be equal $r_{cr}(p)$ iff*

$$r_{cr}(p) = \frac{n}{p} - \frac{n}{p_{cr}} > \frac{n}{p} - \frac{4}{s} - \frac{n}{\max\{p,s\}}. \tag{2.6}$$

3. The general results in the spaces $A^{\gamma}_{\theta,q}(T)$

First we state the local results for (1.1), (1.2) with singular initial data.

Theorem 3.1. *Let* $(q,\theta) \in \dot{\Theta}(n)$ *and* $u^0 \in \mathcal{B}_q^{-\theta}(\Omega)$. *Then there exists a non-increasing positive function* $T(\gamma)$, $\gamma \geq 0$ *such that the IVP (1.1), (1.2) admits a solution* $u \in \bigcap\limits_{\gamma\geq 0} A^{\gamma}_{\frac{\theta}{4},q}(T(\gamma))$. *The solution is unique in* $C_{\frac{\theta}{4}}(L^q(\Omega);T(0))$. *Assume now that* $(q,\theta(q)) \in \partial\Theta(n)$. *We claim that there exists* $C' = C'_q > 0$ *such that if* $u^0 \in \mathcal{B}_q^{-\theta(q)}(\Omega)$ *satisfies* $\lim\limits_{T\searrow 0} \|E^{\Omega}[u^0]\|_{C_{\frac{\theta(q)}{4}}(L^q;T)} \leq C'$ *then the IVP (1.1), (1.2) has a unique solution* $u \in C_{\frac{\theta(q)}{4}}(L^q;T')$ *for certain* $T' > 0$. *Moreover,* $u \in A^{\gamma}_{\frac{\theta(q)}{4},q}(T'_\gamma)$ *for some* $T'_\gamma \in]0,T']$, $0 < \gamma \ll 1$.

Now we state the results for global solutions.

Theorem 3.2. *Let* $(q,\theta) \in \Theta(n)$. *If* $\mathcal{P}_c > 0$ *there exists* $\varepsilon = \varepsilon_q > 0$ *such that, for every* $u^0 \in \dot{\mathcal{B}}_q^{-\theta}(\Omega)$ *satisfying* $\|E^{\Omega}[u^0]\|_{C_{\frac{\theta}{4}}(L^q(\Omega))} < \varepsilon$, *the IVP (1.1), (1.2) admits a unique global solution* $u \in C_{\frac{\theta}{4}}(L^q(\Omega))$ *and* $u \in A^{\gamma}_{\frac{\theta}{4},q}$ *for* $0 < \gamma \ll 1$.

Remark 3.3. *As far as we know the results on the analytic regularity in the references cited in the introduction guarantee at best the estimate* $\liminf\limits_{t\to+\infty} \rho_{[u]}(t) > 0$. *Under the assumptions of Theorem 3.2, we have, at least for* $0 < \gamma \ll 1$, *that* $\rho_{[u]}(t) \geq \gamma t^{\frac{1}{4}}$ *for all* $t > 0$. *The two theorems above and Proposition 2.1 show that if* $u^0 \in H^r_p(\Omega)$ *and* $I_{r,p} \neq \emptyset$ *we can resolve (1.1), (1.2).*

4. Estimates on the fundamental solution

In view of (2.3) for every $b > -\mathcal{P}_c$ we can find $\overline{b} > 0$ such that

$$|\xi|^4 + \mathrm{Re}P(\xi) \geq \overline{b}|\xi|^4 - b, \qquad \xi \in \mathbb{R}^n. \tag{4.1}$$

Next we investigate the analytic regularity in x of E for $t > 0$.

Theorem 4.1. *There exists a constant* $\overline{a} > 0$ *such that for every* $1 \leq r \leq +\infty$,

$d \in \mathbb{Z}_+$ one can find positive constant $C = C_{r,d}$ such that

$$\|\partial_x^\alpha \partial_x^\beta E(t)\|_{L^r} \leq C e^{bt} t^{-\frac{d}{4}-\frac{n}{4}(1-\frac{1}{r})} \overline{a}^{|\alpha|} t^{-\frac{|\alpha|}{4}} (\alpha!)^{1/4} \tag{4.2}$$

for all $\alpha, \beta \in \mathbb{Z}_+^n$, $|\beta| = d$ and $t > 0$. Moreover, (4.2) implies that $\partial_x^\beta E \in A^\gamma_{\frac{d}{4}+\frac{n}{4}(1-\frac{1}{r}),r}(T)$ and for some $C_1 = C_1(r,d) > 0$

$$\|\partial_x^\beta e^{-b\cdot} E\|_{A^\gamma_{\frac{d}{4}+\frac{n}{4}(1-\frac{1}{r}),r}(T)} \leq C_1 \exp(c\gamma^{\frac{3}{4}}), \qquad \gamma \geq 0, T > 0 \tag{4.3}$$

where $c > 0$ depends only on $\overline{a}$ and n. The same result is true if we replace ∂_x^β with a homogeneous pseudodifferential operator $\kappa(D)$ of order $d > 0$.

Proof. Set $\kappa(\xi) = \xi^\beta$ for given $\beta \in \mathbb{Z}_+^n$, $|\beta| = d$. For each $\alpha \in \mathbb{Z}_+^n$ we can write

$$E_\kappa^\alpha(t,x) := D_x^\alpha \kappa(D) E(t,x) = \int_\Omega e^{ix\cdot\xi - t|\xi|^4 - tP(\xi)} \kappa(\xi) \xi^\alpha \, đ\xi. \tag{4.4}$$

Here $đ\xi = (2\pi)^{-n} d\xi$. After the dilation $\xi = \dfrac{\eta}{t^{1/4}}$ we obtain that

$$E_\kappa^\alpha(t,x) = t^{-\frac{d+|\alpha|}{4}-\frac{n}{4}} \psi^\alpha(\frac{x}{t^{1/4}}; t), \tag{4.5}$$

where $\psi^\alpha(z;t) := \int_{\mathbb{R}^n} e^{iz\cdot\eta - \Phi(\eta;t)} \kappa(\eta) \eta^\alpha \; đ\eta$, $\Phi(\eta;t) := |\eta|^4 - tP(\frac{\eta}{t^{\frac{1}{4}}})$. Since $\|\psi^\alpha(\frac{\cdot}{t^{1/4}}; t)\|_{L^r} = t^{n/(4r)} \|\psi^\alpha(\cdot;t)\|_{L^r}$, $t > 0$ we are reduced to the proof of the estimate of $\|\psi^\alpha(\cdot;t)\|_{L^r}$. We observe that

$$Re(\Phi(\eta;t)) \geq \overline{b}|\eta|^4 - bt, \qquad \eta \in \mathbb{R}^n, t > 0. \tag{4.6}$$

