SOME RESULTS AND QUESTIONS ON NIL RINGS

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Abstract

The purpose of this paper is to survey some old and new results on associative nil rings. Several of them are connected with Koethe's problem. We also formulate some open problems and make several comments and remarks.

1. Introduction and Preliminaries

Many open problems in the theory of associative rings concern nil rings. The most famous is Koethe's problem (it asks whether the two-sided ideals generated by left nil ideals must be nil) and Levitzki's problem (which asks whether there exist simple nil rings). Many other questions are also extensively studied. Recently several new interesting and deep results were obatined. In this paper we survey and discuss them making some comments and remarks and raising new questions. We also present several relevant older results.

In [7, 31] one can find many other related questions, in particular on tensor products of nil algebras. We do not discuss them here because no progress in this area was made in the last years.

All considered rings are associative but not necessarily have identities. The ring (or algebra if we consider algebras) R with an identity adjoined will be denoted by R^*.

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To denote that I is a two-sided ideal (left ideal) of a ring R we write $I \lhd R$ ($I <_l R$). Obviously if L is a left ideal of R, then the two-sided ideal of R generated by L is equal LR^*.

The prime and the Jacobson radicals will be denoted by β and $\mathcal{J}$, respectively.

2. Koethe's Problem

It is not difficult to check that if $L <_l R$ and L is locally nilpotent, then so is LR^*. In some classes of rings, e.g. PI-rings or rings with Krull dimension (in particular Noetherian rings), left nil ideals are locally nilpotent. Hence in these classes of rings Koethe's problem has a positive solution. For some time it was hoped that all nil rings might be locally nilpotent. This obviously would solve Koethe's problem positively. However the famous examples constructed by Golod and Shafarevich [9, 10] show that this is not true.

Another approach to Koethe's problem was applied in [1] by Amitsur. He proved

Theorem 2.1. *If R is an algebra over a field F and $dim_F R < cardF$, then the Jacobson radical $\mathcal{J}(R)$ of R is nil.*

This theorem easily implies that Koethe's problem has a positive solution in the classes of algebras over uncountable fields. Indeed, let L be a left nil ideal of an algebra A over an uncountable field F. For every $a \in LA^*$, there are $l_1, ..., l_n \in L$ and $a_1, ..., a_n \in A^*$ such that $a = l_1a_1 + ... + l_na_n$. Let B be the subalgebra of A^* generated by $l_1, ...l_n, a_1, ..., a_n$. Clearly $L \cap B$ is a left nil ideal of B, so $a \in (L \cap B)B^* \subseteq \mathcal{J}(B)$. Since B is finitely generated, $dim_F B < cardF$. Hence the theorem implies that a is nilpotent.

The following result due to Krempa shows that studying Koethe's problem one can concentrate on algebras over fields only.

Theorem 2.2. (Krempa [22]). *Koethe's problem has a positive solution if and* *only if it has a positive solution for F-algebras over each field F.*

This result and a reasoning similar to that for uncountable fields show that Koethe's problem would have a positive solution if we knew that finitely generated algebras over fields necessarily have nil Jacobson radicals. It was hoped that it might be true. However this is not always the case. As it was noted by Beidar [4] a counterexample can be easily constructed by employing the following general result due to Markov (Beidar gave also another example based on Markov's theorem). We sketch the argument.

Theorem 2.3. (Markov [25]). *Suppose that $\Re$ is an isomorphically closed class of algebras over a field F such that*

(i) $\Re$ contains an algebra A with $dim_F A \leq \aleph_0$;

(ii) If $I \lhd R$, $I^2 = 0$ and $R/I \in \Re$ then $R \in \Re$.

Then there exists an F-algebra with an identity generated by two elements which has a non-zero left ideal in $\Re$.

Proof. Let $T = F\{X, Y\}$ be the algebra of polynomials in non-commuting indeterminates X and Y with coefficients in F and let $L = TX$. It is not difficult to check that L is isomorphic to the free F-algebra P in indeterminates $x_1, x_2, \ldots$. Since $dim_F A \leq \aleph_0$, P can be homomorphically mapped onto A. Hence there exists an ideal I in L such that $L/I \simeq A$.

