LOCAL TOPOLOGY OF ELEMENTARY WAVES FOR SYSTEMS OF TWO CONSERVATION LAWS

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Abstract

For shocks, generic shock foliations (shock curves) are known for quadratic flux functions. The same is true for rarefactions. However, certain submanifolds naturally associated to shocks and rarefaction are not transversal at certain points of the wave manifold where both these foliations are embedded. We show that generic cubic perturbations of the flux functions is sufficient to restore transversality at these points. Such transversality is an important ingredient to obtain stability of Riemann solutions for systems of conservation laws.

Resumo

Quando se estudam, na variedade de onda, choques com funções de fluxo quadráticas, obtém-se folhações de choque genéricas. O mesmo ocorre para rarefações. Entretanto, há certas subvariedades unidimensionais da superfície característica , naturalmente associadas a choques e rarefações, que não se cortam transversalmente. Neste artigo mostra-se que uma perturbação cúbica genérica é suficiente para restabelecer a transversalidade nestes pontos. Tal transversalidade é um importante ingrediente para a obtenção de estabilidade de soluções de Riemann para sistemas de leis de conservação.

1. Introduction

Rarefaction and shock curves in the fundamental wave manifold for systems of two conservation laws with quadratic flux functions were described in [3] and [2], respectively. For rarefaction curves (in [3]) and shock curves (in [2]) it was shown, among other things, that the configurations obtained were stable under C^3 perturbations of the flux functions in the Whitney topology. When one considers both shock and rarefaction curves simultaneously however, the following coincidence appears. The shock one-dimensional foliation is singular along a one-dimensional set r. Associated to r , we have two invariant manifolds which are formed by all shock curves through r. These invariant manifolds are two-dimensional surfaces Σ_1 and Σ_2, which intersect transversely along r (fig.1) Each of them intersects, also transversely, the characteristic surface $\mathcal{C}$ (fig.2). In the characteristic surface we have the family of rarefaction curves , which is singular exactly at $r \cap \mathcal{C}$ (fig. 3). Through each of these singular points there are two invariant 1-dimensional manifolds ϕ_1 and ϕ_2. It turns out that the curve ϕ_2 coincides with the curve $\Sigma_2 \cap \mathcal{C}$, as opposed to the behavior of ϕ_1 and $\Sigma_1 \cap \mathcal{C}$ which are transversal (fig. 4). Let us note that it is possible that r is formed by three disjoint lines, and this will generate three pairs of surfaces (Σ_1^i,Σ_2^i) and three pairs of curves (ϕ_1^i,ϕ_2^i) such that the situation just described occurs for each $i = 0, 1, 2$, but it is important to notice that for all i, j ,k, l, Σ_j^i, Σ_k^l are transversal.

Here we show that, by adding cubic terms to the flux functions, one generically destroys this coincidence (obtaining actually transversality at the singular points considered, fig. 5), indicating that it might be possible to obtain a stability theorem for rarefaction and shock foliations jointly. Such a stability theorem would be a step toward the proof of global stability of solutions of Riemann problems for systems of two conservation laws. The reason for this is that, as it was shown in [2] and [3], the rarefaction foliation and the shock foliation, for quadratic flux functions, are separately stable and the only lack of transversality, preventing a joint local stability in the neighborhood of the singularities of the rarefaction foliation, is the one removed in this paper. We also remark that adding higher order terms does not change the configurations. Again, this is due to transversality.

Consider a system of two partial differential equations $W_t + F(W)_x = 0$, where $W = (u, v)$; $W(x, t) \in \mathbf{R}^2$ and the flux function $F = (f, g) : \mathbf{R}^2 \to \mathbf{R}^2$ is given by

$$\begin{aligned} f(u,v) &= a_1u + a_2v + \tfrac{v^2}{2} + (b_1+1)\tfrac{u^2}{2} + sf_1(u,v) \\ g(u,v) &= a_3u + a_4v + uv - b_2\tfrac{v^2}{2} + sg_1(u,v), \end{aligned} \tag{1}$$

where $b_1 \neq 0$, $b_1 \neq 1$, $b_1 \neq 1 + \frac{b_2^2}{4}$, $a_2 - a_3 \neq 0$ and f_1 and g_1 are homogeneous polynomials of degree 3, i.e.,

$$f_1(u,v) = \alpha_1u^3 + \alpha_2u^2v + \alpha_3uv^2 + \alpha_4v^3$$

$$g_1(u,v) = \alpha_5u^3 + \alpha_6u^2v + \alpha_7uv^2 + \alpha_8v^3$$

Let $\mathcal{C}$ be the characteristic manifold contained in the wave manifold M see [2]. Given a singularity of the rarefaction foliation, see [2], let ϕ_1 and ϕ_2 be the invariant 1-dimensional manifolds at the singularity.

