VIRTUALLY FREE PRO-P GROUPS

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In this paper we overview recent results on virtually free pro-p groups, i.e., pro-p groups having open free subgroups. We describe also applications of these results to the study of automorphisms of finite order of free pro-p groups. Some of the interest in this topic comes from number theory; for instance, virtually free pro-2 groups have been studied in the contex of Galois theory in [Haran 93], [Engler 95].

Well-known result on virtually free pro-p groups was obtained by Serre in a seminal paper [Serre 65]:

Theorem 1. [Serre 65] *Let G be a pro-p-group and F an open free pro-p subgroup of G. If G is torsion free, then G is also a free pro-p group.*

The next result is the pro-p analogue of the Dyer-Scott theorem [Dyer-Scott 75]. It has been proved by Scheiderer in the finitely generated case and extended by W.Herfort, L.Ribes and P. Zalesskii to the general case.

Theorem 2. [Scheiderer 98], [Herfort-Ribes-Zalesskii 98]. *If G is a pro-p group having a free pro-p subgroup F of index p then*

$$G \cong (\coprod_{x\in X} (C_p \times H_x) \coprod H,$$

is a free pro-p product, where C_p denotes the group of order p, H_x, H are free pro-p subgroups of F and X is the space of conjugacy classes of subgroups of order p in G.

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A characterization of cyclic extensions of free pro-p groups as the fundamental groups of a graph of groups in the category of pro-p groups is given in the following

Theorem 3. [Herfort-Zalesskii 99] *Let G be a cyclic extension of a free pro-p group, i.e., there is an exact sequence*

$$1 \longrightarrow F \longrightarrow G \longrightarrow C_{p^n} \longrightarrow 1,$$

where F is a free pro-p group and C_{p^n} is a cyclic group of order p^n for $n \in \mathbb{N}$. Then G is the fundamental group of a profinite graph of finite p groups of bounded order.

Furthermore, free-by-cyclic pro-p groups admit an internal description as a free product along the lines of Theorem 2.

Theorem 4. [Herfort-Zalesskii 99] *Let G be a cyclic extension of a free pro-p group F. Then*

$$G \cong \coprod_{x \in X} N_G(C_x) \coprod H,$$

is a free pro-p product, where $\{C_x \mid x \in X\}$ is a system of representatives of conjugacy classes of subgroups of order p in G, $N_G(C_x)$ is the normalizer of C_x in G, and H is a free pro-p subgroup of F.

It turns out that Theorem 3 is the best possible result one can obtain without restrictions on the rank of F. An example of a semidirect product $F \rtimes (C_2 \times C_2)$ which cannot be represented as the fundamental group of a profinite graph of finite p groups is given. The reason for this seems to be purely topological, arising from the fact that for a pro-p group acting on a profinite space X of large cardinality (for example $\aleph_2$) the quotient map $X \longrightarrow X/G$ does not always admit a continuous section. It is reasonable to believe that similar characterization is likely to be true if the rank of the free subgroup F is not so large, in particular, when it is finite. The following result provides strong support for this.

Theorem 5. [Herfort-Ribes-Zalesskii (1) 98] *Let G be a finite extension of free pro-p group of $rank(F) < 3$. Then G is the fundamental pro-p group of a finite graph of finite p-groups.*

In fact, the information obtained in [Herfort-Ribes-Zalesskii (1) 98] is very precise. For example the next theorem gives a very explicit description of finite extensions of a free group of rank 2 having trivial center.

Theorem 6. [Herfort, Ribes, Zalesskii (1) 98] *Let G be a pro-p-group with trivial center having an open normal free subgroup F of rank 2. Then G has one of the following structures:*

1) G is a free pro-p group of finite rank.

2) $p = 3$ and $G \simeq C_3 \coprod C_3$

3) $p = 2$ and G has one of the following forms:

a) $G \simeq C_2 \coprod C_2 \coprod C_2$;

b) $G \simeq C_2 \coprod \mathbb{Z}_2$;

c) $G \simeq C_2 \coprod (C_2 \times \mathbb{Z}_2)$;

d) $G \simeq C_4 \coprod C_2$;

e) $G \simeq (C_2 \times C_2) \coprod C_2$;

f) $G \simeq (\mathbb{Z}_2 \times C_2) \coprod_{C_2} (C_2 \times C_2) \coprod_{C_2} (C_2 \times \mathbb{Z}_2)$

The counterexample mentioned above provides also a counterexample to a pro-p analogue of the Kurosh Subgroup Theorem. In the case of countably based pro-p groups such an analogue was proved by D.Haran [Haran 87] and O.V.Melnikov [Melnikov 90] independently.

