GLOBAL HYPOELLIPTICITY FOR SUMS OF SQUARES OF VECTOR FIELDS OF INFINITE TYPE

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Abstract

We prove global C^∞ regularity for a family of sums of squares operators on the torus, which may not satisfy the bracket condition. Instead, an independence condition on the coefficients is shown to be necessary and sufficient for global hypoellipticity.

Resumo

Neste trabalho provamos a regularidade C^∞ no toro para uma família de operadores na forma de uma soma de quadrados, os quais podem não satisfazer a condição do colchete. Mostramos que uma condição de independência sobre os coeficientes é necessária e suficiente para a hipoeliticidade global.

1. Introduction and Results

Let $\mathcal{M}^n$ be a C^∞ manifold and $X = \{X_1, \ldots, X_\nu\}$ be a collection of real vector fields with C^∞ coefficients on $\mathcal{M}^n$. The "sum of squares operator" or sublaplacian associated to the vector fields X is the second order operator $P = \Delta_X \doteq -(X_1^2 + \cdots + X_\nu^2)$. P is said (locally) hypoelliptic if for any open set $V \subset \mathcal{M}^n$ the conditions $Pu = f, u \in \mathcal{D}'(V)$, and $f \in C^\infty(V)$ imply $u \in C^\infty(V)$. P is said globally hypoelliptic if the above implication holds for $V = \mathcal{M}^n$. Observe that hypoellipticity implies global hypoellipticity. A point $x_0 \in \mathcal{M}^n$ is said to be of finite type if the dimension of the Lie algebra generated by X at x_0 is equal to the dimension of the manifold. Otherwise it is said to be

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of infinite type. By the celebrated theorem of Hörmander [10] (see also Kohn [11], Oleinik-Radkevich [13], and Rothschild-Stein [14]) Δ_X is hypoelliptic if all points of $\mathcal{M}^n$ are of finite type. However the finite type condition is not necessary for local or global hypoellipticity (see Bell-Mohammed [2], Fedii [5], Kusuoka -Strook [12]). For example, the operator $P = -\partial_t^2 - a^2(t)\partial_x^2$ is hypoelliptic if a is even, nonnegative, nondecreasing on $[0, \infty)$, and $a(t) = 0$ only for $x = 0$ (see [5]). Notice if 0 is a zero of a of infinite order then $t = 0$ is of infinite type. If $\mathbb{T}^2$ denotes the 2-dimensional torus, then P is globally hypoelliptic on $\mathbb{T}^2$ if and only if $a(t_0) \neq 0$ for some $t_0 \in \mathbb{T}^1$, (see [6]). Observe that a can be chosen so that most points (except for points in a set of arbitrarily small area) on $\mathbb{T}^2$ are of infinite type.

In this work we consider a class of operators Δ_X on the torus where the finite type condition and/or the reachability condition (see Sussman [15]) may fail but global hypoellipticity holds. This paper consists of an extension of theorem 1.2 in [9]. While no satisfactory characterization of global hypoellipticity exists in the literature, it is hoped that our result provides some insight into this open problem.

Theorem 1. *Let $\mathbb{T}^{n+2m}$ be the $(n+2m)$-dimensional torus with variables $(t, x, y) = (t_1, \ldots, t_n, x_1, \ldots, x_m, y_1, \ldots, y_m)$, and P be the operator defined by*

$$P = -\Delta_t - \sum_{j=1}^{m} \left(a_j(t)\partial_{x_j} + b_j(t)\partial_{y_j} \right)^2,$$

where a_j, b_j are in $C^\infty(\mathbb{T}^n)$ and real valued. Then P is globally hypoelliptic on $\mathbb{T}^{n+2m}$ if and only if for each fixed j, the coefficient b_j is not identically equal to zero and $a_j \neq \lambda b_j$ for any $\lambda \in Q \cup L$, where Q is the rationals and L is the set of Liouville numbers.

Remark. The conditions $b_j \not\equiv 0$ and $a_j \neq \lambda b_j$ for any $\lambda \in Q \cup L$ are equivalent to $a_j \not\equiv 0$ and $b_j \neq \mu a_j$ for any $\mu \in Q \cup L$. This follows from the fact that $\lambda \notin Q \cup L$ is equivalent to $\lambda^{-1} \notin Q \cup L$. We recall that an irrational number λ is *Liouville* if it can be approximated to any order by rational numbers. That is, for any $C > 0$ and $K > 0$ there exists $(p, q) \in \mathbb{Z}^2 - (0,0)$ such that

$$\left|\lambda - \frac{p}{q}\right| < \frac{C}{q^K}.$$

For a partial list of results on global hypoellipticity we refer the reader to Amano [1], Bergamasco, Cordaro and Malagutti [3], Cordaro and Himonas [4], Gramchev, Popivanov and Yoshino [7], Greenfield and Wallach [8], Taira [16], and Tartakoff [17].