Observe that $LIT \lhd T$ and $I^2 \subseteq (LIT) \cap L$. Moreover, since $T = F + TX + TY$, we have $(LIT) \cap L = LI(F + TX + TY) \cap L \subseteq (LI + LIL + LITY) \cap L \subseteq (I + TY) \cap L$. However $I \subset L$ and $TY \cap L = TY \cap TX = 0$, so $(I + TY) \cap L = I + ((TY) \cap L) = I$. Consequently

$$I^2 \subseteq LIT \cap L \subseteq I.$$

Now $(L/(L \cap LIT))/(I/(L \cap LIT)) \simeq L/I \in \Re$ and $I/(L \cap LIT) \lhd L/(L \cap LIT)$, $(I/(L \cap LIT))^2 = 0$, so by (ii), $L/(L \cap LIT) \in \Re$. Moreover $L/(L \cap LIT) \simeq (L + LIT)/LIT <_l T/LIT$. Thus T/LIT is an F-algebra with an identity generated by two generators (images of X and Y) which contains the non-zero left ideal $(L + LIT)/LIT$ in $\Re$. $\square$

Now let F be a countable field and let $\Re$ be the class of all Jacobson radical F-algebras which are not nil. It is clear that $\Re$ satisfies the second condition of Markov's theorem. To see that it also satisfies the condition (i) one can take the Jacobson radical of the ring of polynomials in one indeteminate x over F localized at the ideal generated by x. Thus Markov's theorem implies that there exists an algebra A with an identity generated by two generators and containing a non-zero left ideal $L \in \Re$. Clearly the Jacobson radical of A is not nil.

Remark. Note that to solve Koethe's problem it is enough to prove that sums of left nil algebras over fields are nil. Indeed, let L be a left nil ideal of an algebra A over a field F. Then for each $a \in A^*$ and every $l \in L$, $(la)^{n+1} = l(al)^n a$. Hence La is a left nil ideal of A. Now for every $x \in LA^*$ there are $a_1, ..., a_m \in A^*$ such that $x \in La_1 + ... + La_m$. Hence if sums of left nil ideals are nil, then x is nilpotent. Consequently LA^* is nil.

Clearly to check that sums of left nil ideals are nil it suffices to do this for two such ideals. Thus let L_1 and L_2 be left nil ideals of an algebra A. Pick $l_1 \in L_1$ and $l_2 \in L_2$ and denote by B the subalgebra of A generated by l_1 and l_2. Now $l_1 \in L_1 \cap B$ and $l_2 \in L_2 \cap B$ and both $L_1 \cap B$ and $L_2 \cap B$ are left nil ideals of B. Hence $l_1, l_2 \in \mathcal{J}(B)$, so B is Jacobson radical. Therefore $L_1 + L_2$ would be nil if we could prove that finitely generated Jacobson radical algebras over fields are nil. However this is also not true. A counterexample based on Markov's and Golod-Shafarevich's ideas was recently constructed by Smoktunowicz (unpublished).

$\square$

The following question remains open.

Question 1. Do the Jacobson and nil radicals coincide for finitely presented algebras, i.e., factor algebras of finitely generated free algebras modulo finitely generated ideals?

We conclude this section with a result due to Krempa which gives two equivalent formulations of Koethe's problem. This result inspired many further studies not only connected with Koethe's problem.

Given a ring R, we shall denote by $M_2(R)$ the ring of 2×2-matrices over R and by $R[x]$ the ring of polynomials in an indeterminate x over R.

Theorem 2.4. ([22]). *The following are equivalent*

(i) Koethe's problem has a positive solution;

(ii) if R is a nil ring, then $M_2(R)$ is a nil ring;

(iii) if R is a nil ring, then $R[x]$ is a Jacobson radical ring.

Other proofs of Theorem 2.4 can be found in [3] and [23]. The equivalence of (i) and (ii) was independently obtained by Sands [37].