We check easily that for all $a > 0$, $s > 0$ we have

$$\sup_{z \geq 0} (z^s e^{-az^4}) = (\frac{s}{4ae})^{s/4}. \tag{4.7}$$

Next, we recall the Stirling formula $\ell! \approx (\frac{\ell}{e})^\ell \sqrt{2\pi\ell}$, as $\ell \to +\infty$. Here $\ell! = \Gamma(\ell+1)$, where $\Gamma(z)$ stands for the Gamma function. As a consequence, we show that there exist $0 < \gamma_1 < 1 < \gamma_2$ such that

$$\gamma_1^\ell (\ell!)^{1/4} \leq (\frac{\ell}{4})! \leq \gamma_2^\ell (\ell!)^{1/4}, \quad \ell \in \mathbb{R}_+; \tag{4.8}$$

and if $\delta > -1$ there exists $B > 0$ such that

$$\Gamma(\frac{z}{4} + \delta + 1) \leq B^z (z!)^{\frac{1}{4}}, \qquad z \geq 0. \tag{4.9}$$

We will prove (4.2) for $r = \infty$ and $r = 1$. Then the interpolation in the L^p spaces yields the general case. We have, by using (4.6) and (4.7):

$$\begin{aligned} \|\psi^\alpha\|_{L^\infty} &\leq (2\pi)^{-n} \|\widehat{\psi^\alpha}\|_{L^1} \leq \int_{\mathbb{R}^n} e^{-Re(\Phi(\eta;t))} |\eta|^{|\alpha|+d}\, đ\eta \\ &\leq e^{bt} \int_{\mathbb{R}^n} e^{-\overline{b}|\eta|^4} |\eta|^{|\alpha|+d}\, đ\eta \\ &= e^{bt} \omega_n \int_0^\infty e^{-\overline{b}\rho^4} \rho^{d+|\alpha|+n-1} d\rho \\ &\leq e^{bt} \frac{\omega_n}{4\overline{b}^{\frac{|\alpha|+d+n}{4}}} \Gamma(\frac{|\alpha|+d+n}{4}), \end{aligned} \tag{4.10}$$

where ω_n is the area of the unit sphere S^{n-1}, which implies (4.2) for $r = +\infty$, in view of (4.9), (4.8) and the fact that one can find $C = C(n) > 0$ having the property $|\alpha|! \leq C^{|\alpha|}\alpha!$, $\alpha \in \mathbb{Z}_+^n$.

Set $B_n(\delta) = \{x \in \mathbb{R}^n : |x| \leq \delta\}$, $\delta > 0$. The Hölder inequality implies

$$\|g\|_{L^1(\mathbb{R}^n)} \leq mes(B_n(1)) \|g\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^1(\mathbb{R}^n \setminus B_n(1))}, \tag{4.11}$$

and therefore it is enough to show (4.2) for the $L^1(\mathbb{R}^n \setminus B_n(1))$-norm of $\psi^\alpha(\cdot; t)$.

If $|z| \geq 1$ we introduce the linear operator $\mathcal{L} = \mathcal{L}(z, \partial_\eta) = -i|z|^{-1} \sum_{j=1}^n z_j \partial_{\eta_j}$. We note that $|z|^{-1}\mathcal{L}(e^{iz\cdot\eta}) = e^{iz\cdot\eta}$ and $\mathcal{L}^t = -\mathcal{L}$, with $\mathcal{L}^t$ standing for the transposed operator of $\mathcal{L}$. Furthermore

$$(-\mathcal{L})^M = \sum_{|\beta|=M} \frac{M!}{\beta!} \frac{i^M z^\beta}{|z|^{|\beta|}} \partial_\eta^\beta = \sum_{|\beta|=M} \frac{M!}{\beta!} c_\beta(z) \partial_\eta^\beta, \tag{4.12}$$

with $c_\beta(z) = i^{|\beta|} z^\beta |z|^{-|\beta|}$, $\beta \in \mathbb{Z}_+^N \setminus 0$, $z \in \mathbb{R}^n \setminus 0$. Evidently $|c_\beta(z)| \leq 1$.

Integration by parts M times gives $\psi^\alpha(z;t) = |z|^{-M} \widetilde{\psi^\alpha}(z;t)$ where

$$\widetilde{\psi^\alpha}(z;t) = \int_{\mathbb{R}^n} e^{iz\cdot\eta} (-\mathcal{L})^M \left(e^{-\Phi(\eta;t)} \kappa(\eta) \eta^\alpha \right) đ\eta. \tag{4.13}$$

The definition of $\Phi(\eta; t)$ implies that there is $C > 0$ such that

$$|\partial_\eta^\mu \Phi(\eta;t)| \leq C(1 + t^{\frac{3}{4}} + |\eta|)^{4-|\mu|}, \quad t \geq 0, \eta \in \mathbb{R}^n, \mu \in \mathbb{Z}_+^n, |\mu| \leq 4. \tag{4.14}$$

Then, since $\kappa(\eta)$ and $\Phi(\eta;t)$ are polynomials in η, straightforward computations show that for each $M \in \mathbb{N}$ there exists $C = C(M, \kappa, P) > 0$ such that

$$\begin{aligned} |(-\mathcal{L})^M(e^{-\Phi(\eta;t))}\kappa(\eta)\eta^\alpha)| &\leq Ce^{bt}(1+t^{\frac{3}{4}})^M|\alpha|^M e^{-\overline{b}|\eta|^4} \\ &\times (|\eta|^{(|\alpha|+d-M)_+} + |\eta|^{|\alpha|+d+3M}) \end{aligned} \tag{4.15}$$

for all $\eta \in \mathbb{R}^n$, $\alpha \in \mathbb{Z}^n_+$, $t > 0$. Here $r+ := \max\{r, 0\}$. Hence for $M = n+1$ we obtain, taking into account (4.13) and (4.14),

$$\|\psi^\alpha(\cdot;t)\|_{L^1(\mathbb{R}^n\setminus B_n(1))} \leq \left(\int_{|z|\geq 1} \frac{1}{|z|^{n+1}}dz\right) \|\widetilde{\psi^\alpha}(\cdot;t)\|_{L^\infty(\mathbb{R}^n)}. \tag{4.16}$$

We estimate $\|\widetilde{\psi^\alpha}(\cdot;t)\|_{L^\infty(\mathbb{R}^n)}$ exactly in the same way as $\|\psi^\alpha(\cdot;t)\|_{L^\infty(\mathbb{R}^n)}$ and conclude the proof of (4.2) for $\Omega = \mathbb{R}^n$.