Considering the shock foliation in the wave manifold, let r be the connected component of the secondary bifurcation see [2], which contains the above mentioned singularity of the rarefaction foliation, and let Σ_1 and Σ_2 be its invariant 2-dimensional manifolds. The main result in this work is the following

Theorem. *Let $f, g, f_1, g_1, \mathcal{C}, \Sigma_1, \Sigma_2, \phi_1, \phi_2$ be defined as above. Then there exists a polynomial in $\alpha = (\alpha_1; \alpha_2; \alpha_3; \alpha_4; \alpha_5; \alpha_6; \alpha_7; \alpha_8)$ such that for α away from its set of zeros and for all s sufficiently small, ϕ_1, ϕ_2, $\Sigma_1 \cap \mathcal{C}$ and $\Sigma_2 \cap \mathcal{C}$ are transversal.*

Remark 1. In [2] it is shown that Σ_2 is a plane $Z = z_i, i = 0, 1, 2$ and Σ_1 is a surface whose equation is computed (they are called there P_i and S_i). In [3] the invariant manifolds ϕ_1 and ϕ_2 of the rarefaction foliation at each singular point are computed; ϕ_2 is a line which is the intersection of $\mathcal{C}$ and Σ_2; ϕ_1 is a curve transversal to $\Sigma_1 \cap \mathcal{C}$.

Remark 2. The fact that the singular set of the shock foliation intersects the characteristic manifold exactly at the singularities of the rarefaction foliation is true for any F, as can be seen in [1], theorem 6.3.

The proof of the theorem will be given in three parts. In Section 2 we will study rarefactions and compute the tangent directions to ϕ_1 and ϕ_2. In Section

3 we will consider shock curves and compute the tangent directions to $\Sigma_1 \cap \mathcal{C}$ and $\Sigma_2 \cap \mathcal{C}$. In Section 4 we will perform the final computations which show that the angle between ϕ_2 and $\Sigma_2 \cap \mathcal{C}$ is non zero for generic α. We refer the interested reader to [4] for general background on systems of conservation laws.

2. Rarefactions

We will follow the same strategy of [3] to construct the characteristic surface. Eliminating the eigenvalue λ between the equations which define the eigenvectors of the matrix DF and putting $z = \frac{dv}{du}$, we obtain the equation

$$f_v z^2 + (f_u - g_v)z - g_u = 0.$$

Substituting all derivatives involved and introducing the new variables

$$z = \frac{dv}{du}$$
$$\tilde{U} = b_1 u + b_2 v + a_1 - a_4,$$
$$\tilde{V} = v + a_2,$$
$$c = a_3 - a_2,$$

we obtain the system

$$\begin{cases} G(\tilde{U}, \tilde{V}, z) = \tilde{V}(z^2 - 1) + \tilde{U}z - c + sh(\tilde{U}, \tilde{V}, z) = 0 \\ z d\tilde{U} - (b_1 + b_2 z) d\tilde{V} = 0 \end{cases} \tag{2}$$

where $h(\tilde{U}, \tilde{V}, z)$ is found by substituting $u(\tilde{U}, \tilde{V})$ and $v(\tilde{U}, \tilde{V})$ in the expression

$$\frac{\partial f_1}{\partial v} z^2 + (\frac{\partial f_1}{\partial u} - \frac{\partial g_1}{\partial v})z - \frac{\partial g_1}{\partial u},$$

which is an easy, but long, computation.