2. Proof of Theorem 1.

Necessity: Assume that the condition in Theorem 1 does not hold. Then for some j, $1 \leq j \leq m$, either $b_j \equiv 0$ or $a_j = \lambda b_j$ for some $\lambda \in Q \cup L$. If b_j is identically zero for some j, $1 \leq j \leq m$, then P is not globally hypoelliptic since any function $u = u(y_j)$ which depends only on the variable y_j is a solution to $Pu = 0$. We now suppose that there exists $\lambda \in Q \cup L$ such that $a_j(t) = \lambda b_j(t)$ for all $t \in (-\pi, \pi)^n$. Then

$$P = -\Delta_t - \sum_{\substack{l=1 \\ l \neq j}}^{m} (a_l(t)\partial_{x_l} + b_l(t)\partial_{y_l})^2 - b_j^2(t)\left(\lambda\partial_{x_j} + \partial_{y_j}\right)^2.$$

Since $\lambda \in Q \cup L$ there exists $u \in D'(\mathbb{T}^2_{x_j y_j}) - C^\infty(\mathbb{T}^2_{x_j y_j})$ such that $(\lambda\partial_{x_j} + \partial_{y_j})u \in C^\infty(\mathbb{T}^2_{x_j y_j})$ (see Greenfield-Wallach [8]). Therefore, P is not globally hypoelliptic on $\mathbb{T}^{n+2m}$.

Sufficiency: Let $u \in D'(\mathbb{T}^{n+2m})$ be such that

$$Pu = f, \; f \in C^\infty(\mathbb{T}^{n+2m}). \tag{2.1}$$

By taking partial Fourier transform with respect to $(x, y) \in \mathbb{T}^{2m}$ we obtain

$$\left[-\Delta_t + \sum_{j=1}^{m} (a_j(t)\xi_j + b_j(t)\eta_j)^2\right] \hat{u}(t, \xi, \eta) = \hat{f}(t, \xi, \eta), \tag{2.2}$$

for all $t \in T^n$ and $(\xi, \eta) \in \mathbb{Z}^{2m}$. Note that by the elliptic theory for each fixed

$(\xi,\eta)\in\mathbb{Z}^{2m}$ the solution $\hat{u}(\cdot,\xi,\eta)$ to equation (2.2) is in $C^\infty(\mathbb{T}^n)$. If we multiply (2.2) by $\bar{\hat{u}}(t,\xi,\eta)$ and integrate by parts with respect to $t\in\mathbb{T}^n$, then we obtain

$$\begin{aligned}\sum_{k=1}^{n}\|\hat{u}_{t_k}(\cdot,\xi,\eta)\|^2_{L^2(\mathbb{T}^n)} &+\int_{\mathbb{T}^n}\sum_{j=1}^{m}(a_j(t)\xi_j+b_j(t)\eta_j)^2|\hat{u}(t,\xi,\eta)|^2dt\\ &=\int_{\mathbb{T}^n}\hat{f}(t,\xi,\eta)\bar{\hat{u}}(t,\xi,\eta)dt.\end{aligned}\tag{2.3}$$

For each $j=1,2,\dots,m$ and $(\xi_j,\eta_j)\in\mathbb{Z}^2$ we let

$$\|\varphi\|^2_{C_j}\doteq\sum_{k=1}^{n}\|\varphi_{t_k}\|^2_{L^2(\mathbb{T}^n)}+\int_{\mathbb{T}^n}C_j^2(t,\xi_j,\eta_j)|\varphi(t)|^2dt,\quad \varphi\in C^\infty(\mathbb{T}^n),\tag{2.4}$$

where

$$C_j(t,\xi_j,\eta_j)=a_j(t)\xi_j+b_j(t)\eta_j.\tag{2.5}$$

We shall need the following lemma.