We now turn to automorphisms of free pro-p groups. It is well-known that the automorphism group $\mathrm{Aut}(F)$ of a free pro-p group of finite rank is a profinite group, in fact a virtually pro-p group, i.e. $\mathrm{Aut}(F)$ contains an open pro-p subgroup. It follows that one can consider the order of an automorphism of F as supernatural number rp^t, where r is a natural number relatively prime to p and $0 \leq t \leq \infty$.

By a celebrated theorem of Gersten [Gersten 86] the group of fixed points of an automorphism of a free abstract group of finite rank is finitely generated. The next theorem shows that the pro-p analogue of this theorem does not hold, unless one makes some restrictions on the order of the automorphism.

Theorem 7. [Herfort-Ribes 90] *Let F be a free pro-p group of rank$(F) = n > 1$ and α an automorphism of F. If the order m of α is coprime to p, then the rank of the group of fixed points $Fix_F(\alpha)$ is necessarily infinite.*

This result is less surprising if one considers the holomorph $F \rtimes \langle\alpha\rangle$, and $\mathrm{Fix}_F(\alpha)$ as the centralizer of α. Indeed, the properties of virtually free pro-p groups are in many cases similar to those of abstract free groups, but may change radically if one is at some point forced to go outside the class of pro-p groups. Thus, one might resonably conjecture that the appropriate pro-p analogue of Gersten's theorem will require in addition that the holomorph be a pro-p group, or, equivalently, that the order of α be p^t for $0 \leq t \leq \infty$.

Conjecture [Herfort-Ribes-Zalesskii 95] Let F be a free pro-p group of rank n, and α an automorphism of F of order p^t $(0 \leq t \leq \infty)$. Then the rank of $\mathrm{Fix}_F(\alpha)$ is at most n.

If the order of the automorphism α is finite, an affirmative answer to this conjecture follows from Theorem 2. Indeed, in this case the holomorph $F \rtimes \langle\alpha\rangle$ is a virtually free group, and its subgroup $F \rtimes \langle\alpha^{p^{t-1}}\rangle$ satisfies the assumptions of Theorem 2. Hence by applying that theorem subsequently t times in succession (or using induction on t) one deduces the following result.

Theorem 8. [Scheiderer 98], [Herfort-Ribes-Zalesskii 98] *Suppose F is a free pro-p group and α is an automorphism of F of order p^t ($t < \infty$). Then the set of fixed points $Fix_F(\alpha)$ is a free factor of F. In particular, rank*($\mathrm{Fix}_F(\alpha)$) $\leq$*rank(F).*

It is known that the automorphism group $\mathrm{Aut}(F_n)$ of a free pro-p group of rank $n > 1$ is much more complicate than the automorphism group $\mathrm{Aut}(\Phi_n)$ of the abstract free group Φ_n of rank n. Athough $\mathrm{Aut}(\Phi_n)$ is embedded in $\mathrm{Aut}(F_n)$, it is by no means dense there. In fact, V.Romankov has proved that $\mathrm{Aut}(F_n), n > 1$, is (topologically) infinitely generated! (See [Romankov 93]). Nevertheless, Theorem 4 affords a description of the conjugacy classes of the automorphisms of finite order of a free pro-p group.

Theorem 9. [Herfort-Zalesskii 99] *Let F be a free pro-p group.*

(1) There exists a dense abstract free subgroup Φ of the same rank as F, such that each conjugacy class of automorphisms of order p^n in $Aut(F)$ intersects precisely one conjugacy class of automorphisms of order p^n in $Aut(\Phi)$;

(2) The conjugacy classes of automorphisms of F having order q coprime to p are in one-to-one correspondence to the conjugacy classes of automorphisms of order q of the Frattini quotient F/F^*.

An application of Theorem 6 gives a precise list of the number of conjugacy classes of a given order automorphism of a free pro-p group of rank 2.

Theorem 10. [Herfort, Ribes, Zalesskii (1) 98] *Let S denote the set of all possible orders of torsion elements of the automorphism group* $\mathrm{Aut}(F_2)$ *of a free pro-p group of rank 2. Let $c(s)$ be the number of conjugacy classes of automorphisms of order s. Then*

(1) if s is coprime to p, then $s \mid p^2 - 1$, and one of the following holds:

(a) $s \mid p-1$ and $c(s) = \phi(s)s$

(b) $s \mid p-1$ and $c(s) = \frac{\phi(s)}{2}$,

where ϕ denotes Euler's function;

(2) if $p = 2$, then $S = \{2, 3, 4\}$ and $c(2) = 4$, $c(3) = 1$, $c(4) = 1$;

(3) if $p = 3$, then $S = \{2, 3, 4, 8\}$ and $c(2) = 2$, $c(3) = 1$, $c(4) = 1$, $c(8) = 2$;

(4) if $p > 3$, then any $s \in S$ is coprime to p, so that the formulas in (1) hold.

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