The equation $G(\tilde{U}, \tilde{V}, z) = 0$ defines a 2-dimensional surface $\mathcal{C} \subset \mathbf{R^2} \times \mathbf{P^1}$, which is called the characteristic surface. On this surface we consider the line field defined by the intersection of the kernel of the 1-form

$$\omega = z d\tilde{U} - (b_1 + b_2 z) d\tilde{V}$$

with the tangent space of $\mathcal{C}$ at each point. This line field induces a foliation $\mathcal{F}$ on $\mathcal{C}$ which is singular at points where the matrix

$$\begin{pmatrix} G_{\tilde{U}} & G_{\tilde{V}} & G_z \\ z & -(b_1+b_2z) & 0 \end{pmatrix}$$

has rank less than 2. Thus the foliation $\mathcal{F}$ is singular at points which are solutions of the system

$$\begin{cases} G(\tilde{U},\tilde{V},z)=0 \\ 2z\tilde{V}+\tilde{U}+s\frac{\partial h}{\partial z}=0 \\ z(z^2+b_2z+b_1-1)-s(\frac{\partial h}{\partial \tilde{U}}(b_1+b_2z)-\frac{\partial h}{\partial \tilde{V}})=0 \end{cases}$$

Solving this system for $\tilde{U}$, $\tilde{V}$, z, we see that the singular points are dependent on the parameter s. We are interested in the study of the solutions of this system, for s sufficiently small, which lie in a neighborhood of the solution for $s=0$. As in [3] and [2], we have one or three singular points, depending on whether ${b_2}^2-4(b_1-1)$ is negative or positive.

To study the foliation $\mathcal{F}$ in $\mathcal{C}$ we use $\tilde{U}$, z as coordinates for $\mathcal{C}$, i.e., we use the fact that equation $G(\tilde{U},\tilde{V},z)=0$ can be solved in $\tilde{V}$, for small s, because this is true for $s=0$. However, for $s\neq 0$ we can no longer obtain explicitly the expression of $\tilde{V}$ as a function of z and $\tilde{U}$, since G is polynomial in $\tilde{V}$ of higher degree. Let $\tilde{V}=\psi(\tilde{U},z)$ be the function defined by $G(\tilde{U},\tilde{V},z)=0$. In each region of the $(\tilde{U},z)$-plane where the coordinate system is defined, we have

$$\frac{\partial\psi}{\partial\tilde{U}}=\frac{-s\frac{\partial h}{\partial\tilde{U}}-z}{z^2-1+sn(\tilde{U},z)},$$

$$\frac{\partial\psi}{\partial z}=\frac{-2z\psi(\tilde{U},z)-\tilde{U}-s\frac{\partial h}{\partial z}}{z^2-1+sn(\tilde{U},z)},$$

where $n(\tilde{U},z)$ is found by substituting $\psi(\tilde{U},z)$ in the expression of $\frac{\partial h}{\partial\tilde{V}}$.

In these coordinates, the equation $\omega=0$ becomes

$$n_1(\tilde{U},z)d\tilde{U}-n_2(\tilde{U},z)dz=0,$$

where

$$n_1(\tilde{U},z) = z(z^2+b_2z+b_1-1)+s(zn(\tilde{U},z)-(b_1+b_2z)\frac{\partial h}{\partial \tilde{U}}),$$

$$n_2(\tilde{U},z) = (2z\psi(\tilde{U},z)+\tilde{U})(b_1+b_2z)+s(b_1+b_2z)\frac{\partial h}{\partial z}.$$

In each region of the plane where $\tilde{U}$ and z can be used as coordinates for $\mathcal{C}$, the line field has the same behaviour as the vector field B defined by

$$B = \begin{cases} \dot{\tilde{U}} = n_2(\tilde{U},z), \\ \dot{z} = n_1(\tilde{U},z). \end{cases}$$