Polynomial rings and monomial algebras

In a connection with Theorem 2.4 it was asked [3, 22] whether for every nil ring R, the ring $R[x]$ is nil. Applying Theorem 2.1 and Theorem 2.4 (or direct arguments) it is not difficult to show that the answer is positive if R is an algebra over an uncountable field. Recently Smoktunowicz [38] constructed an example showing that the question has a negative answer for algebras over countable fields. The general idea of the example is similar to that of Golod and Shafarevich applied in [9, 10] (cf. also [28]) in their construction of a finitely generated nil but not nilpotent algebra. Namely, take a finitely generated free algebra A over a countable field F. Obviously A is countable, i.e., $A = \{a_1, a_2, ...\}$. Take the factor algebra A/I, where I is the ideal of A generated by $a_1^{n_1}, a_2^{n_2}, ...$ for a sequence $n_1, n_2, ...$ of natural numbers. Clearly A/I is nil and Smoktunowicz proved that if the sequence $n_1, n_2, ...$ increases quickly enough, then the polynomial algebra $(A/I)[X, Y]$ in two commuting indeterminates over A/I is not nil. Then A/I or $(A/I)[X]$ (it was not clarified which one) is a nil ring, the polynomial ring in one indeterminate over which is not nil. More precisely in [38] the following theorem was proved.

Theorem 3.1. *Let F be a countable field and let $A = \{a_1, a_2, ...\}$ be the free F-*

algebra in indeterminates x, y, z. Let I be the ideal of A generated by $\{a_i^{10m_{i+1}} \mid i = 1, 2, ...\}$, where $m_1, m_2, ...$ are natural numbers satisfying

i) $m_1 > 10^8$ and $m_{i+1} > m_i 2^{i+101}$ for $i \geq 1$;

ii) $m_i > 3^{6 \deg a_i}$ for $i \geq 1$;

iii) each m_i divides m_{i+1}.

Then $(A/I)[X, Y]$ is not nil.

The choice of the exponents in the theorem is so complicated for some technical reasons appearing in the proof (which is quite difficult). It is however not hard to see that one could take any enough quickly increasing sequence n_i and then for the ideal J of A generated as I with exponents n_i, one would get that A/J is nil but $(A/J)[X, Y]$ is not. This case can be just easily reduced to that appearing in the theorem. Note also that if R is a finitely generated nil algebra over a field F such that $R[x]$ is not Jacobson radical, then by the above quoted results, F must be countable. Obviously R is isomorphic to A/I, for a finitely generated free F-algebra A and an ideal I of A. Now A is countable, so $A = \{a_1, a_2, ...\}$. Since R is nil, there are natural numbers $n_1, n_2, ...$ such that $a_i^{n_i} \in I$. Of course if we take the ideal J generated by all $a_i^{n_i}$, then we get that $J \subseteq I$. Clearly A/J is nil and $R[x]$ is a homomorphic image of $(A/J)[x]$, so $(A/J)[x]$ is not Jacobson radical. These remarks show that the idea applied by Smoktunowicz might be useful in constructing a counterexample to Koethe's problem.

Recall that a ring is Brown-McCoy radical if and only if it cannot be homomorphically mapped onto a ring with an identity. The Brown-McCoy radical is larger than the Jacobson radical. Hence a first step in proving that Koethe's problem has a positive solution might be to show that for every nil ring R the ring $R[x]$ is Brown-McCoy radical. The question whether it is so was raised in [31]. A positive answer was recently obtained in [33] as a consequence of the following more general result.

Theorem 3.2. *Given a ring R, the ring $R[x]$ can be homomorphically mapped* *onto a ring with an identity if and only if the ring R can be homomorphically mapped onto a prime ring containing a non-zero central element.*

This theorem implies also the following.