Lemma 1. *If for each fixed j, b_j is not identically equal to zero and $a_j\neq\lambda b_j$ for any $\lambda\in Q\cup L$, then for each $(\xi_j,\eta_j)\in\mathbb{Z}^2-0$ there exist constants $K_j>0, L_j\geq 0$, $\delta_j>0$ independent of (ξ_j,η_j), and there exists an n-dimensional interval $I_j=I_j(\xi_j,\eta_j)$ such that*

$$C_j^2(t,\xi_j,\eta_j)\geq\frac{K_j}{|(\xi_j,\eta_j)|^{L_j}}, t\in I_j, \mathrm{Vol}(I_j)\geq\delta_j.\tag{2.6}$$

Before proving Lemma 1 we shall show that inequality (2.6) implies that the operator P is globally hypoelliptic in $\mathbb{T}^{n+2m}$. By using the fundamental theorem of calculus for $s\in I_j$ and $t\in(-\pi,\pi)^n$ we obtain for any $\varphi\in C^\infty(\mathbb{T}^n)$

$$|\varphi(t)|^2\leq C\left(|\varphi(s)|^2+\sum_{k=1}^{n}\int_{-\pi}^{\pi}|\varphi_{\rho_k}(s_1,\dots,s_{k-1},\rho_k,t_{k+1},\dots,t_n)|^2d\rho_k\right),\tag{2.7}$$

where C is a positive constant. Integrating for $t\in(-\pi,\pi)^n$ and for $s\in I_j=I_j(\xi_j,\eta_j)$ gives

$$(\mathrm{Vol}\ I_j)\|\varphi\|^2_{L^2(\mathbb{T}^n)}\leq K_j\left(\int_{I_j}|\varphi(s)|^2ds+(\mathrm{Vol}\ I_j)\sum_{k=1}^{n}\|\varphi_{t_k}\|^2_{L^2(\mathbb{T}^n)}\right),\tag{2.8}$$

where K_j is a positive constant (in the following K_j will represent several different positive constants). By using (2.6) we have

$$\int_{I_j} |\varphi(s)|^2 ds \leq K_j^{-1} |(\xi_j,\eta_j)|^{L_j} \int_{\mathbb{T}^n} C_j^2(s,\xi_j,\eta_j)|\varphi(s)|^2 ds. \tag{2.9}$$

Then (2.8) and (2.9) give

$$\begin{aligned} \|\varphi\|^2_{L^2(\mathbb{T}^n)} &\leq K_j|(\xi_j,\eta_j)|^{L_j} \textstyle\int_{\mathbb{T}^n} C_j^2(s,\xi_j,\eta_j)^2|\varphi(s)|^2 ds \\ &+K_j \textstyle\sum_{k=1}^n \|\varphi_{t_k}\|^2_{L^2(\mathbb{T}^n)}. \end{aligned} \tag{2.10}$$

Since $(\xi_j,\eta_j) \in \mathbb{Z}^2 - 0$, (2.10) implies that

$$\|\varphi\|^2_{L^2(\mathbb{T}^n)} \leq K_j|(\xi_j,\eta_j)|^{L_j}\|\varphi\|^2_{C_j} \leq K|(\xi,\eta)|^L\|\varphi\|^2_{C_j}, \tag{2.11}$$

where $K = \max\{K_1,\ldots,K_m\}$ and $L = max\{L_1,\ldots,L_m\}$.

Let $(\xi,\eta) \in \mathbb{Z}^{2m} - 0$. Then there exists $j \in \{1,\ldots,m\}$ such that $(\xi_j,\eta_j) \in \mathbb{Z}^2 - 0$. If we apply (2.10) with $\varphi(t) = \hat{u}(t,\xi,\eta)$ then we obtain

$$\|\hat{u}(\cdot,\xi,\eta)\|^2_{L^2(\mathbb{T}^n)} \leq K|(\xi,\eta)|^L\|\hat{u}(\cdot,\xi,\eta)\|^2_{C_j}. \tag{2.12}$$

By (2.3), (2.4), and (2.12), we obtain

$$\|\hat{u}(\cdot,\xi,\eta)\|^2_{L^2(\mathbb{T}^n)} \leq K|(\xi,\eta)|^L \int_{\mathbb{T}^n} \hat{f}(t,\xi,\eta)\bar{\hat{u}}(t,\xi,\eta)dt.$$

This and the Cauchy-Schwarz inequality give

$$\|\hat{u}(\cdot,\xi,\eta)\|_{L^2(\mathbb{T}^n)} \leq K|(\xi,\eta)|^L\|\hat{f}(\cdot,\xi,\eta)\|_{L^2(\mathbb{T}^n)}. \tag{2.13}$$