Now let $\Omega = \mathbb{T}^n$. We have (cf. [17] for similar arguments for the Ginzburg-Landau equation)

$$E^{\mathbb{T}^n}(t,x) = \sum_{\beta\in\mathbb{Z}^n} E(t, x+2\pi\beta), \qquad x \in \mathbb{T}^n,\ t > 0. \tag{4.17}$$

Set $E^{\alpha,\kappa}_{\mathbb{T}^n}(t,x) := \partial^\alpha_x \kappa(D) E^{\mathbb{T}^n}(t,x) = \sum_{\beta\in\mathbb{Z}^n} E^\alpha_\kappa(t, x+2\pi\beta)$. We write, using (4.2)

$$\begin{aligned} \|E^{\alpha,\kappa}_{\mathbb{T}^n}(t,\cdot)\|_{L^1(\mathbb{T}^n)} &\leq \sum_{\beta\in\mathbb{Z}^n} \int_{\mathbb{T}^n} |E^\alpha_\kappa(t, x+2\pi\beta)|dx \\ &= \|E^\alpha_\kappa(t,\cdot)\|_{L^1(\mathbb{R}^n)} \leq C_1^{|\alpha|+1} e^{bt}(\alpha!)^{\frac{1}{4}} t^{-\frac{d+|\alpha|}{4}} \end{aligned}$$

for all $\alpha \in \mathbb{Z}^n_+$, $t > 0$. We recall another expression for $E^{\mathbb{T}^n}(t,x)$, namely

$$E^{\mathbb{T}^n}(t,x) = \frac{1}{(2\pi)^n} \sum_{\xi\in\mathbb{Z}^n} e^{ix\cdot\xi - t(|\xi|^4 + P(\xi))}. \tag{4.18}$$

Hence $E^{\alpha,\kappa}_{\mathbb{T}^n}(t,x) = \frac{1}{(2\pi)^n} \sum_{\xi\in\mathbb{Z}^n} e^{ix\cdot\xi - t(|\xi|^4+P(\xi))} \xi^\alpha \kappa(\xi)$. Next, by using (4.6)-(4.9) and the estimate $\sum_{|\xi|=j} 1 = O(j^{n-1})$, $j \to \infty$, we get for some $C > 0$

$$
\begin{aligned}
\|E^{\alpha,\kappa}_{\mathbb{T}^n}(t,\cdot)\|_{L^\infty(\mathbb{T}^n)} &\leq \frac{e^{bt}}{(2\pi)^n} \sum_{\xi\in\mathbb{Z}^n} e^{-\bar{b}t|\xi|^4}|\xi|^{|\alpha|}|\kappa(\xi)| \\
&\leq \frac{Ce^{bt}}{(2\pi)^n} \sup_{z\geq 0}(z^{|\alpha|+d}e^{-\frac{\bar{b}}{2}tz^4}) \sum_{\xi\in\mathbb{Z}^n} e^{-\frac{\bar{b}}{2}t|\xi|^4} \\
&\leq C^{|\alpha|+1}e^{bt}(\alpha!)^{\frac{1}{4}}t^{-\frac{|\alpha|+d}{4}} \sum_{j=0}^{\infty} e^{-\frac{\bar{b}}{2}tj^4}j^{n-1} \\
&\leq C^{|\alpha|+1}e^{bt}(\alpha!)^{\frac{1}{4}}t^{-\frac{|\alpha|+d}{4}} \sup_{j\geq 1}[e^{-\frac{\bar{b}}{4}tj^4}j^{n-1}] \sum_{j=0}^{\infty} e^{-\frac{\bar{b}}{4}tj^4} \\
&\leq C^{|\alpha|+1}e^{bt}(\alpha!)^{\frac{1}{4}}t^{-\frac{d+n}{4}}(\frac{C}{t^{\frac{1}{4}}})^{|\alpha|}, \qquad \alpha\in\mathbb{Z}^n_+.
\end{aligned}
$$

Finally, as for $\mathbb{R}^n$, the interpolation leads to the desired estimates for $E^{\mathbb{T}^n}$.

□

5. Proofs of Theorems 3.1 and 3.2

Throughout this section (q,θ) will be fixed. Following the standard approach, we reduce (1.1), (1.2) to the integral equation

$$
\begin{aligned}
u(t) &= E[u^0](t) - K[u](t), &(5.1)\\
K[u](t) &:= \int_0^t \nabla E(t-\tau) * F(u(\tau))d\tau &(5.2)
\end{aligned}
$$

where $\nabla E(t-\tau) * F(u(\tau)) := \sum_{\ell=1}^n \partial_{x_\ell}E(t-\tau) * F_\ell(u(\tau))$. For $\alpha\in\mathbb{Z}^n_+$ we set

$$
L^\alpha[u](t) = t^{\frac{\theta}{4}+\frac{|\alpha|}{4}}\|\int_0^t \partial_x^\alpha\nabla E(t-\tau) * F(u(\tau))d\tau\|_{L^q}. \tag{5.3}
$$

We have, using the convolution property $\partial^{\rho+\delta} f * g = \partial^\rho f * \partial^\delta g$, the inequality $(a+b)^\sigma \le a^\sigma + b^\sigma$ if $\sigma \in]0,1]$, $a, b \ge 0$ and the Young inequality,

$$\begin{aligned}
L^\alpha[u](t) &\le t^{\frac{\theta}{4}} \int_0^t ((t-\tau)^{1/4} + \tau^{1/4})^{|\alpha|} \|\partial^\alpha \nabla E(t-\tau) * F(u(\tau))\|_{L^q} d\tau \\
&= \alpha! t^{\frac{\theta}{4}} \sum_{\alpha'+\alpha''=\alpha} \int_0^t \| \frac{(t-\tau)^{\frac{|\alpha'|}{4}} \partial^{\alpha'} \nabla E(t-\tau)}{\alpha'!} * \frac{\tau^{\frac{|\alpha''|}{4}} \partial^{\alpha''} F(u(\tau))}{\alpha''!} \|_{L^q} d\tau \\
&\le \alpha! t^{\frac{\theta}{4}} \int_0^t \frac{e^{b(t-\tau)} d\tau}{(t-\tau)^{\frac{1}{4}+\frac{n(s-1)}{4q}} \tau^{\frac{\theta s}{4}}} \\
&\times \sum_{\alpha'+\alpha''=\alpha} \sup_{0<t<T} \left(\frac{(t^{\frac{|\alpha'|}{4}+\frac{1}{4}+\frac{n(s-1)}{4q}} \|\partial^{\alpha'} e^{-bt} \nabla E(t)\|_{L^{\frac{q}{q-s+1}}}}{\alpha'!} \right) \\
&\times \sup_{0<\tau<T} \left(\frac{\tau^{\frac{|\alpha''|}{4}+\frac{\theta s}{4}} \|\partial^{\alpha''} F(u(\tau))\|_{L^{\frac{q}{s}}}}{\alpha''!} \right).
\end{aligned} \tag{5.4}$$