Corollary 3.3. *If $L <_l R$ and L is nil, then $(LR^*)[x]$ is Brown-McCoy radical.*

Proof. If $(LR^*)[x]$ is not Brown-McCoy radical, then by Theorem 3.2 there is a prime ideal I of LR^* such that LR^*/I contains a non-zero central element a. It is not hard to see that (since I is a prime ideal of LR^*) I is an ideal of R^*. Moreover if J is an ideal of R^* maximal with respect to $J \cap (LR^*) = I$, then J is a prime ideal of R^* and $LR^*/I = LR^*/(J \cap LR^*) \simeq (LR^* + J)/J = ((L+J)/J)(R^*/J)$ and that the image of a in $((L+J)/J)(R^*/J)$ is a regular element of R^*/J. Factoring out the ideal J we can assume that R^* is a prime ring. Then a is a regular element of R^* contained in LR^*. Let R^*_a be the ring R^* localized at the set $\{a^n \mid n = 0, 1, 2...\}$, where a^0 denotes the identity of R^*. Now $L_a = \{la^{-n} \mid l \in L, n = 0, 1, 2, ...\}$ is a left nil ideal of R^*_a, so the two-sided ideal T of R^* generated by L_a is a Jacobson radical ring. This is impossible because $a \in LR^*$ implies that T is a ring with an identity.

□

Remark. Note that it is not true that if $L <_l R$ and $L[x]$ is Brown-McCoy radical, then $(LR^*)[x]$ is Brown-McCoy radical. Indeed, let R be a simple domain with an identity which is not a division ring. Then R contains a non-zero proper left ideal L. One can check that L is a simple ring without an identity, so L cannot be homomorphically mapped onto a ring containing a non-zero central element. Consequently by Theorem 3.2, $L[x]$ is Brown-McCoy radical. However $LR^* = R$, so $(LR^*)[x]$ is not Brown-McCoy radical.

□

To make a further progress one can try to answer the following questions.

Question 2. Suppose that R is a nil ring.

a) Is it true that $R[x]$ cannot be homomorphically mapped onto a ring containing a non-zero idempotent?;

b) Is the ring $R[X]$ of polynomials in a set X of two or more commuting indeterminates Brown-McCoy radical?;

c) Is the polynomial ring $R\{X\}$ in a set X of two or more non-commuting indeterminates Brown-McCoy radical?

As we have seen the problem of describing the Jacobson radical of polynomial rings in commuting indeterminates is difficult and strictly connected with Koethe's problem. It is much easier to describe the Jacobson radical of polynomial rings in non-commuting indeterminates. Namely [29] the Jacobson radical of the polynomial ring $R\{X\}$ in a set X of at least two non-commuting indeterminates is locally nilpotent and is equal $L(R)\{X\}$, where $L(R)$ is the locally nilpotent radical of R. Related are some recent results on the Jacobson radical of monomial algebras.

Recall that *monomial algebras* are defined as the factor algebras $F\{X\}/I$, where $F\{X\}$ is the polynomial algebra over a field F in a set X of non-commuting indeterminates and I is an ideal of $F\{X\}$ generated by monomials.

In [26, 27] Okninski asked whether the Jacobson radical of monomial algebras is locally nilpotent. He proved that in the characteristic zero case the answer is positive if and only if the Jacobson radical of the monomial algebra $F\{X\}/I$ regarded as an algebra graded in the natural way by the free group generated by X is homogeneous. Homogenity of the Jacobson radical of monomial algebras was proved in [13]. These answered the question for monomial algebras of characteristic zero. Next in [5] Belov and Gateva-Ivanova proved that the Jacobson radical of monomial algebras is nil. Finally in [6] Beidar and Fong answerd Okninski's question in the positive.

4. Power series rings

It is not difficult to prove [29] that the power series ring $R\{\{X\}\}$ in a set of at least two non-commuting indeterminates is nil if and only if the ring R is nilpotent and it is nil semisimple if and only if $\beta(R) = 0$.

Recently [32] the following result, giving in particular a complete description of the nil radical of power series rings in non-commuting indeterminates, was obtained. It answers positively a question of [31]. This characterization was in [30] obtained for uncountable sets of indeterminates.