The invariant directions of the singularities are given by the eigenspaces of the matrix DB evaluated at each singular point. A straightforward computation shows that the invariant directions at each singular point are

$$w_1(s) = (w_1^1(s), w_1^2(s)),$$

and

$$w_2(s) = (w_2^1(s), w_2^2(s)),$$

where

$$w_1^1(s) = w_2^1(s) = 2\frac{\partial n_2}{\partial z},$$

$$\begin{aligned} w_1^2(s) &= \left(\tfrac{\partial n_1}{\partial z} - \tfrac{\partial n_2}{\partial \tilde{U}}\right) + \sqrt{\left(\tfrac{\partial n_1}{\partial z} - \tfrac{\partial n_2}{\partial \tilde{U}}\right)^2 + 4\left(\tfrac{\partial n_2}{\partial z}\right)\left(\tfrac{\partial n_1}{\partial \tilde{U}}\right)}, \\ w_2^2(s) &= \left(\tfrac{\partial n_1}{\partial z} - \tfrac{\partial n_2}{\partial \tilde{U}}\right) - \sqrt{\left(\tfrac{\partial n_1}{\partial z} - \tfrac{\partial n_2}{\partial \tilde{U}}\right)^2 + 4\left(\tfrac{\partial n_2}{\partial z}\right)\left(\tfrac{\partial n_1}{\partial \tilde{U}}\right)}. \end{aligned} \tag{3}$$

We now turn our attention to shock curves.

3. Shocks

We follow [2] in the study of the shock curves for the flux function F defined by the equations (1). Shock curves for a fixed state (u,v) consist of the set of pairs (u', v') for which there exists λ such that

$$\begin{aligned} f(u,v) - f(u',v') &= \lambda(u-u') \\ g(u,v) - g(u',v') &= \lambda(v-v'). \end{aligned} \tag{4}$$

Equation (4) is called the Rankine-Hugoniot condition.

Eliminating λ in (4) we obtain

$$(f(u,v) - f(u',v'))(v-v') - (g(u,v) - g(u',v'))(u-u') = 0. \tag{5}$$

The set P defined by (5) is the union of the plane $u = u'$; $v = v'$ with a 3-manifold $\mathbf{M}^3$. In this manifold we consider the shock curves defined by $du = dv = 0$. The manifold $\mathbf{M}^3$ will be foliated by an auxiliary 2-dimensional regular foliation, where each leaf is formed by shock curves. On each auxiliary leaf the shock curves are the level curves of a certain Morse function. We will determine also the loci where the shock curves bifurcate in $\mathbf{M}^3$. They turn out to be curves intersecting transversally each auxiliary leaf. These intersection points are saddle singularities of the shock curves in the auxiliary leaf.

As in [2], the manifold $\mathbf{M}^3$ is given by the quadratic polynomial

$$Bz^2 + (A-D)z - C = 0, \tag{6}$$

with:

$$\begin{aligned} z &= \frac{v-v'}{u-u'} \\ A &= \int_0^1 \frac{\partial f}{\partial u} dt \\ B &= \int_0^1 \frac{\partial f}{\partial v} dt \\ C &= \int_0^1 \frac{\partial g}{\partial u} dt \\ D &= \int_0^1 \frac{\partial g}{\partial v} dt \end{aligned}$$

where the integrands are evaluated at $(tu + (1-t)u', tv + (1-t)v')$. By straightforward computations, we obtain

$$A = (b_1 + 1)(\frac{u+u'}{2}) + a_1 + sf_2(u, u', v, v'),$$

$$B = \frac{v+v'}{2} + a_2 + sf_3(u,u',v,v'),$$
$$C = \frac{v+v'}{2} + a_3 + sg_2(u,u',v,v'),$$
$$D = \frac{u+u'}{2} - b_2(\frac{v+v'}{2}) + a_4 + sg_3(u,u',v,v')$$

where

$$f_2 = \int_0^1 \frac{\partial f_1}{\partial u}(tu+(1-t)u', tv+(1-t)v')dt,$$
$$f_3 = \int_0^1 \frac{\partial f_1}{\partial v}(tu+(1-t)u', tv+(1-t)v')dt,$$
$$g_2 = \int_0^1 \frac{\partial g_1}{\partial u}(tu+(1-t)u', tv+(1-t)v')dt,$$
$$g_3 = \int_0^1 \frac{\partial g_1}{\partial v}(tu+(1-t)u', tv+(1-t)v')dt.$$

Substituting A, B, C, D in equation (5) and considering

$$U = \frac{u+u'}{2},$$
$$V = \frac{v+v'}{2},$$
$$X = u-u',$$
$$\tilde{U} = b_1U - b_2V + a_1 - a_4,$$
$$\tilde{V} = V + a_2,$$
$$z = \frac{v-v'}{u-u'}$$

we obtain the following equation for the manifold $\mathbf{M}^3$:

$$\mathcal{G}(\tilde{U},\tilde{V},z,X) = \tilde{V}(z^2-1) + \tilde{U}z - c + sH(\tilde{U},\tilde{V},X,z) = 0, \tag{7}$$

where $H(\tilde{U},\tilde{V},X,z)$ is obtained by substituting $\tilde{U}$, $\tilde{V}$, X, z in the expression $f_3(u,u',v,v')z^2 + (f_2(u,u',v,v') - g_3(u,u',v,v'))z + g_2(u,u',v,v')$.