Now, (2.1), (4.3), (5.3) and (5.4) imply that for some $C > 0$

$$\begin{aligned}
&\|K^0[u]\|_{A^\gamma_{\frac{\theta}{4},q}(T)} \le \sum_{\alpha \in \mathbb{Z}^n_+} \frac{\gamma^{|\alpha|}}{\alpha!} \sup_{0<t<T} L^\alpha[u](t) \\
&\le C\Psi(T) \|\nabla e^{-b\cdot} E\|_{A^\gamma_{\frac{1}{4}+\frac{n(s-1)}{4q}, \frac{q}{q-s+1}}(T)} \|F(u)\|_{A^\gamma_{\frac{\theta s}{4}, \frac{q}{s}}(T)} \\
&\le C\Psi(T) \exp(c\gamma^{\frac{3}{4}}) \|F(u)\|_{A^\gamma_{\frac{\theta s}{4}, \frac{q}{s}}(T)}, \qquad \gamma \ge 0,\ T \ge 0
\end{aligned} \tag{5.5}$$

where $c > 0$ is the constant appearing in (4.3) and

$$\Psi(T) = \Psi^{n,b}_{q,\theta,s}(T) := \sup_{0<t<T} \left(t^{\frac{\theta}{4}} \int_0^t \frac{e^{b(t-\tau)} d\tau}{(t-\tau)^{\frac{1}{4}+\frac{n(s-1)}{4q}} \tau^{\frac{\theta s}{4}}} \right). \tag{5.6}$$

In view of (5.6), $\Psi(T) = O(T^{\rho(q,\theta)})$ as $T \searrow 0$ ($\rho(q,\theta)$ given in (2.2)) and

$$\psi(+\infty) := \sup_{T>0} \Psi(T) < +\infty \quad \text{provided } b < 0 \text{ or } b = 0,\ \rho(q,\theta) = 0. \tag{5.7}$$

In the same way as above we show that

$$\|K[u_1] - K[u_2]\|_{A^\gamma_{\frac{\theta}{4},q}(T)} \le 2C\Psi(T) e^{c\gamma^{\frac{3}{4}}} \|F(u_1) - F(u_2)\|_{A^\gamma_{\frac{\theta s}{4}, \frac{q}{s}}(T)}, \tag{5.8}$$

for all $u_\ell \in A^\gamma_{\frac{\theta}{4},q}(T)$, $\ell = 1, 2$, $\gamma \ge 0$, $T > 0$.

Next we extend the generalized Hölder inequality in the spaces $A^\gamma_{\theta,q}(T)$.

Lemma 5.1. *Let* $\gamma \geq 0$, $1 \leq q_\nu \leq \infty$, $\theta_\nu \geq 0$, $\nu = 1, \dots, \ell$, $\overline{\theta} = \theta_1 + \dots + \theta_\ell$ *and let* $\dfrac{1}{\overline{q}} = \displaystyle\sum_{\nu=1}^{\ell} \frac{1}{q_\nu} \leq 1$. *Then for every* $h_\nu \in A^\gamma_{\theta_\nu,q_\nu}(T)$, $1 \leq \nu \leq \ell$ *we have*

$$\|\prod_{\nu=1}^{\ell} h_\nu\|_{A^\gamma_{\overline{\theta},\overline{q}}(T)} \leq \prod_{\nu=1}^{\ell} \|h_\nu\|_{A^\gamma_{\theta_\nu,q_\nu}(T)}. \tag{5.9}$$

Proof. We obtain, for $0 < \tau < T$,

$$\begin{aligned}
&\sum_\beta \frac{\gamma^{|\beta|}\tau^{\overline{\theta}+\frac{|\beta|}{4}}}{\beta!}\|\partial^\beta(h_1(\tau)\dots h_\ell(\tau))\|_{L^{\overline{q}}} \\
&\leq \sum_\beta \gamma^{|\beta|}\tau^{\overline{\theta}+\frac{|\beta|}{4}} \sum_{\beta_1+\dots+\beta_\ell=\beta} \frac{1}{\beta_1!\dots\beta_\ell!}\|\partial^{\beta_1}h_1(\tau)\dots\partial^{\beta_\ell}h_\ell(\tau)\|_{L^{\overline{q}}} \\
&\leq \sum_{\beta_1} \frac{\gamma^{|\beta_1|}\tau^{\theta_1+\frac{|\beta_1|}{4}}}{\beta_1!}\|\partial^{\beta_1}h_1(\tau)\|_{L^{q_1}} \dots \sum_{\beta_\ell} \frac{\gamma^{|\beta_\ell|}\tau^{\theta_\ell+\frac{|\beta_\ell|}{4}}}{\beta_\ell!}\|\partial^{\beta_\ell}h_\ell(\tau)\|_{L^{q_\ell}} \\
&= \prod_{\nu=1}^{\ell} \|h_\nu\|_{A^\gamma_{\theta_\nu,q_\nu}(T)}
\end{aligned}$$

which implies (5.9).