If $u \in D'(\mathbb{T}^{n+2m})$, $Pu = f$, $f \in C^\infty(\mathbb{T}^{n+2m})$ then by (2.13) we obtain that for any positive integer N there exists $C_N > 0$ such that

$$\|\hat{u}(\cdot,\xi,\eta)\|_{L^2(\mathbb{T}^n)} \leq C_N |(\xi,\eta)|^{-N}, (\xi,\eta) \in \mathbb{Z}^{2m} - 0.$$

Since

$$\hat{u}(\tau,\xi,\eta) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} e^{-it\cdot\tau} \hat{u}(t,\xi,\eta) dt,$$

by the last inequality and Cauchy-Schwarz inequality we obtain

$$|\hat{u}(\tau,\xi,\eta)| \leq C'_N |(\xi,\eta)|^{-N}, (\tau,\xi,\eta) \in \mathbb{Z}^{n+2m}, \quad (\xi,\eta) \neq 0.$$

If $(\tau_0,\xi_0,\eta_0) \in \mathbb{Z}^{n+2m}$, with $(\xi_0,\eta_0) \neq 0$, then there exist $c > 0$ such that $(\tau_0,\xi_0,\eta_0) \in \Gamma \doteq \{(\tau,\xi,\eta) \in \mathbb{Z}^{n+2m} : |\tau| < c\ |(\xi,\eta)|\}$. Then the last inequality gives

$$|\hat{u}(\tau,\xi,\eta)| \leq C_N^{\Gamma} \frac{1}{(|\tau| + |(\xi,\eta)|)^N},\ (\tau,\xi,\eta) \in \Gamma. \tag{2.14}$$

By the elliptic theory we obtain similar estimates near directions (τ_0,ξ_0,η_0) with $(\xi_0,\eta_0) = 0$. Therefore $u \in C^\infty(\mathbb{T}^{n+2m})$, which shows that P is globally hypoelliptic in $\mathbb{T}^{n+2m}$.

3. Proof of Lemma 2.1

By our hypothesis we know that a_j and b_j are not identically equal to zero. Since $a_j \not\equiv 0$ there exists an interval $I_j \subset [-\pi,\pi]^n$ and a constant $K_j > 0$ such that

$$a_j^2(t) \geq K_j > 0, \text{ for all } t \in I_j.$$

Thus, if $\eta_j = 0$ and $\xi_j \neq 0$ then

$$C_j^2(t,\xi_j,0) = a_j^2(t)\xi_j^2 \geq a_j^2(t) \geq K_j > 0,\ t \in I_j.$$

Similarly, if $\eta_j \neq 0$ and $\xi_j = 0$ then there exists an interval $I_j \subset [-\pi,\pi]^n$ and $K_j > 0$ such that

$$C_j^2(t,0,\eta_j) = b_j^2(t)\eta_j^2 \geq K_j > 0,\ t \in I_j,\ \eta_j \neq 0.$$

Now we assume $\xi_j \neq 0$ and $\eta_j \neq 0$, and consider several cases. Set

$$A_j = \{t \in [-\pi, \pi]^n : a_j(t) \neq 0\}, \text{ and } B_j = \{t \in [-\pi, \pi]^n : b_j(t) \neq 0\}. \quad (3.1)$$

Therefore, supp $a_j = \bar{A}_j$ and supp $b_j = \bar{B}_j$.

Case 1. supp $a_j \neq$ supp b_j.

We suppose that there exists $t_0 \in$ supp a_j such that $t_0 \notin$ supp b_j (the other case is similar). Since $t_0 \notin$ supp b_j there exists an interval J such that $t_0 \in J$ and $b_j \equiv 0$ on J. Since $t_0 \in$ supp a_j we have $J \cap A_j \neq \emptyset$. Let $t_1 \in J \cap A_j$, then there exists an interval I_j such that $t_1 \in I_j \subset J \cap A_j$ and $a_j^2(t) \geq K_j > 0$ for all $t \in I_j$. Thus

$$C_j^2(t, \xi_j, \eta_j) = a_j^2(t)\xi_j^2 \geq a_j^2(t) \geq K_j, \ t \in I_j.$$

Case 2. supp $a_j =$ supp b_j.