Thus the set P in the new variables is given by the union of the plane $X=0$ (reflecting the fact that $(u,v)=(u',v')$ satisfies (5)) with the 3-manifold $\mathbf{M}^3 \subset \mathbf{R}^3 \times \mathbf{P}^1$, given by

$$\mathcal{G}(\tilde{U},\tilde{V},z,X) = 0.$$

Embedded in $\mathbf{M}^3$, which is called the wave manifold, we have the characteristic surface $\mathcal{C}$, defined by $X = 0$. In this surface, rarefaction curves are naturally defined (as was done in the previous section).For a more precise definition, see section 3 of [1].

One family of shock curves in $\mathbf{M}^3$ is defined by making u and v constant, or equivalently by $du = dv = 0$. Using the variables introduced above, these equations become, respectively

$$\begin{array}{l} dK = 0 \\ dL = 0, \end{array} \tag{8}$$

where

$$K = b_1 X + 2\tilde{U} - 2b_2\tilde{V}$$

$$L = Xz + 2\tilde{V}.$$

These curves are singular at points where $d\mathcal{G}$, dK, dL are linearly dependent. The singular set of the shock curves is the solution set of the system

$$\begin{cases} \mathcal{G}(\tilde{U}, \tilde{V}, z, X) = 0 \\ \mathcal{H}(\tilde{U}, \tilde{V}, z, X) = 0 \\ \mathcal{W}(\tilde{U}, \tilde{V}, z, X) = 0, \end{cases} \tag{9}$$

where $\mathcal{G} = 0$ is just the fact that the point lies in $\mathbf{M}^3$, and $\mathcal{H} = 0$ and $\mathcal{W} = 0$ are the conditions obtained from the linear dependence of $d\mathcal{G}$, dK, dL. The expressions of $\mathcal{H}$ and $\mathcal{W}$ are given below.

$$\mathcal{H}(\tilde{U}, \tilde{V}, z, X) = z(z^2 + b_2 z + b_1 - 1) - sH_1(\tilde{U}, \tilde{V}, X, z),$$

$$\mathcal{W}(\tilde{U}, \tilde{V}, z, X) = (2z\tilde{V} + \tilde{U}) - \frac{X}{2}(z^2 + b_2 z - 1) + sH_2(\tilde{U}, \tilde{V}, X, z),$$

where H_1 and H_2 are given by:

$$H_1(\tilde{U}, \tilde{V}, X, z) = 2\frac{\partial H}{\partial X} - b_1\frac{\partial H}{\partial \tilde{U}} - (\frac{\partial H}{\partial \tilde{V}} + b_2\frac{\partial H}{\partial \tilde{U}})z$$

and

$$H_2(\tilde{U}, \tilde{V}, X, z) = \frac{1}{2}(2\frac{\partial H}{\partial z} - (\frac{\partial H}{\partial \tilde{V}} + b_2\frac{\partial H}{\partial \tilde{U}})X).$$

The set defined by equation (9) is called the *secondary bifurcation*. Since $d\mathcal{G}$, dK are everywhere linearly independent for $s = 0$, the same will be true for small s. Thus the level surfaces of K define a 2-dimensional regular foliation in $\mathbf{M}^3$. Let $\mathbf{M}_k$ be the level surface defined by $K = k$. $\mathbf{M}_k$ is defined by two equations

$$\begin{cases} \mathcal{G}(\tilde{U}, \tilde{V}, z, X) = 0 \\ K = k, \end{cases} \tag{10}$$

The shock curves on each $\mathbf{M}_k$ are the level curves of the function L, i.e. the shock curves are defined by $L = l$ for different values of l.