□

Next, taking into account the identity $u_1^s - u_2^s = \sum_{j=0}^{s}(u_1-u_2)u_1^{s-1-j}u_2^j$ and applying to its right-hand side Lemma 5.1, we get the following inequality for $s \geq 2$, $u_1, u_2 \in A^\gamma_{\frac{\theta}{4},q}(T)$

$$\|u_1^s - u_2^s\|_{A^\gamma_{\frac{\theta s}{4},\frac{q}{s}}(T)} \leq \|u_1-u_2\|_{A^\gamma_{\frac{\theta}{4},q}(T)}(\|u_1\|_{A^\gamma_{\frac{\theta}{4},q}(T)} + \|u_2\|_{A^\gamma_{\frac{\theta}{4},q}(T)})^{s-1}. \tag{5.10}$$

We note that, since $E(t) * f = E(\frac{t}{2}) * E(\frac{t}{2}) * f$, the properties of the convolution and the Young inequality show that for some $C_0 > 0$

$$\begin{aligned}
\|E[u^0]\|_{A^\gamma_{\frac{\theta}{4},q}(T)} &\leq \max\{1, e^{b\frac{T}{2}}\}\|e^{-b\cdot}E\|_{A^\gamma_{0,1}(\frac{T}{2})}U^0_{q,\theta}(T) \\
&\leq C_0\max\{1, e^{b\frac{T}{2}}\}e^{c\gamma^{\frac{3}{4}}}U^0_{q,\theta}(T),
\end{aligned} \tag{5.11}$$

for all $T > 0$, $\gamma \geq 0$, $u^0 \in \mathcal{B}^{-\theta}_q(\Omega)$ with $U^0_{q,\theta}(T) := \|E[u^0]\|_{A^0_{\frac{\theta}{4},q}(\frac{T}{2})}$.

Define $B^{\gamma}_{\theta,q}(R;T) = \{u \in A^{\gamma}_{\frac{\theta}{4},q}(T);\ \|u\|_{A^{\gamma}_{\frac{\theta}{4},q}(T)} \leq R\}$, $0 < T \leq +\infty$, $R > 0$. The estimates (5.5), (5.8), (5.10) and (5.11) show that if, for given u^0, we can find $R > 0$ and $T > 0$ satisfying, for some $C > 0$ independent of $\gamma \geq 0$,

$$\begin{aligned} C_0 e^{c\gamma^{\frac{3}{4}}} \max\{1, e^{b\frac{T}{2}}\} U^0_{q,\theta}(T) + C e^{c\gamma^{\frac{3}{4}}} \max\{1, e^{bT}\}\Psi(T) R^s &\leq R, && (5.12) \\ 2C e^{c\gamma^{\frac{3}{4}}} \max\{1, e^{bT}\}\Psi(T) R^{s-1} &< 1, && (5.13) \end{aligned}$$

then we can apply the FPT in $B^{\gamma}_{\theta,q}(R;T)$ and solve uniquely the integral equation (5.1). If $b < 0$ and (5.12), (5.13) hold for $T = +\infty$ and some $R = R(U^0_{q,\theta}(+\infty))$, the FPT yields global solution to (5.1).

We observe that, plugging (5.13) in (5.12), straightforward computations show that if the estimates

$$C_0 e^{c\gamma^{\frac{3}{4}}} \max\{1, e^{b\frac{T}{2}}\} U^0_{q,\theta}(T) \leq \frac{R}{2} \tag{5.14}$$

and (5.13) are satisfied, then (5.12) holds as well.

We show first the local results. Assume $0 < T \leq 1$. If $(\theta, q) \in \dot{\Theta}(n)$ we have $\rho = \rho(\theta, q) > 0$. In view of (5.6), (5.7) and $\rho > 0$ we obtain that $\Psi(T)$ is stricly increasing for $0 < T \ll 1$. Thus if we choose R by assuming equality in (5.14) i.e. $R = 2C_0 e^{c\gamma^{\frac{3}{4}}} \max\{1, e^{\frac{b}{2}}\} U^0_{q,\theta}(T)$ we obtain that (5.13) will hold if

$$\Psi(T)(U^0_{q,\theta}(T))^{s-1} \leq \Psi(T)(U^0_{q,\theta}(1))^{s-1} < \frac{1}{2^s C C_0^{s-1} e^{cs\gamma^{\frac{3}{4}}} (\max\{1, e^{\frac{b}{2}}\})^s}. \tag{5.15}$$

In view of the monotonicity of $\Psi(T)$ and the fact that $\Psi(T) = O(T^{\rho})$ as $T \searrow 0$ we resolve (5.13) and obtain the following estimate for T

$$T \geq T(\gamma) := \min\{1, C_1 e^{-\frac{s}{\rho}\gamma^{\frac{3}{4}}} (U^0_{q,\theta}(1))^{-\frac{s-1}{\rho}}\} \tag{5.16}$$

with $C_1 > 0$ depending only on C, C_0, b and Ψ.

If $(\theta, q) \in \partial\Theta(n)$ we have $\rho = 0$. In that case $\Psi(T) = O(1)$ for $T \searrow 0$ and we could not proceed as above. However, choosing R by assuming equality in (5.14), the estimate (5.13) is true for small T provided $\lim_{T \searrow 0} U^0_{q,\theta}(T)$ is small enough. Theorem 3.1 is proved.

Let us show now the global results in Theorem 3.2. The hypotheses imply that $U^0_{q,\theta} := U^0_{q,\theta}(+\infty)$ is finite, we can choose $b < 0$ and therefore we can apply the FPT in $B^{\gamma}_{\theta,q}(R;+\infty)$ provided we choose R by assuming equality in (5.14) with $T = +\infty$ and the next inequality is satisfied:

$$\Psi(+\infty)(U^0_{q,\theta}(+\infty))^{s-1} < \frac{1}{2^s C C_0^{s-1} e^{cs\gamma^{\frac{3}{4}}}}. \tag{5.17}$$

Clearly (5.17) is satisfied if $U^0_{q,\theta}(+\infty)$ is small enough. This concludes the proof of Theorem 3.2.

6. Regularity and global well-posedness

First we estimate the regularity down to $t = 0$ due to the dissipation by Δ^2.