We consider the function

$$r_j(t) \doteq \frac{b_j}{a_j} : A_j \longrightarrow \mathbb{R}, \quad (3.2)$$

and distinguish two cases.

Case 2.1. We assume that there exists $t_0 \in A_j$ such that $r_j'(t_0) \neq 0$. Therefore there exists an interval J_j such that $t_0 \in J_j \subset A_j$, $r_j'(t) \neq 0$ for all $t \in J_j$, and $a_j^2(t) \geq K_j > 0$ for all $t \in J_j$. Now we let

$$\alpha_j = M_j - m_j, \text{ where } M_j = \max_{t \in J_j} r_j(t), \quad m_j = \min_{t \in J_j} r_j(t).$$

Since $r_j' \not\equiv 0$ we have $\alpha_j > 0$. Then by the uniform continuity of r_j on $\bar{J}_j$ there exists $\delta > 0$ independent of (ξ_j, η_j), and a subinterval I_j of J_j such that

$$(\xi_j + r_j(t)\eta_j)^2 > \mu_j > 0, \quad t \in I_j, \, \mathrm{Vol}(I_j) \geq \left(\frac{\delta}{2}\right)^n,$$

where $\mu_j = \min\left\{1, \frac{\alpha_j^2}{64}\right\}$. For the details of this argument we refer the reader to Lemma 3.2 in [9]. Thus we obtain

$$C_j^2(t, \xi_j, \eta_j) = a_j^2(t)\left(\xi_j + \frac{b_j(t)}{a_j(t)}\eta_j\right)^2 \geq \mu_j K_j, \ t \in I_j.$$

Case 2.2. We now assume that

$$r_j' \equiv 0 \text{ on } A_j.$$

We may write $A_j = \bigcup_{l=1}^{\infty} G_{jl}$ where G_{jl} are open and connected sets with $G_{jl_1} \cap G_{jl_2} = \emptyset$ if $jl_1 \neq jl_2$. Then

$$r_j = \lambda_{jl} = \text{ constant on } G_{jl},\ l = 1, 2, \ldots$$

Case 2.2.1. We assume that $\lambda_{jl} = \lambda_j$ for all $l = 1, 2, \ldots$. Then it follows that $b_j(t) - \lambda_j a_j(t) = 0$ for all $t \in [-\pi, \pi]^n$, and λ_j must be a non-Liouville number. Therefore there exists an interval $I_j \subset [-\pi, \pi]^n$ and constants $K_j > 0$ and $L_j > 0$ such that

$$C_j^2(t, \xi_j, \eta_j) \geq K_j |(\xi_j, \eta_j)|^{-L_j},\ t \in I_j.$$

Case 2.2.2. We suppose that there are $\lambda_{jl_1} \neq \lambda_{jl_2}$. For simplicity we write $\lambda_{jl_1} = \lambda_1$ and $\lambda_{jl_2} = \lambda_2$. Then we have

$$b_j(t) = \lambda_1 a_j(t),\ t \in G_1,$$

and

$$b_j(t) = \lambda_2 a_j(t),\ t \in G_2,$$

with $\lambda_1 \neq \lambda_2$ and $G_1 \cap G_2 = \emptyset$. We will analyze only the case where $\lambda_2 > \lambda_1 > 0$ (the other cases are similar). Let $c = \frac{\lambda_1 + \lambda_2}{2}$ and consider the region

$$R_1 = \{(\xi_j, \eta_j) \in \mathbb{Z}^2 :\ \eta_j > 0,\ \xi_j \geq -c\eta_j\}.$$

In R_1 we have $\xi_j + \lambda_2 \eta_j \geq -c\eta_j + \lambda_2 \eta_j = \frac{(\lambda_2 - \lambda_1)}{2} \eta_j > 0$. Therefore

$$|\xi_j + \lambda_2 \eta_j| > \frac{(\lambda_2 - \lambda_1)}{2} \eta_j \geq \frac{\lambda_2 - \lambda_1}{2} = K_j > 0. \tag{3.3}$$

In the region

$$R_2 = \{(\xi_j, \eta_j) \in \mathbb{Z}^2 : \eta_j < 0, \xi_j \leq -c\eta_j\}$$

we also can prove that

$$|\xi_j + \lambda_2 \eta_j| \geq \frac{\lambda_2 - \lambda_1}{2} = K_j > 0. \tag{3.4}$$