Theorem 4.1. *An element a of the power series ring $R\{\{X\}\}$ in a set X of at least two non-commuting indeterminates belongs to a right nil ideal of $R\{\{X\}\}$ if and only if the right ideal of R generated by the coefficients of a is nilpotent.*

The problem concerning a description of the nil radical of power series rings in one indeterminate is much more complicated. In [19] Klein proved that if a ring R is nil of bounded index, then so is $R[x]$. From this result it easily follows that if R is nil of bounded index, then also the ring $R\{\{x\}\}$ of power series in an indeterminate x over R is nil of bounded index. In [30] it was proved that if $R\{\{x\}\}$ is nil, then R is nil of bounded index. Consequently $R\{\{x\}\}$ is nil if and only if R is nil of bounded index. It was natural to expect that an element a of $R\{\{x\}\}$ belongs to the nil radical of $R\{\{x\}\}$ if and only if the ideal of R generated by the coefficients of a is nil of bounded index. This question was raised in [31]. It turns out that it was known ealier that it has a negative answer even for commutative rings. An example showing that was constructed in another context by Hamann and Swan in [11].

In [32] the following characterization of the nil radical of $R\{\{x\}\}$ was obtained.

Theorem 4.2. *The nil radical of $R\{\{x\}\}$ coincides with $\{a \in R\{\{x\}\} \mid aR\{\{x\}\}$ is nil of bounded index$\}$.*

Rings which are sums of two subrings

In [14] Kegel proved that a ring which is a sum of two nilpotent subrings is nilpotent and in [15] he showed that a ring which is a sum of a nilpotent subring and a locally nilpotent subring is locally nilpotent. Some extensions of Kegel's results were found in [8]. In particular it was there proved that Koethe's problem is equivalent to the problem of whether a ring which is a sum of a nilpotent subring (or even a subring with zero multiplication) and a nil subring must be nil. These results led to many natural questions (cf. [15, 31]), e.g., whether a ring which is a sum of two nil (locally nilpotent, β-radical) subrings must be nil. It was expected that at least some of these questions have positive answers. For instance in [12] it was conjectured that rings which are sums of two locally nilpotent subrings must be locally nilpotent and the same was expected in [31] for Wedderburn radical subrings (recall that a ring is called Wedderburn radical if it is equal to the sum of its nilpotent ideals). However it turned out that the situation is quite opposite. In [16] Kelarev showed that rings which are sums of two locally nilpotent subrings need not be nil. In his construction he presented the free semigroup W on letters x, y as the union of the subsemigroups $W_1 = \{w \in W \mid deg_x w < deg_y w\}$ and $W_2 = W \setminus W_1$ and found an ideal I in W such that for a field F the contracted semigroup algebra $F_0[W/I]$ was not nil but its subalgebras $F_0[(W_1 \cup I)/I]$ and $F_0[(W_2 \cup I)/I]$ were locally nilpotent. Clearly $F_0[W/I] = F_0[(W_1 \cup I)/I] + F_0[(W_2 \cup I)/I]$. The construction of the ideal I as well as the proofs were quite complicated. In [35] Salwa found simpler and stronger examples. They were based on semigroups of partial translations of intervals of the real line. He presented them also as some monomial algebras which was useful in studying their further properties.

Now we shall present one of the most extreme of Salwa's examples in a bit modified form.

Example. Let α, β be positive real numbers such that α/β is irrational and let F be a field. Let f be the homomorphism of the free semigroup W on letters x, y into the additive group of real numbers such that $f(x) = -\alpha$ and $f(y) = \beta$. Clearly $W_1 = \{w \in W \mid f(w) < 0\}$ and $W_2 = \{w \in W \mid f(w) > 0\}$ are subsemigroups of W and $W = W_1 \cup W_2$ (since α/β is irrational, $f(w) \neq 0$ for $w \in W$). Let I be the ideal of W generated by the set $M = \{w \in W \mid |f(w)| \geq$

$\alpha + \beta\}$.

Observe that if $w \in W \setminus I$ and $wx \in I$, then there are $u', u \in W$ such that $w = u'u$ and $ux \in M$. Obviously $u \notin M$, so $f(u) < \alpha + \beta$ and $f(ux) = f(u) - \alpha < \beta < \alpha + \beta$. Hence since $ux \in M$, we have $f(u) - \alpha = f(ux) \leq -\alpha - \beta$, so $f(u) \leq -\beta$. Similarly if $wy \in I$, then there are $v', v \in W$ such that $w = v'v$ and $f(v) \geq \alpha$. However wx and wy cannot belong to I simultaneously. Namely if they both were in I, then we would have that $u = tv$ or $v = tu$ for some $t \in W$. In the former case $f(t) + \alpha \leq f(t) + f(v) = f(tv) = f(u) \leq -\beta$, so $f(t) \leq -\alpha - \beta$. This implies that $w \in I$, a contradiction. Similarly if $v = tu$, then $f(t) \geq \alpha + \beta$ and again $w \in I$ which is impossible. This shows that the image of $x + y$ in $R = F_0[W/I]$ is left regular. Symmetric arguments show that it is right regular.