Theorem 6.1. *Let $s \in \mathbb{N}$, $2 \le s \le 1 + \frac{6}{n}$. Fix $q = s$ if $s < 1 + \frac{6}{n}$ and $q = s + \varepsilon$ with $0 < \varepsilon \ll 1$ when $s = 1 + \frac{6}{n}$. Let $u \in C([0,T[; L^2(\Omega)) \cap C_{\frac{n}{4}(\frac{1}{2}-\frac{1}{q})}(L^q(\Omega);T)$ be a weak solution to (1.1), (1.2) for some $T > 0$. Then we claim that*

$$\begin{aligned} \partial_x^\alpha u &\in C_{\frac{|\alpha|}{4}+\frac{n}{4}(\frac{1}{2}-\frac{1}{p})}(L^p(\Omega);T), \quad \text{for every } \alpha \in \mathbb{Z}^n_+,\ 2 \le p \le \infty, && (6.1)\\ u &\in C([0,T[: H^r_p(\Omega)) \quad \text{if } u^0 \in H^r_p(\Omega) \text{ for some } r \in \mathbb{Z}_+,\ p \ge 2. && (6.2) \end{aligned}$$

Proof. We write $K[u](t) = K'[u](t) + K''[u](t)$ with $K'[u](t)$ (respectively $K''[u](t)$) being defined as $K[u]$ replacing $\int_0^t$ by $\int_0^{\delta t}$ (respectively by $\int_{\delta t}^t$) and where $0 < \delta < 1$. We will show by induction with respect to ν that

$$W^p_\nu(t) := \max_{|\alpha|=\nu} \sup_{0<t<T} (t^{\frac{|\alpha|}{4}+\frac{n}{4}(\frac{1}{2}-\frac{1}{p})} \|\partial^\alpha u(t)\|_{L^p}), \qquad p \ge 2 \tag{6.3}$$

which will imply (6.1).

First we note that the estimates on E and the definition of $K'[u]$ yield that for every fixed $\mu \in \mathbb{Z}_+$, $2 \le p \le +\infty$ and r_p, defined by $1 + \dfrac{1}{p} = \dfrac{1}{r_p} + \dfrac{s}{q}$, the following estimates hold for some $C = C_T > 0$, $\tilde{C} = \tilde{C}_{T,\delta} > 0$:

$$\begin{aligned} V^0_{p,\mu}(T) &:= \max_{|\alpha|=\mu} \|\partial^\alpha E[u^0]\|_{C_{\frac{\mu}{4}+\frac{n}{4}(\frac12-\frac1p)}(L^p;T)} < +\infty, && (6.4)\\ \|\partial^\alpha K'[u](t)\|_{L^p} &\le \int_0^{\delta t} \|\partial^\alpha \nabla E(t-\tau)\|_{L^{r_p}} \|F(u(\tau))\|_{L^{\frac{q}{s}}} d\tau \\ &\le Ct^{-\frac{|\alpha|}{4}} \int_0^{\delta t} \frac{(W_0^q(\tau))^s}{(t-\tau)^{\frac14+\frac{n}{4}(1-\frac{1}{r_p})}\tau^{\frac{sn}{4}(\frac12-\frac1q)}} d\tau \\ &\le \tilde{C} \frac{t^\rho (W_0^q(t))^s}{t^{\frac{|\alpha|}{4}+\frac{n}{4}(\frac12-\frac1p)}}, \qquad 0<t<T && (6.5)\end{aligned}$$

where $\rho := \frac{6+n-sn}{8} \ge 0$ in view of the condition on s and n.

The most delicate part of the proof concerns the L^p estimates of $K''[u](t)$. We will prove (6.3) for $\nu = 0$ in the following way: we will find $M \in \mathbb{N}$ and M indices $p_0 = q < p_1 < \ldots < p_M = +\infty$ such that if, for some $j = 0, 1, \ldots, M-1$,

$$u \in C_{\frac{n}{4}(\frac12-\frac{1}{p_j})}(L^{p_j}(\Omega);T), \quad \text{then} \quad u \in C_{\frac{n}{4}(\frac12-\frac{1}{p_{j+1}})}(L^{p_{j+1}}(\Omega);T). \tag{6.6}$$

The key in proving (6.6) is the next chain of estimates with r_j defined by $1 + \dfrac{1}{p_{j+1}} = \dfrac{1}{r_j} + \dfrac{s}{p_j}$:

$$\begin{aligned} \|K''[u](t)\|_{L^{p_{j+1}}} &\le \int_{\delta t}^t \|\nabla E(t-\tau)\|_{L^{r_j}} \|F(u(\tau))\|_{L^{\frac{p_j}{s}}} d\tau \\ &\le C\int_{\delta t}^t \frac{(\tau^{\frac{n}{4}(\frac12-\frac{1}{p_j})}\|u(\tau)\|_{L^{p_j}})^s}{(t-\tau)^{\frac14+\frac{n}{4}(\frac{s}{p_j}-\frac{1}{p_{j+1}})}\tau^{\frac{sn}{4}(\frac12-\frac{1}{p_j})}} d\tau \\ &\le \tilde{C}\frac{t^\rho (W_0^{p_j}(t))^s}{t^{\frac{n}{4}(\frac12-\frac{1}{p_{j+1}})}}, \qquad 0<t<T && (6.7)\end{aligned}$$

provided

$$\frac14 + \frac{n}{4}\Big(\frac{s}{p_j} - \frac{1}{p_{j+1}}\Big) < 1 \ \Leftrightarrow \frac{1}{p_j} - \frac{1}{p_{j+1}} < \frac{3}{n} - \frac{s-1}{p_j} \tag{6.8}$$

The inequalities (6.8) hold if we fix $\delta_0 \in]0, \frac{3}{n} - \frac{s-1}{q}[$ and define $\frac{1}{p_j} = \frac1q - j\delta_0$, $j = 0, 1, \ldots, M-1$, with M being the first positive integer j such that $\frac1q - j\delta_0 \le 0$. The choice of $\{p_j\}_{j=0}^M$, the estimates (6.7), (6.6), (6.4), (6.5) and the integral equation (5.1) imply (6.3) for $\nu = 0$.

Assume the validity of (6.3) for $0 \leq \nu \leq \mu - 1$. Choose and fix $\alpha \in \mathbb{Z}^n_+$, $|\alpha| = \mu$. Using the Leibnitz rule for the higher order derivatives of products of s functions, we get the following estimates for $0 < t < T$ and some $C > 0$:

$$\begin{aligned}
\|\partial^\alpha K''[u](t)\|_{L^p} &\leq C\int_{\delta t}^t \|\nabla E(t-\tau)\|_{L^1}(\|u^{s-1}(\tau)\partial^\alpha u(\tau)\|_{L^p})d\tau \\
&+ C\sum_{(\beta_1,\dots,\beta_s)\in Z'(\alpha)} \int_{\delta t}^t \|\nabla E(t-\tau)\|_{L^1}\|\prod_{i=1}^s \partial^{\beta_i}u(\tau)\|_{L^p}d\tau \\
&\leq C\int_{\delta t}^t \frac{(\tau^{\frac{n}{8}}\|u(\tau)\|_{L^\infty})^{s-1}\tau^{\frac{|\alpha|}{4}+\frac{n}{4}(\frac{1}{2}-\frac{1}{p})}\|\partial^\alpha u(\tau)\|_{L^p}}{(t-\tau)^{\frac{1}{4}}\tau^{\frac{|\alpha|}{4}+\frac{sn}{8}-\frac{n}{4p}}}d\tau \\
&+ C\sum_{(\beta_1,\dots,\beta_s)\in Z'(\alpha)} \int_{\delta t}^t \frac{(\prod_{i=1}^{s-1}\|\partial^{\beta_i}u(\tau)\|_{L^\infty})\|\partial^{\beta_s}u(\tau)\|_{L^p}}{(t-\tau)^{\frac{1}{4}}}d\tau \\
&\leq \frac{C(W_0^\infty(t))^{s-1}}{t^{\frac{|\alpha|}{4}+\frac{n}{4}(\frac{1}{2}-\frac{1}{p})}}\int_{\delta t}^t \frac{W_\mu^p(\tau)}{(t-\tau)^{\frac{1}{4}}\tau^{\frac{(s-1)n}{8}}}d\tau + \frac{F_\mu(T,\delta)}{t^{\frac{|\alpha|}{4}+\frac{n}{4}(\frac{1}{2}-\frac{1}{p})}} \qquad (6.9)
\end{aligned}$$

where $(\beta_1,\dots,\beta_s) \in Z'(\alpha)$ means $\beta_i \in \mathbb{Z}^n_+$, $|\beta_i| < |\alpha|$, $\beta_1 + \dots + \beta_s = \alpha$; $F_\mu(T,\delta)$ is a polynomial of degree s of $W_\nu^\infty(T)$, $W_\nu^p(T)$, $\nu = 0, 1, \dots, \mu-1$ and in view of the inductive assumption $F_\mu(T,\delta)$ is bounded. Multiplying (5.1) with $t^{\frac{\mu}{4}+\frac{n}{4}(\frac{1}{2}-\frac{1}{p})}$, $|\alpha| = \mu$ and using (6.4), (6.5) and (6.9) we get

$$W_\mu^p(t) \leq F_\mu(T,\delta) + C(W_0^\infty(t))^{s-1}\int_{\delta t}^t \frac{W_\mu^p(\tau)}{(t-\tau)^{\frac{1}{4}}\tau^{\frac{(s-1)n}{8}}}d\tau, \qquad 0 < t < T. \qquad (6.10)$$

Now, if $s < 1 + \frac{6}{n}$, we can apply the singular Gronwall inequality (see [14], p. 190) and obtain by (6.10) the validity of (6.3) for $\nu = \mu$. If $s - 1 = \frac{6}{n}$, the sum of the exponents in the denominator of the integral in (6.10) is 1 and we could not apply the singular Gronwall inequality. However, we have the freedom to choose δ close to 1 so that

$$\int_{\delta t}^t \frac{W_\mu^p(\tau)}{(t-\tau)^{\frac{1}{4}}\tau^{\frac{3}{4}}}d\tau \leq \mathcal{C}(\delta)W_\mu^p(t), \qquad \mathcal{C}(\delta) = \int_\delta^1 \frac{1}{(1-\tau)^{\frac{1}{4}}\tau^{\frac{3}{4}}}d\tau \underset{\delta\to 1}{\longrightarrow} 0$$

and we obtain (6.3) directly from (6.10).

Now we prove (6.2) by induction, namely

$$\widetilde{W}_\nu^p(t) := \max_{|\alpha|=\nu}\sup_{0<t'<t}\|\partial^\alpha u(t')\|_{L^p} < +\infty, \qquad \nu = 0, 1, \dots, r. \qquad (6.11)$$

In order to simplify the presentation we suppose that F is quadratic in u i.e. $s=2$. If $\nu=0$ we get (6.11) from the uniqueness and the assumption $u^0 \in H^r_p(\Omega) \hookrightarrow L^p(\Omega)$. Assume the validity of (6.11) for $\nu=0,1,\dots,\mu-1$ with $\mu \in \mathbb{N}$, $\mu \le r-1$. Let $\alpha \in \mathbb{Z}^n_+$, $|\alpha|=\mu$. The Leibnitz rule for differentiation and the Hölder inequality imply that $J_\alpha(t) := \int_0^t \|\partial^\alpha(\nabla E(t-\tau) * F(u(\tau))\|_{L^p} d\tau$ is estimated as follows:

$$\begin{aligned} J_\alpha(t) &\le \int_0^t \|\nabla E(t-\tau)\|_{L^{\frac{p}{p-1}}} \|\partial^\alpha F(u(\tau))\|_{L^{\frac{p}{2}}} d\tau \\ &\le C\Bigg(\sum_{\substack{\beta<\alpha \\ \beta\neq 0}} \int_0^t \frac{\|\partial^{\alpha-\beta} u(\tau)\|_{L^p} \|\partial^\beta u(\tau)\|_{L^p}}{(t-\tau)^{\frac14+\frac{n}{4p}}} d\tau \\ &\quad + \int_0^t \frac{\|u(\tau)\|_{L^p}\|\partial^\alpha u(\tau)\|_{L^p}}{(t-\tau)^{\frac14+\frac{n}{4p}}} d\tau\Bigg) \end{aligned} \tag{6.12}$$

provided $\frac14 + \frac{n}{4p} < 1$, which is true except for the case $n=6$, $p=2$. We get then, in view of (5.1), (6.12) and the induction hypothesis,

$$\begin{aligned} \|\partial^\alpha u(t)\|_{L^p} &\le C\left(\|u^0\|_{H^\mu_p} + \sum_{i=1}^{\mu} \widetilde{W}^p_i(t)\widetilde{W}^p_{\mu-i}(t)\right) \\ &+ \widetilde{W}^p_0(t) \int_0^t \frac{\|\partial^\alpha u(\tau)\|_{L^p} d\tau}{(t-\tau)^{\frac14+\frac{n}{4p}}}. \end{aligned} \tag{6.13}$$

Using again the singular Gronwall inequality we deduce the validity of (6.11) for $\nu=\mu$. The case $n=6$, $p=2$ is settled by similar arguments to those used for the proof of (6.1). The proof of Theorem 6.1 is complete.