In the regions

$$R_3 = \{(\xi_j, \eta_j) \in \mathbb{Z}^2 : \xi_j < 0,\ \eta_j > 0,\ \xi_j \leq -c\eta_j\}$$

and

$$R_4 = \{(\xi_j, \eta_j) \in \mathbb{Z}^2 : \xi_j > 0,\ \eta_j < 0,\ \xi_j \geq -c\eta_j\}$$

we can prove that

$$|\xi_j + \lambda_1 \eta_j| \geq \frac{\lambda_2 - \lambda_1}{2} = K_j > 0. \tag{3.5}$$

We observe that $\mathbb{Z}^2 - \{(\xi_j, 0)\} = \bigcup_{j=1}^{4} R_j$. Since $a_j \neq 0$ for all $t \in G_2$ it follows from (3.3) and (3.4) that there exists an interval $I_j \subset G_2$ such that

$$C_j^2(t, \xi_j, \eta_j) = a_j^2(t)(\xi_j + \lambda_2 \eta_j)^2 \geq K_j > 0, t \in I_j \text{ and } (\xi_j, \eta_j) \in R_1 \cup R_2.$$

Similarly it follows from (3.5) that there exist an interval $I_j \subset G_1$ and $K_j > 0$ such that

$$C_j^2(t, \xi_j, \eta_j) = a_j^2(t)(\xi_j + \lambda_1 \eta_j)^2 \geq K_j > 0, t \in I_j \text{ and } (\xi_j, \eta_j) \in R_3 \cup R_4.$$

The proof of Lemma 1 is complete.

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References

[1] Amano, K., *The Global Hypoellipticity of a class of degenerate elliptic-parabolic operators*, Proc. Japan Acad. 60, Ser. 4, (1984), 312-314.

[2] Bell, D. R. and Mohammed, S. A., *An extension of Hörmander's theorem for infinitely degenerate second-order operators*, Duke Math. J. 78, (1995), 453-475.

[3] Bergamasco, A. P., Cordaro, P. D. and Malagutti, P. A., *Globally hypoelliptic systems of vector fields*, J. Funct. Anal. 114, no.2, (1993), 267-285.

[4] Cordaro, P. D. and Himonas, A. A., *Global analytic hypoellipticity for sums of squares of vector fields*, to appear in the Trans. Amer. Math. Soc.

[5] Fedii, V. S., *Estimates in H^s norms and hypoellipticity*, Dokl. Akad. Nauk SSSR, 193, No. 2, (1970), 940-942.

[6] Fujiwara, D. and Omori, H., *An example of a globally hypoelliptic operator*, Hokkaido Math. J. 12, (1983), 293-297.

[7] Gramchev, T., Popivanov, P. and Yoshino, M., *Global properties in spaces of generalized functions on the torus for second order differential operators with variable coefficients*, Rend. Sem. Mat. Pol. Torino 51, 2, (1993), 145-172.

[8] Greenfield, S. J. and Wallach, N. R., *Global hypoelliptic and Liouville numbers*, Proc. Amer. Math. Soc. 31, (1972), 112-114.

[9] Himonas, A. A., *On degenerate elliptic operators of infinite type*, Math. Z. 220, (1995), 449-460.

[10] Hörmander, L., *Hypoelliptic second order differential equations*, Acta Math. 119, (1967), 147-171.

[11] Kohn, J. J., *Pseudo-differential operators and hypoellipticity*, Proc. of Symposia in Pure Math. XXIII, (1973), 61-70.

[12] Kusuoka, S. and Strook, D., *Applications of the Malliavin calculus, Part II*, J. Fac. Sci. Tokio, Sect. IA, Math. 32, (1985), 1-76.

[13] Oleinik, O. A. and Radkevic, E. V., *Second order equations with nonnegative characteristic form*, Amer. Math. Soc. and Plenum Press, (1973).

[14] Rothschild, L. P. and Stein, E. M., *Hypoelliptic differential operators and nilpotent groups*, Acta Math. 137, (1977), 247-320.

[15] Sussmann, H. J., *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. 180 (1973), 171-188.

[16] Taira, K., *Le principe du maximum et l'hypoellipticite globale*, Seminaire Bony-Söstrand-Meyer, Expose no. I 348, No. 7 (1984).

[17] Tartakoff, D. S., *Global (and local) analyticity for second order operators constructed from rigid vector fields on products of tori*, Trans. Amer. Math. Soc. 348, No 7, (1996), 2577-2583.

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