In order to study the shock curves on each $\mathbf{M}_k$, we use $\tilde{U}$ and z as coordinates and obtain the shock curves as an equation relating $\tilde{U}$, z, k, l. For k and l fixed, we have a single shock curve in $\mathbf{M}_k$. Keeping k constant and letting l vary, we obtain all shock curves in $\mathbf{M}_k$. For fixed k, we can solve (9) in X, $\tilde{V}$, obtaining $X = X(\tilde{U}, z)$, $\tilde{V} = \nu(\tilde{U}, z)$. In each region of the $(\tilde{U}, z)$-plane where the coordinate system is defined, we have

$$\frac{\partial \nu}{\partial \tilde{U}} = -\frac{z + s\frac{\partial H}{\partial \tilde{U}}}{z^2 - 1 + sm(\tilde{U}, z)}$$

and

$$\frac{\partial \nu}{\partial z} = -\frac{2z\nu(\tilde{U}, z) + \tilde{U} + s\frac{\partial H}{\partial z}}{z^2 - 1 + sm(\tilde{U}, z)},$$

where $m(\tilde{U}, z)$ is obtained by substituting $\nu(\tilde{U}, z)$ in the expression of $\frac{\partial H}{\partial \tilde{V}}$.

Remembering that $L = Xz + 2\tilde{V}$, this equation becomes

$$L = L_k(\tilde{U}, z),$$

where

$$L_k(\tilde{U}, z) = \frac{z(k - 2\tilde{U}) + 2(b_1 + b_2 z)\nu(\tilde{U}, z)}{b_1}.$$

The singular points of the shock curves are given by

$$\frac{\partial L_k}{\partial \tilde{U}} = \frac{\partial L_k}{\partial z} = 0.$$

L_k which cross it. Each singularity in $\mathbf{M}_k$ is a saddle point, since it is shown in [2] that it is so for $s = 0$, and we are considering small s. When k varies, the separatrices of the singularities generate two surfaces of dimension 2, which we call Σ_1 and Σ_2. These surfaces intersect (transversally) along the secondary bifurcation.For each $i = 1, 2$, $\Sigma_i \cap \mathcal{C}$ is a curve passing through the corresponding singular point of the rarefaction foliation. When $s = 0$, Σ_2 becomes a plane which intersects $\mathcal{C}$ along separatrices of the rarefaction foliation at the singular point in question, while Σ_1 remains transversal to both separatrices.

4. Transversality

Let p be a singular point of the rarefaction foliation in $\mathcal{C}$; we will now compute the vectors $t_1(s)$ and $t_2(s)$ which generate $T_p(\Sigma_1 \cap \mathcal{C})$ and $T_p(\Sigma_2 \cap \mathcal{C})$, respectively, and the vectors r_1 and r_2 which are tangent to the invariant 1-dimensional manifolds of the rarefaction foliation. For $s = 0$, we know that r_1, r_2 and t_1 are already independent and r_2 coincides with t_2. The vectors r_1 and r_2 are obtained from w_1 and w_2 by applying to them the matrix C below.

$$C = \begin{pmatrix} 1 & 0 \\ \frac{\partial \psi}{\partial \tilde{U}} & \frac{\partial \psi}{\partial z} \\ 0 & 1 \end{pmatrix},$$

evaluated at singular points, obtaining

$$r_1(s) = (w_1^1(s), w_1^1(s)\frac{\partial \psi}{\partial \tilde{U}}; w_1^2(s)),$$

$$r_2(s) = (w_2^1(s), w_2^1(s)\frac{\partial \psi}{\partial \tilde{U}}; w_2^2(s)).$$

This is so because w_1 and w_2 were computed in coordinates $\tilde{U}$ and z . Now we are regard them as vectors in the $(\tilde{U}, \tilde{V}, z)$ space.