□

Remark 6.2. *If $\mu \ge 0$ is not integer, and $u^0 \in H^\mu_p(\Omega)$ for some $1<p<+\infty$, we can show $C([0,T]:H^\mu_p(\Omega))$ regularity of the solution u. The proof is more involved and it is based on $L^p - L^q$ estimates, the use of fractional derivatives and Sobolev embedding theorems.*

Now we will study the equation (1.1) with initial data

$$u(0,\cdot) = u^0 \in L^2(\Omega), \tag{6.14}$$

under the hypothesis that the nonlinearity is conservative, namely

$$Re\Big(\int_\Omega < \nabla\overline{u(x)}, F(u(x)) > dx\Big) = 0 \qquad \forall u \in H^\mu(\Omega),\, \mu > \frac{n}{2}. \tag{6.15}$$

Theorem 6.3. *Let (6.15) be true.*

1. *If $s < 1+\frac{6}{n}$, $(q, \dfrac{n}{4}(\dfrac{1}{2}-\dfrac{1}{q})) \in \Theta(n)$ then the IVP (1.1), (6.14) has a unique global solution $u \in X_q(\Omega) := C([0,\infty); L^2(\Omega)) \cap C_{\frac{n}{4}(\frac{1}{2}-\frac{1}{q})}(L^q(\Omega))$.*
2. *Suppose now that $s = 1+\frac{6}{n}$, $\mathcal{P}_c \geq 0$ and $q = s+\varepsilon$, $0 < \varepsilon \ll 1$. Then we can find $c_q > 0$ such that for every $u^0 \in L^2(\Omega)$ satisfying $\|u^0\|_{L^2} < c_q$ the IVP (1.1), (6.14) admits a unique global solution $u \in X_q(\Omega)$.*
3. *Let $s = 1+\frac{6}{n}$, $\Omega = \mathbb{T}^n$, $\mathcal{P}_d \geq 0$ and $q = s+\varepsilon$, $0 < \varepsilon \ll 1$. Then we can find $c_q > 0$ such that if $u^0 \in L^2(\mathbb{T}^n)$ satisfies $\|u^0\|_{L^2} < c_q$ and the mean value of u^0 is zero, there exists a unique $u \in X_q(\mathbb{T}^n)$ solving (1.1), (6.14).*

Proof. First we construct local solution via the FPT as in section 5. We note that the estimate (4.3) on the fundamental solution E implies that, under the hypotheses of Theorem 6.3, we get for some absolute constant $C_0 > 0$

$$\|E[u^0]\|_{C_{\frac{n}{4}(\frac{1}{2}-\frac{1}{q})}(L^q(\Omega);T)} \leq C_0\|u^0\|_{L^2}, \qquad u^0 \in L^2(\Omega), 0 < T \leq 1. \tag{6.16}$$

Hence, in view of (5.12), (5.13) for $\gamma = 0$ and (6.16), we can apply the FPT in $Y_q(T;R) := B^0_{q,\frac{n}{4}(\frac{1}{2}-\frac{1}{q})}(T;R)$ if for suitable $C_1 > 0$

$$C_0\|u^0\|_{L^2} + C_1 T^\rho R^s \leq R, \qquad 2C_1 T^\rho R^{s-1} < 1. \tag{6.17}$$

If $s < 1+\frac{6}{n}$ i.e. $\rho > 0$, we get readily from (6.17) that the lifespan $T_{max}(u^0)$ for the local solution of (1.1), (6.14) satisfies $T_{max}(u^0) \geq \min\{1, C_2(\|u^0\|_{L^2})^{-\frac{s-1}{\rho}}\}$, with $C_2 > 0$ being an absolute constant, while in the case $s = 1+\frac{6}{n}$ i.e. $\rho = 0$, such estimate from below could not be derived from (6.17) for all $u^0 \in L^2(\Omega)$. However, from arguments used in the proof of the local results in the critical case we get that there exist $c_q > 0$ and $\overline{T} > 0$ such that $T_{max}(u^0) \geq \overline{T}$ provided $\|u^0\|_{L^2} \leq c_q$. Next, we follow the approach in [22]. Indeed, taking into account the regularity result (6.1), multiplying (1.1) by $\overline{u}$ and integrating in $x \in \Omega$ and then from 0 to t, we get, by using (6.15)

$$\|u(t)\|_{L^2}^2 = \|u^0\|_{L^2}^2 + \int_0^t Re(< -(\Delta^2 + P(D))u(\tau), u(\tau) >_{L^2})d\tau \tag{6.18}$$

where $< \cdot, \cdot >_{L^2}$ stands for the scalar product in $L^2(\Omega)$. Since $\mathcal{P}_c \leq \mathcal{P}_d$, the Parseval identity for $\mathbb{R}^n$ and $\mathbb{T}^n$ and the Gronwall inequality imply that, for all $t > 0$ for which the local solution exists, we have $\|u(t)\|_{L^2}^2 \leq e^{-\mathcal{P}_c t}\|u^0\|_{L^2}^2$. If $s < 1 + \frac{6}{n}$, the estimate of the lifespan by means of $\|u^0\|_{L^2}$ and the energy estimate allow us to use the same arguments as in [22] and to construct global in time solution by "patching" together the local solutions. The main difficulties arise in the critical case $s = 1 + \frac{6}{n}$, when, although according to the results in section 4 we can always construct local solution with initial data in $L^2(\Omega)$, we can control the lifespan by $\|u^0\|_{L^2}$ only for $\|u^0\|_{L^2}$ small enough. In fact, in this case the difference between $\mathbb{R}^n$ and $\mathbb{T}^n$ occurs. We note that the following slightly more precise a priori energy estimate is true

$$\|u(t)\|_{L^2}^2 \leq e^{-\rho_\Omega t}\|u^0\|_{L^2}^2 \tag{6.19}$$

with ρ_Ω equals $\mathcal{P}_c$ (respectively $\mathcal{P}_d$) if $\Omega = \mathbb{R}^n$, $u^0 \in L^2(\mathbb{R}^n)$ (respectively $\Omega = \mathbb{T}^n$, $u^0 \in L^2(\mathbb{T}^n)$ and the mean value of u^0 is zero). We have used the fact that the mean value zero is preserved for the solutions of (1.1) if $\Omega = \mathbb{T}^n$. Now, since in the critical case we have $\mathcal{P}_c \geq 0$ (respectively $\mathcal{P}_d \geq 0$) if $\Omega = \mathbb{R}^n$ (respectively $\Omega = \mathbb{T}^n$), the $L^2(\Omega)$ norm of $u(t)$ does not increase and we can patch together again the local solutions into a global one in view of the fact that on each step we construct a local solution on a time interval of length at least $\overline{T}$.

$\square$

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