If S is a finite subset of W_1 and $i_1, ..., i_n$ belong to the ideal of W_1 generated by S, then $f(i_1 ... i_n) \leq n max\{f(s) \mid s \in S\}$. This implies that every element of $R_1 = F_0[(W_1 \cup I)/I]$ belongs to a nilpotent ideal of R_1. Similarly every element of $R_2 = F_0[(W_2 \cup I)/I]$ belongs to a nilpotent ideal of R_2. Thus both R_1 and R_2 are Wedderburn radical.

Obvioulsy $R = R_1 + R_2$. Thus R is a ring containing a regular element (in particular it is not a nil ring) which is the sum of Wedderburn radical subrings R_1 and R_2.

□

Salwa proved that this example has the following further properties:

1. R is primitive,
2. R is nilpotent modulo each non-zero ideal,
3. The Gelfand-Kirillov dimension of R is 2.

In a connection with the last of these properties Salwa proved the following two results. The former of them shows that in some sense his example is minimal.

Theorem 5.1. *Assume that A is a finitely generated algebra over a field. If A is a sum of two nil subalgebras and the Gelfand-Kirillov dimension of A is < 2,* *then A is nilpotent.*

Theorem 5.2. *If an algebra A over a field is a sum of two nil subalgebras and the Gelfand-Kirillov dimension of A is 0, then A is locally nilpotent.*

Remark. It is clear that given a real number $\gamma \geq \alpha + \beta$ the construction of the example applied to $M = \{w \in W \mid |f(w)| > \gamma\}$ gives an algebra containing a regular element which is a sum of two Wedderburn radical subalgebras. However such algebras have higher Gelfand-Kirillov dimension. All of them can be homomorphically mapped onto so constructed algebra for $\gamma = \alpha + \beta$ and this algebra has, as it was proved by Salwa, the Gelfand-Kirillov dimension equal 3.

In [17] Kelarev considered the particular case of such algebras for $\alpha = \sqrt{2}$, $\beta = 1$ and $\gamma = 3$. He found simpler arguments to show that in characteristic zero the obtained algebra is not nil.

□

Remark. During his talk at the XV Escola de Álgebra Prof. Y. Bakhturin announced that Fukshansky showed that rings which are sums of two locally nilpotent subrings can contain free subrings. Let us observe that applying Salwa's example it is easy to show that such rings can be also found among rings which are sums of two Wedderburn radical subrings. Indeed, it is not hard to check that if r is a non-nilpotent element of a ring R, then the subring of the polynomial ring $R\{x,y\}$ in two non-commuting indeterminates x, y over R generated by rx, ry is a free ring in two indeterminates. Obviously if R is a sum of two Wedderburn radical subring, then so is $R\{x,y\}$.

□

In [18] it was proved that rings which are sums of two nil subrings of bounded index are also nil of bounded index. However no relation between the indices is known. Klein obtained the same result [20] when the subrings are left ideals but also did not find such a relation.

We conclude with some questions.

Question 3. Can a ring which is a sum of a nilpotent ring and a nil ring contain a regular element?

From the quoted above result of [8] it follows that such a ring would give a counterexample to Koethe's problem. Thus one would rather expect a negative answer to this question.

Question 4. Can a ring which is a sum of two nil (locally nilpotent, β-radical, etc.) subrings contain a non-zero idempotent?

Question 5. Is every finitely generated nil ring which is a sum of two Wedderburn radical subrings nilpotent?

Question 6. Does there exist a simple ring which is a sum of two nil (locally nilpotent, β-radical, etc.) subrings?

In [18] it was proved that there exists a simple ring which is a sum of a nilpotent subring and a nil subring if and only if there exists a simple nil ring.

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