The first step is to obtain the tangent directions to the invariant manifold of the shock foliation restricted to $\mathbf{M}_k$ at point p. To do this we use $\tilde{U}$ and z as coordinates for $\mathbf{M}_k$ and we consider the second order Taylor expansion of L_k in a neighborhood of p:

$$\mathcal{L}_k(\tilde{U},z)=(\frac{1}{2}\frac{\partial^2 L_k}{\partial \tilde{U}^2})\tilde{U}^2+(\frac{\partial^2 L_k}{\partial \tilde{U}\partial z})\tilde{U}z+(\frac{1}{2}\frac{\partial^2 L_k}{\partial z^2})z^2,$$

where all derivatives are evaluated at the projection $\bar{p}$ of p in $(\tilde{U},z)$-plane. Solving the equation $\mathcal{L}_k(\tilde{U},z)=0$ in z, we have

$$\mathcal{L}_k(\tilde{U},z)=(z+d_1(s)\tilde{U})(z+d_2(s)\tilde{U}),$$

where

$$d_1(s)=\frac{\frac{\partial^2 L_k}{\partial \tilde{U}\partial z}-\sqrt{(\frac{\partial^2 L_k}{\partial \tilde{U}\partial z})^2-(\frac{\partial^2 L_k}{\partial \tilde{U}^2})(\frac{\partial^2 L_k}{\partial z^2})}}{\frac{\partial^2 L_k}{\partial z^2}},$$

and

$$d_2(s)=\frac{\frac{\partial^2 L_k}{\partial \tilde{U}\partial z}+\sqrt{(\frac{\partial^2 L_k}{\partial \tilde{U}\partial z})^2-(\frac{\partial^2 L_k}{\partial \tilde{U}^2})(\frac{\partial^2 L_k}{\partial z^2})}}{\frac{\partial^2 L_k}{\partial z^2}}$$

It follows that the tangent directions to the invariant manifolds in coordinates $\tilde{U}$ and z are given by

$$v_1(s)=(v_1^1(s),v_1^2(s))=(-1,d_1(s))$$

$$v_2(s)=(v_2^1(s),v_2^2(s))=(-1,d_2(s)).$$

Let A be the matrix

$$A=\begin{pmatrix}1 & 0\\ \frac{\partial\nu}{\partial\tilde{U}} & \frac{\partial\nu}{\partial z}\\ 0 & 1\\ \frac{\partial X}{\partial\tilde{U}} & \frac{\partial X}{\partial z}\end{pmatrix}. \tag{11}$$

One obtains the pre-image of $v_1(s)$, $v_2(s)$, in $\mathbf{M}^3$ by computing the vectors $V_1(s)=A(p).v_1(s)$ and $V_2(s)=A(p).v_2(s)$. We consider $\mathbf{M}^3$ as a surface in $(\tilde{U},\tilde{V},z,X)$ space. We have:

$$\begin{array}{c}V_1(s)=(V_1^1(s),V_1^2(s),V_1^3(s),V_1^4(s))\\ V_2(s)=(V_2^1(s),V_2^2(s),V_2^3(s),V_2^4(s)),\end{array} \tag{12}$$

where

$$V_1^1(s) = v_1^1(s),$$
$$V_1^2(s) = v_1^1(s)\frac{\partial \nu}{\partial \tilde{U}} + v_1^2(s)\frac{\partial \nu}{\partial z},$$
$$V_1^3(s) = v_1^2(s),$$
$$V_1^4(s) = v_1^1(s)\frac{\partial X}{\partial \tilde{U}} + v_1^2(s)\frac{\partial X}{\partial z},$$
$$V_2^1(s) = v_2^1(s),$$
$$V_2^2(s) = v_1^2(s)\frac{\partial \nu}{\partial \tilde{U}},$$
$$V_2^3(s) = v_2^2(s),$$
$$V_2^4(s) = v_2^1(s)\frac{\partial X}{\partial \tilde{U}},$$

and where all derivatives are evaluated at $\bar{p}$.

Let S_B be the matrix

$$S_B = \begin{pmatrix} \nabla \mathcal{G} \\ \nabla \mathcal{H} \\ \nabla \mathcal{W} \end{pmatrix},$$

where $\mathcal{G}$, $\mathcal{H}$, $\mathcal{W}$ are defined in (9). The tangent space to the secondary bifurcation at the singular point p is given by the kernel of S_B evaluated at p. Taking $X = 0$ and computing the kernel of S_B, we obtain a vector of the form

$$b(s) = (b_1(s), b_2(s), b_3(s), b_4(s)).$$

A vector in $T_p\Sigma_i \cap \mathcal{C}$, $i = 1, 2$, is a linear combination of each vector V_i, $i = 1, 2$ and $b(s)$. Thus a vector in the tangent space $T_p\Sigma_i \cap \mathcal{C}$, $i = 1, 2$ has coordinates

$$\begin{aligned} \tilde{U} &= rV_i^1 + tb_1(s) \\ \tilde{V} &= rV_i^2 + tb_2(s) \\ z &= rV_i^3 + tb_3(s) \\ X &= rV_i^4 + tb_4(s). \end{aligned}$$

For $X = 0$ we solve this system, obtaining

$$t_1(s) = (b_4(s)V_1^1(s) - b_1(s)V_1^4(s), b_4(s)V_1^2(s) - b_2(s)V_1^4(s), b_4(s)V_1^3(s) - V_1^4(s)b_3(s), 0)$$

$t_2(s) = (b_4(s)V_2^1(s) - b_1(s)V_2^4(s), b_4(s)V_2^2(s) - b_2(s)V_2^4(s), b_4(s)V_2^3(s) - V_2^4(s)b_3(s), 0).$

In order to compare $t_i(s)$ with vectors $r_i(s)$ we consider the inclusion of $r_i(s)$ in $T_p\mathbf{M}^3$, i.e., we consider the vectors

$$R_1(s) = (w_1^1(s), w_1^1(s)\frac{\partial\psi}{\partial\tilde{U}}, w_1^2(s), 0)$$

$$R_2(s) = (w_2^1(s), w_2^1(s)\frac{\partial\psi}{\partial\tilde{U}}, w_2^2(s), 0),$$

as vectors in $(\tilde{U}, \tilde{V}, z, X)$-space.

It is easy to see that the invariant manifolds of the shock and rarefaction foliations which coincide for $s = 0$ are $V_2(s)$ and $r_2(s)$. Thus, we must compare the vectors $t_2(s)$ and $R_2(s)$ in $T_p\mathbf{M}^3$ and show that they are linearl y independent.

What we have to show now is that the vectors R_2 and t_2, which are linearly dependent for $s = 0$, become independent for small non zero s, and generic α. Let $\theta(s, \alpha)$ be the angle between $R_2(s, \alpha)$ and $t_2(s, \alpha)$. We have $\theta(0, \alpha) = 0$ for all α. It is sufficient to show that, generically in α, $\frac{\partial\theta}{\partial s}(0, \alpha) \neq 0$, since this implies that the graph $s \longmapsto \theta_\alpha(s)$ cuts the s axis transversely at the point $s = 0$, $\theta_\alpha = 0$, where we write $\theta_\alpha(s)$ for $\theta(s, \alpha)$.

Differentiating

$$cos\theta_\alpha(s) = \frac{\langle t_2(s), R_2(s)\rangle}{\|t_2(s)\|\|R_2(s)\|} \tag{13}$$

twice, we obtain

$$-cos\theta_\alpha(s)(\theta'_\alpha(s))^2 - sin\theta_\alpha(s)(\theta'_\alpha(s)) = \Big(\frac{\langle t_2(s), R_2(s)\rangle}{\|t_2(s)\|\|R_2(s)\|}\Big)'', \tag{14}$$

and making $s = 0$ we obtain

$$-\theta'_\alpha(0)^2 = \Big(\frac{\langle t_2, R_2\rangle}{\|t_2\|\|R_2\|}\Big)''(0). \tag{15}$$

It is easy to see that $\theta'(0)^2$ is the quotient of two expressions which involve polynomials and square roots of polynomials in α, so we can rationalize its numerator, obtaining a new polynomial which has the same set of zeros as $\theta'(0)^2$. This concludes the proof.

Final remark: A natural question is whether the transversality we have just obtained between ϕ_2 and $\Sigma_2 \cap \mathcal{C}$ is preserved by projection in (u, v)-space, or which is the same, $(\tilde{U}, \tilde{V})$-space. It is just a matter of considering, in the above computations, only the first two coordinates of the vectors R_2, and t_2. We will obtain a new polynomial in α, whose set of zeros must be avoided. By multiplying this polynomial and the one we obtained before, we get a new one such that if α is not in its set of zeros, not only R_2 and t_2 will be transversal, but their projections in the original (u, v)-space will also be transversal.

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